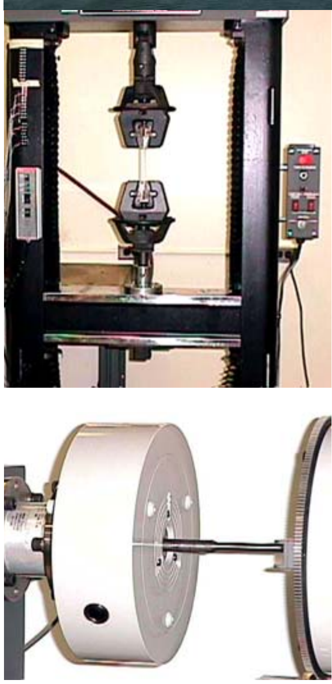
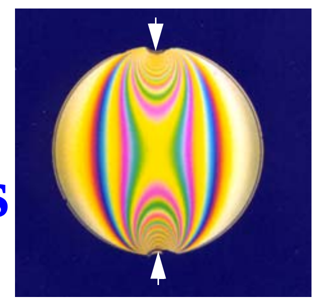


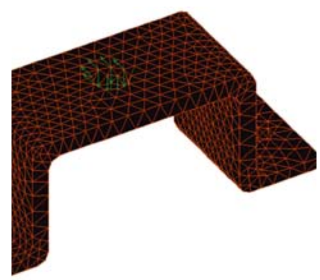


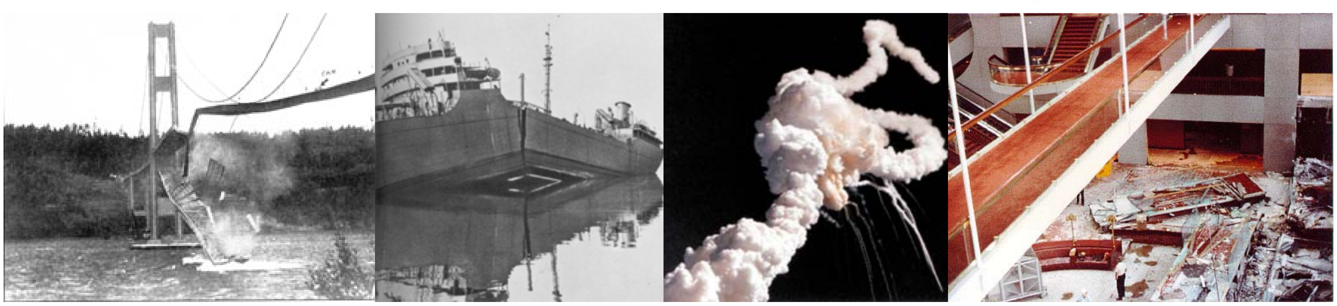
Mechanics of Materials

Second Edition

Madhukar Vable

Michigan Technological University





DEDICATED TO MY FATHER
Professor Krishna Rao Vable
(1911--2000)

AND MY MOTHER
Saudamini Vable
(1921--2006)

CONTENTS

CHAPTER SIX SYMMETRIC BENDING OF BEAMS

CHAPTER SEVEN DEFLECTION OF SYMMETRIC BEAMS

CHAPTER TEN DESIGN AND FAILURE

CHAPTER ELEVEN STABILITY OF COLUMNS

PREFACE

Mechanics is the body of knowledge that deals with the relationships between forces and the motion of points through space, including the material space. Material science is the body of knowledge that deals with the properties of materials, including their mechanical properties. Mechanics is very deductive—having defined some variables and given some basic premises, one can logically deduce relationships between the variables. Material science is very empirical—having defined some variables one establishes the relationships between the variables experimentally. Mechanics of materials synthesizes the empirical relationships of materials into the logical framework of mechanics, to produce formulas for use in the design of structures and other solid bodies.

There has been, and continues to be, a tremendous growth in mechanics, material science, and in new applications of mechanics of materials. Techniques such as the finite-element method and Moiré interferometry were research topics in mechanics, but today these techniques are used routinely in engineering design and analysis. Wood and metal were the preferred materials in engineering design, but today machine components and structures may be made of plastics, ceramics, polymer composites, and metal-matrix composites. Mechanics of materials was primarily used for structural analysis in aerospace, civil, and mechanical engineering, but today mechanics of materials is used in electronic packaging, medical implants, the explanation of geological movements, and the manufacturing of wood products to meet specific strength requirements. Though the principles in mechanics of materials have not changed in the past hundred years, the presentation of these principles must evolve to provide the students with a foundation that will permit them to readily incorporate the growing body of knowledge as an extension of the fundamental principles and not as something added on, and vaguely connected to what they already know. This has been my primary motivation for writing this book.

Often one hears arguments that seem to suggest that intuitive development comes at the cost of mathematical logic and rigor, or the generalization of a mathematical approach comes at the expense of intuitive understanding. Yet the icons in the field of mechanics of materials, such as Cauchy, Euler, and Saint-Venant, were individuals who successfully gave physical meaning to the mathematics they used. Accounting of shear stress in the bending of beams is a beautiful demonstration of how the combination of intuition and experimental observations can point the way when self-consistent logic does not. Intuitive understanding is a must—not only for creative engineering design but also for choosing the marching path of a mathematical development. By the same token, it is not the heuristic-based arguments of the older books, but the logical development of arguments and ideas that provides students with the skills and principles necessary to organize the deluge of information in modern engineering. Building a complementary connection between intuition, experimental observations, and mathematical generalization is central to the design of this book.

Learning the course content is not an end in itself, but a part of an educational process. Some of the serendipitous development of theories in mechanics of materials, the mistakes made and the controversies that arose from these mistakes, are all part of the human drama that has many educational values, including learning from others' mistakes, the struggle in understanding difficult concepts, and the fruits of perseverance. The connection of ideas and concepts discussed in a chapter to advanced modern techniques also has educational value, including continuity and integration of subject material, a starting reference point in a literature search, an alternative perspective, and an application of the subject material. Triumphs and tragedies in engineering that arose from proper or improper applications of mechanics of materials concepts have emotive impact that helps in learning and retention of concepts according to neuroscience and education research. Incorporating educational values from history, advanced topics, and mechanics of materials in action or inaction, without distracting the student from the central ideas and concepts is an important complementary objective of this book.

The achievement of these educational objectives is intricately tied to the degree to which the book satisfies the pedagogical needs of the students. The Note to Students describes some of the features that address their pedagogical needs. The Note to the Instructor outlines the design and format of the book to meet the described objectives.

I welcome any comments, suggestions, concerns, or corrections you may have that will help me improve the book. My e-mail address is mavable@mtu.edu.

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Thirty reviewers looked at my manuscript and checked the numerical examples. Thanks to the following and anonymous reviewers whose constructive criticisms have significantly improved this book.

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Figure Number	Description	Web Address
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7.25*b*	Taipei 101	http://commons.wikimedia.org/wiki/File:31-January-2004-Taipei101-Complete.jpg
7.25*c*	Joint construction.	http://commons.wikimedia.org/wiki/File:Old_timer_structural_worker2.jpg
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A NOTE TO STUDENTS

Some of the features that should help you meet the learning objectives of this book are summarized here briefly.

- A course in statics is a prerequisite for this course. Appendix A reviews the concepts of statics from the perspective of this course. If you had statics a few terms ago, then you may need to review your statics textbook before the brevity of presentation in Appendix A serves you adequately. If you feel comfortable with your knowledge of statics, then you can assess for yourself what you need to review by using the *Statics Review Exams* given in Appendix A.
- All internal forces and moments are printed in ***bold italics.*** This is to emphasize that the internal forces and moments must be determined by making an imaginary cut, drawing a free-body diagram, and using equilibrium equations or by using methods that are derived from this approach.
- Every chapter starts by listing the major learning objective(s) and a brief description of the motivation for studying the chapter.
- Every chapter ends with *Points and Formulas to Remember,* a one-page synopsis of non-optional topics. This brings greater focus to the material that must be learned.
- Every *Example* problem starts with a *Plan* and ends with *Comments*, both of which are specially set off to emphasize the importance of these two features. Developing a plan before solving a problem is essential for the development of analysis skills. Comments are observations deduced from the example, highlighting concepts discussed in the text preceding the example, or observations that suggest the direction of development of concepts in the text following the example.
- *Quick Tests* with solutions are designed to help you diagnose your understanding of the text material. To get the maximum benefit from these tests, take them only after you feel comfortable with your understanding of the text material.
- After a major topic you will see a box called *Consolidate Your Knowledge*. It will suggest that you either write a synopsis or derive a formula. *Consolidate Your Knowledge* is a learning device that is based on the observation that it is easy to follow someone else's reasoning but significantly more difficult to develop one's own reasoning. By deriving a formula with the book closed or by writing a synopsis of the text, you force yourself to think of details you would not otherwise. When you know your material well, writing will be easy and will not take much time.
- Every chapter has at least one module called *MoM in Action*, describing a triumph or a tragedy in engineering or nature. These modules describe briefly the social impact and the phenomenological explanation of the triumph or tragedy using mechanics of materials concept.
- Every chapter has a section called *Concept Connector,* where connections of the chapter material to historical development and advanced topics are made. History shows that concepts are not an outcome of linear logical thinking, but rather a struggle in the dark in which mistakes were often made but the perseverance of pioneers has left us with a rich inheritance. Connection to advanced topics is an extrapolation of the concepts studied. Other reference material that may be helpful in the future can be found in problems labeled "Stretch yourself."
- Every chapter ends with *Chapter Connector,* which serves as a connecting link to the topics in subsequent chapters. Of particular importance are chapter connector sections in Chapters 3 and 7, as these are the two links connecting together three major parts of the book.
- A *glossary* of all the important concepts is given in Appendix C.7 for easy reference.Chapters number are identified and in the chapter the corresponding word is highlighted in bold.
- At the end is a *Formula Sheet* for easy reference. Only equations of non-optional topics are listed. There are no explanations of the variables or the equations in order to give your instructor the option of permitting the use of the formula sheet in an exam.

A NOTE TO THE INSTRUCTOR

The best way I can show you how the presentation of this book meets the objectives stated in the Preface is by drawing your attention to certain specific features. Described hereafter are the underlying design and motivation of presentation in the context of the development of theories of one-dimensional structural elements and the concept of stress. The same design philosophy and motivation permeate the rest of the book.

Figure 3.15 (page 93) depicts the logic relating displacements—strains—stresses—internal forces and moments—external forces and moments. The logic is intrinsically very modular—equations relating the fundamental variables are independent of each other. Hence, complexity can be added at any point without affecting the other equations. This is brought to the attention of the reader in Example 3.5, where the stated problem is to determine the force exerted on a car carrier by a stretch cord holding a canoe in place. The problem is first solved as a straightforward application of the logic shown in Figure 3.15. Then, in comments following the example, it is shown how different complexities (in this case nonlinearities) can be added to improve the accuracy of the analysis. Associated with each complexity are post-text problems (numbers written in parentheses) under the headings "Stretch yourself" or "Computer problems," which are well within the scope of students willing to stretch themselves. Thus the central focus in Example 3.5 is on learning the logic of Figure 3.15, which is fundamental to mechanics of materials. But the student can appreciate how complexities can be added to simplified analysis, even if no "Stretch yourself" problems are solved.

This philosophy, used in Example 3.5, is also used in developing the simplified theories of axial members, torsion of shafts, and bending of beams. The development of the theory for structural elements is done rigorously, with assumptions identified at each step. Footnotes and comments associated with an assumption directs the reader to examples, optional sections, and "Stretch yourself" problems, where the specific assumption is violated. Thus in Section 5.2 on the theory of the torsion of shafts, Assumption 5 of linearly elastic material has a footnote directing the reader to see "Stretch yourself" problem 5.52 for nonlinear material behavior; Assumption 7 of material homogeneity across a cross section has a footnote directing the reader to see the optional "Stretch yourself" problem 5.49 on composite shafts; and Assumption 9 of untapered shafts is followed by statements directing the reader to Example 5.9 on tapered shafts. Table 7.1 gives a synopsis of all three theories (axial, torsion, and bending) on a single page to show the underlying pattern in all theories in mechanics of materials that the students have seen three times. The central focus in all three cases remains the simplified basic theory, but the presentation in this book should help the students develop an appreciation of how different complexities can be added to the theory, even if no "Stretch yourself" problems are solved or optional topics covered in class.

Compact organization of information seems to some engineering students like an abstract reason for learning theory. Some students have difficulty visualizing a continuum as an assembly of infinitesimal elements whose behavior can be approximated or deduced. There are two features in the book that address these difficulties. I have included sections called *Prelude to Theory* in 'Axial Members', 'Torsion of Circular Shafts' and 'Symmetric Bending of Beams.' Here numerical problems are presented in which discrete bars welded to rigid plates are considered. The rigid plates are subjected to displacements that simulate the kinematic behavior of cross sections in axial, torsion or bending. Using the logic of Figure 3.15, the problems are solved—effectively developing the theory in a very intuitive manner. Then the section on theory consists essentially of formalizing the observations of the numerical problems in the prelude to theory. The second feature are actual photographs showing nondeformed and deformed grids due to axial, torsion, and bending loads. Seeing is believing is better than accepting on faith that a drawn deformed geometry represents an actual situation. In this manner the complementary connection between intuition, observations, and mathematical generalization is achieved in the context of one-dimensional structural elements.

Double subscripts[1] are used with all stresses and strains. The use of double subscripts has three distinct benefits. (i) It provides students with a procedural way to compute the direction of a stress component which they calculate from a stress formula. The procedure of using subscripts is explained in Section 1.3 and elaborated in Example 1.8. This procedural determination of the direction of a stress component on a surface can help many students overcome any shortcomings in intu-

[1]Many authors use double subscripts with shear stress but not for normal stress. Hence they do not adequately elaborate the use of these subscripts when determining the direction of stress on a surface from the sign of the stress components.

itive ability. (ii) Computer programs, such as the finite-element method or those that reduce full-field experimental data, produce stress and strain values in a specific coordinate system that must be properly interpreted, which is possible if students know how to use subscripts in determining the direction of stress on a surface. (iii) It is consistent with what the student will see in more advanced courses such as those on composites, where the material behavior can challenge many intuitive expectations.

But it must be emphasized that the use of subscripts is to complement not substitute an intuitive determination of stress direction. Procedures for determining the direction of a stress component by inspection and by subscripts are briefly described at the end of each theory section of structural elements. Examples such as 4.3 on axial members, 5.6 and 5.9 on torsional shear stress, and 6.8 on bending normal stress emphasize both approaches. Similarly there are sets of problems in which the stress direction must be determined by inspection as there are no numbers given—problems such as 5.23 through 5.26 on the direction of torsional shear stress; 6.35 through 6.40 on the tensile and compressive nature of bending normal stress; and 8.1 through 8.9 on the direction of normal and shear stresses on an inclined plane. If subscripts are to be used successfully in determining the direction of a stress component obtained from a formula, then the sign conventions for drawing internal forces and moments on free-body diagrams must be followed. Hence there are examples (such as 6.6) and problems (such as 6.32 to 6.34) in which the signs of internal quantities are to be determined by sign conventions. Thus, once more, the complementary connection between intuition and mathematical generalization is enhanced by using double subscripts for stresses and strains.

Other features that you may find useful are described briefly.

All optional topics and examples are marked by an asterisk (*) to account for instructor interest and pace. Skipping these topics can at most affect the student's ability to solve some post-text problems in subsequent chapters, and these problems are easily identifiable.

Concept Connector is an optional section in all chapters. In some examples and post-text problems, reference is made to a topic that is described under concept connector. The only purpose of this reference is to draw attention to the topic, but knowledge about the topic is not needed for solving the problem.

The topics of stress and strain transformation can be moved before the discussion of structural elements (Chapter 4). I strived to eliminate confusion regarding maximum normal and shear stress at a point with the maximum values of stress components calculated from the formulas developed for structural elements.

The post-text problems are categorized for ease of selection for discussion and assignments. Generally speaking, the starting problems in each problem set are single-concept problems. This is particularly true in the later chapters, where problems are designed to be solved by inspection to encourage the development of intuitive ability. Design problems involve the sizing of members, selection of materials (later chapters) to minimize weight, determination of maximum allowable load to fulfill one or more limitations on stress or deformation, and construction and use of failure envelopes in optimum design (Chapter 10)—and are in color. "Stretch yourself" problems are optional problems for motivating and challenging students who have spent time and effort understanding the theory. These problems often involve an extension of the theory to include added complexities. "Computer" problems are also optional problems and require a knowledge of spreadsheets, or of simple numerical methods such as numerical integration, roots of a nonlinear equation in some design variable, or use of the least-squares method. Additional categories such as "Stress concentration factor," "Fatigue," and "Transmission of power" problems are chapter-specific *optional* problems associated with optional text sections.

CHAPTER ONE
STRESS

Learning objectives

1. Understanding the concept of stress.
2. Understanding the two-step analysis of relating stresses to external forces and moments.

On January 16th, 1943 a World War II tanker S.S. Schenectady, while tied to the pier on Swan Island in Oregon, fractured just aft of the bridge and broke in two, as shown in Figure 1.1. The fracture started as a small crack in a weld and propagated rapidly overcoming the strength of the material. But what exactly is the strength? How do we analyze it? To answer these questions, we introduce the concept of *stress*. Defining this variable is the first step toward developing formulas that can be used in strength analysis and the design of structural members.

Figure 1.1 Failure of S.S. Schenectady.

Figure 1.2 shows two links of the logic that will be fully developed in Section 3.2. What motivates the construction of these two links is an idea introduced in Statics—analysis is simpler if any distributed forces in the free-body diagram are replaced by equivalent forces and moments before writing equilibrium equations (see Appendix A.6). Formulas developed in mechanics of materials relate stresses to internal forces and moments. Free-body diagrams are used to relate internal forces and moments to external forces and moments.

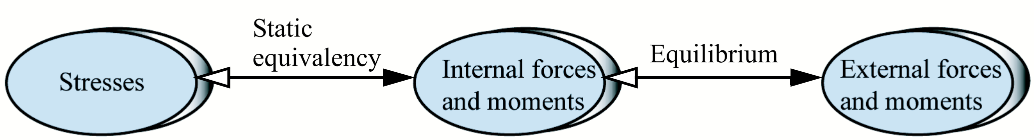


Figure 1.2 Two-step process of relating stresses to external forces and moments.

1.1 STRESS ON A SURFACE

The stress on a surface is an internally distributed force system that can be resolved into two components: normal (perpendicular) to the imaginary cut surface, called *normal* stress, and tangent (parallel) to the imaginary cut surface, called *shear* stress.

1.1.1 Normal Stress

In Figure 1.3, the cable of the chandelier and the columns supporting the building must be strong enough to support the weight of the chandelier and the weight of the building, respectively. If we make an imaginary cut and draw the free-body diagrams, we see that forces normal to the imaginary cut are needed to balance the weight. The *internal* normal force $\boldsymbol{N}$ divided by the area of the cross section A exposed by the imaginary cut gives us the average intensity of an internal normal force distribution, which we call the average **normal** stress:

$$\boxed{\sigma_{\text{av}} = \frac{\boldsymbol{N}}{A}} \tag{1.1}$$

where σ is the Greek letter sigma used to designate normal stress and the subscript av emphasizes that the normal stress is an average value. We may view σ_{av} as a uniformly distributed normal force, as shown in Figure 1.3, which can be replaced by a statically equivalent internal normal force. We will develop this viewpoint further in Section 1.1.4. Notice that $\boldsymbol{N}$ is in boldface italics, as are all internal forces (and moments) in this book.

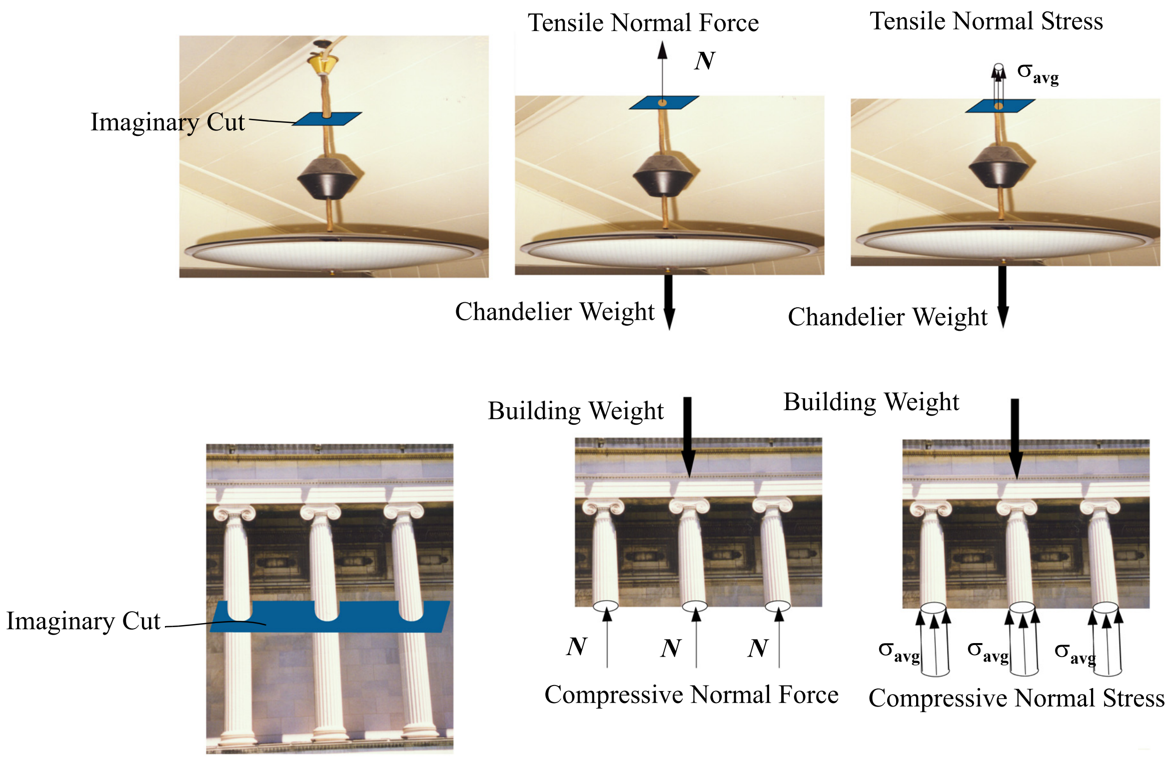


Figure 1.3 Examples of normal stress distribution.

Equation (1.1) is consistent with our intuitive understanding of strength. Consider the following two observations. (i) We know that if we keep increasing the force on a body, then the body will eventually break. Thus we expect the quantifier for strength (stress) to increase in value with the increase of force until it reaches a critical value. In other words, we expect stress to be directly proportional to force, as in Equation (1.1). (ii) If we compare two bodies that are identical in all respects except that one is thicker than the other, then we expect that the thicker body is stronger. Thus, for a given force, as the body gets thicker (larger cross-sectional area), we move away from the critical breaking value, and the value of the quantifier of strength should decrease. In other words, stress should vary inversely with the cross-sectional area, as in Equation (1.1).

Equation (1.1) shows that the unit of stress is force per unit area. Table 1.1 lists the various units of stress used in this book. It should be noted that 1 psi is equal to 6.895 kPa, or approximately 7 kPa. Alternatively, 1 kPa is equal to 0.145 psi, or approximately 0.15 psi. Normal stress that pulls the imaginary surface away from the material is called **tensile** stress, as shown on the cable of the chandelier in Figure 1.3. Normal stress that pushes the imaginary surface into the material is called **compressive** stress, as shown on the column. In other words, tensile stress acts in the direction of the outward normal whereas compressive stress is opposite to the direction of the outward normal to the imaginary surface. Normal stress is usually reported as tensile or compressive and not as positive or negative. Thus $\sigma = 100$ MPa (T) or $\sigma = 10$ ksi (C) are the preferred ways of reporting tensile or compressive normal stresses.

TABLE 1.1 Units of stress

Abbreviation	Units	Basic Units
psi	Pounds per square inch	lb/in.^{2}
ksi	Kilopounds (kips) per square inch	10^3 lb/in.^{2}
Pa	Pascal	N/m^2
kPa	Kilopascal	10^3 N/m^2
MPa	Megapascal	10^6 N/m^2
GPa	Gigapascal	10^9 N/m^2

The normal stress acting in the direction of the axis of a slender member (rod, cable, bar, column) is called **axial** stress. The compressive normal stress that is produced when one real surface presses against another is called the **bearing** stress. Thus, the stress that exist between the base of the column and the floor is a bearing stress but the compressive stress inside the column is not a bearing stress.

An important consideration in all analyses is to know whether the calculated values of the variables are reasonable. A simple mistake, such as forgetting to convert feet to inches or millimeters to meters, can result in values of stress that are incorrect by orders of magnitude. Less dramatic errors can also be caught if one has a sense of the limiting stress values for a material. Table 1.2 shows fracture stress values for a few common materials. Fracture stress is the experimentally measured value at which a material breaks. The numbers are approximate, and $\pm$ indicates variations of the stress values in each class of material. The order of magnitude and the relative strength with respect to wood are shown to help you in acquiring a feel for the numbers.

TABLE 1.2 Fracture stress magnitudes

Material	ksi	MPa	Relative to Wood
Metals	$90 \pm 90\%$	$630 \pm 90\%$	7.0
Granite	$30 \pm 60\%$	$210 \pm 60\%$	2.5
Wood	$12 \pm 25\%$	$84 \pm 25\%$	1.0
Glass	$9 \pm 90\%$	$63 \pm 90\%$	0.89
Nylon	$8 \pm 10\%$	$56 \pm 10\%$	0.67
Rubber	$2.7 \pm 20\%$	$19 \pm 20\%$	0.18
Bones	$2 \pm 25\%$	$14 \pm 25\%$	0.16
Concrete	$6 \pm 90\%$	$42 \pm 90\%$	0.03
Adhesives	$0.3 \pm 60\%$	$2.1 \pm 60\%$	0.02

EXAMPLE 1.1

A girl whose mass is 40 kg is using a swing set. The diameter of the wire used for constructing the links of the chain is 5 mm. Determine the average normal stress in the links at the bottom of the swing, assuming that the inertial forces can be neglected.

Figure 1.4 Girl in a swing, Example 1.2.

PLAN

We make an imaginary cut through the chains, draw a free-body diagram, and find the tension ***T*** in each chain. The link is cut at two imaginary surfaces, and hence the internal normal force ***N*** is equal to ***T***/2 from which we obtain the average normal stress.

SOLUTION

The cross-sectional area and the weight of the girl can be found as

$$A = \frac{\pi d^2}{4} = \frac{\pi(0.005\ \text{m})^2}{4} = 19.6(10^{-6})\ \text{m}^2 \qquad W = (40\ \text{kg})(9.81\ \text{m/s}^2) = 392.4\ \text{N} \tag{E1}$$

Figure 1.5 shows the free body diagram after an imaginary cut is made through the chains. The tension in the chain and the normal force at each surface of the link can be found as shown in Equations (E2) and (E3).

$$\boldsymbol{T} = 2\boldsymbol{N} \tag{E2}$$

$$2\boldsymbol{T} = 392.4\ \text{N} \qquad \text{or} \qquad 4\boldsymbol{N} = 392.4\ \text{N} \qquad \text{or} \qquad \boldsymbol{N} = 98.1\ \text{N} \tag{E3}$$

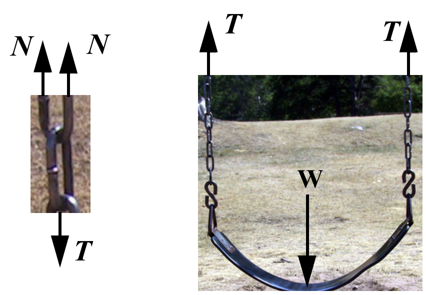


Figure 1.5 Free-body diagram of swing.

The average normal stress can be found as shown in Equation (E4).

$$\sigma_{av} = \frac{\boldsymbol{N}}{A} = \frac{98.1\ \text{N}}{(19.6 \times 10^{-6}\ \text{m}^2)} = 4.996 \times 10^6\ \text{N/m}^2 \tag{E4}$$

ANS. $\sigma_{av} = 5.0\ \text{MPa (T)}$

COMMENTS

1. The stress calculations had two steps. First, we found the internal force by equilibrium; and second we calculated the stress from it.
2. An alternative view is to think that the total material area of the link in each chain is $2A = 39.2 \times 10^{-6}\ \text{m}^2$. The internal normal force in each chain is ***T*** = 196.2 N thus the average normal stress is $\sigma_{av} = \frac{\boldsymbol{T}}{2A} = (196.2/39.2 \times 10^{-6}) = 5 \times 10^6\ \text{N/m}^2$, as before.

1.1.2 Shear Stress

In Figure 1.6*a* the double-sided tape used for sticking a hook on the wall must have sufficient bonding strength to support the weight of the clothes hung from the hook. The free-body diagram shown is created by making an imaginary cut at the wall surface. In Figure 1.6*b* the paper in the ring binder will tear out if the pull of the hand overcomes the strength of the paper. The free-body diagram shown is created by making an imaginary cut along the path of the rings as the paper is torn out. In both free-body diagrams the internal force necessary for equilibrium is parallel (tangent) to the imaginary cut surface. The *internal* shear force V divided by the cross sectional area A exposed by the imaginary cut gives us the average intensity of the internal shear force distribution, which we call the average **shear** stress:

$$\boxed{\tau_{av} = \frac{V}{A}} \tag{1.2}$$

where τ is the Greek letter tau used to designate shear stress and the subscript av emphasizes that the shear stress is an average value. We may view τ_{av} as a uniformly distributed shear force, which can be replaced by a statically equivalent internal normal force V. We will develop this viewpoint further in Section 1.1.4.

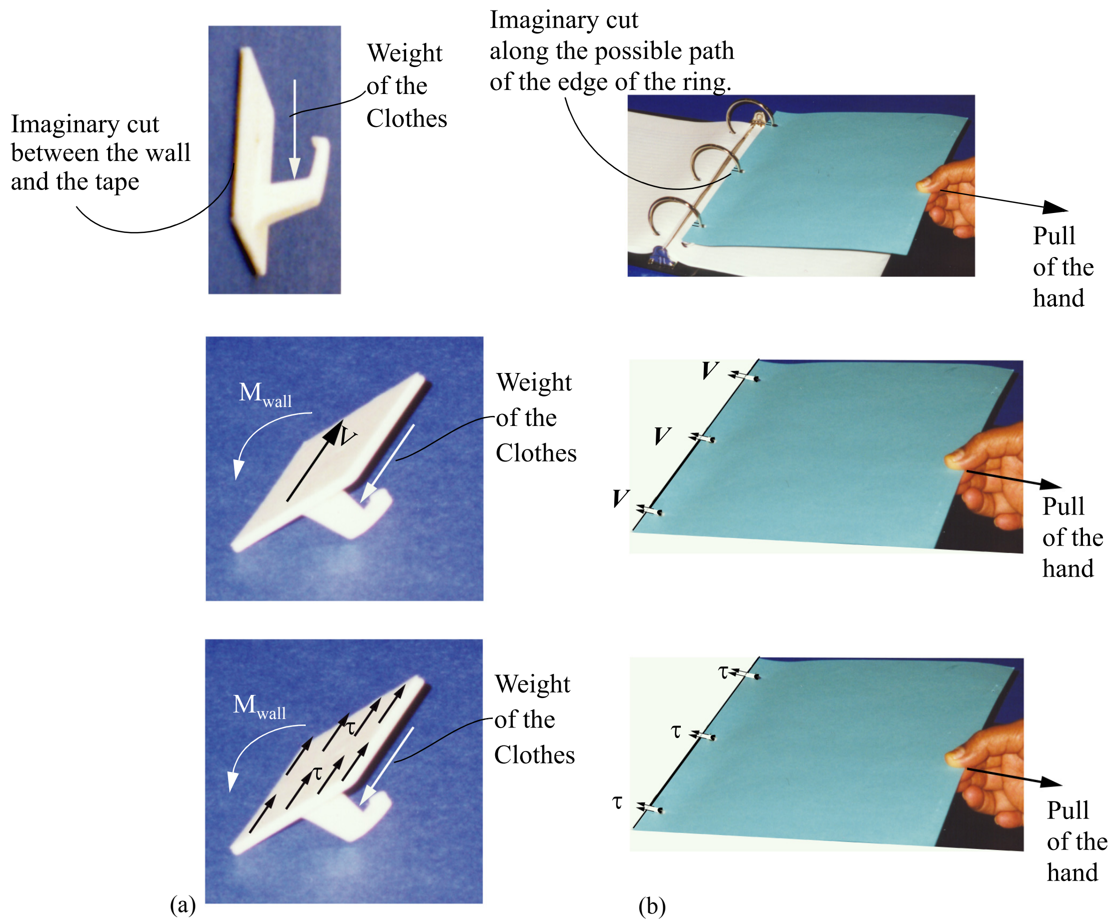


Figure 1.6 Examples of shear stress distribution.

1.1.3 Pins

Pins are one of the most common example of a structural member in which shear stress is assumed uniform on the imaginary surface perpendicular to the pin axis. Bolts, screws, nails, and rivets are often approximated as pins if the *primary* function of these mechanical fasteners is the transfer of shear forces from one member to another. However, if the *primary* function of these mechanical fasteners is to press two solid bodies into each other (seals) then these fasteners cannot be approximated as pins as the forces transferred are normal forces.

Shear pins are mechanical fuses designed to break in shear when the force being transferred exceeds a level that would damage a critical component. In a lawn mower shear pins attach the blades to the transmission shaft and break if the blades hit a large rock that may bend the transmission shaft.

Figure 1.7 shows magnified views of two types of connections at a support. Figure 1.7*a* shows pin in *single* shear as a single cut between the support and the member will break the connection. Figure 1.7*b* shows a pin in *double* shear as two cuts are needed to break the connection. For the same reaction force, the pin in double shear has a smaller shear stress.

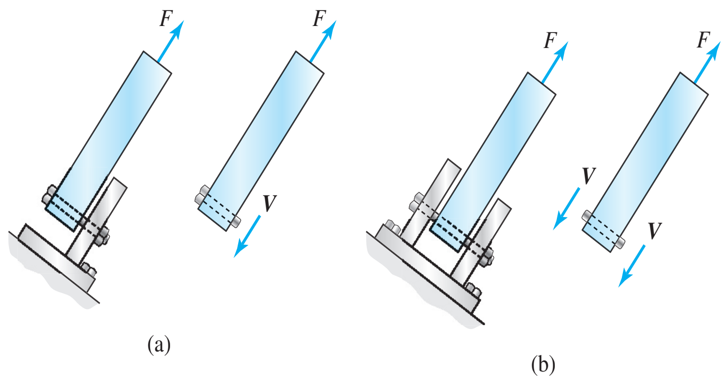


Figure 1.7 Pins in (*a*) single and (*b*) double shear.

When more than two members (forces) are acting on a pin, it is important to visualize the imaginary surface on which the shear stress is to be calculated. Figure 1.8*a* shows a magnified view of a pin connection between three members. The shear stress on the imaginary cut surface 1 will be different from that on the imaginary cut surface 2, as shown by the free-body diagrams in Figure 1.8*b*.

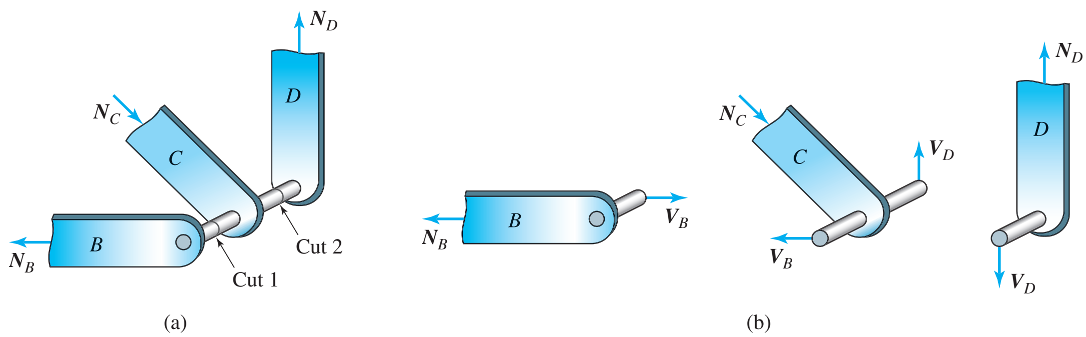


Figure 1.8 Multiple forces on a pin.

EXAMPLE 1.2

Two possible configurations for the assembly of a joint in a machine are to be evaluated. The magnified view of the two configurations with the forces in the members are shown in Figure 1.9. The diameter of the pin is 1 in. Determine which joint assembly is preferred by calculating the maximum shear stress in the pin for each case.

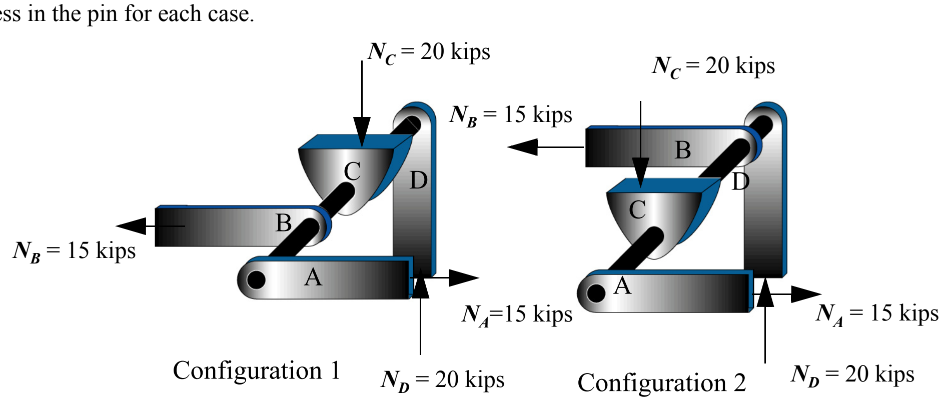


Figure 1.9 Forces on a joint and different joining configurations.

PLAN

We make imaginary cuts between individual members for the two configurations and draw free-body diagrams to determine the shear force at each cut. We calculate and compare the shear stresses to determine the maximum shear stress in each configuration.

SOLUTION

The area of the pin is $A = \pi(0.5 \text{ in.})^2 = 0.7854 \text{ in.}^2$. Making imaginary cuts between members we can draw the free-body diagrams and calculate the internal shear force at the imaginary cut, as shown in Figure 1.10.

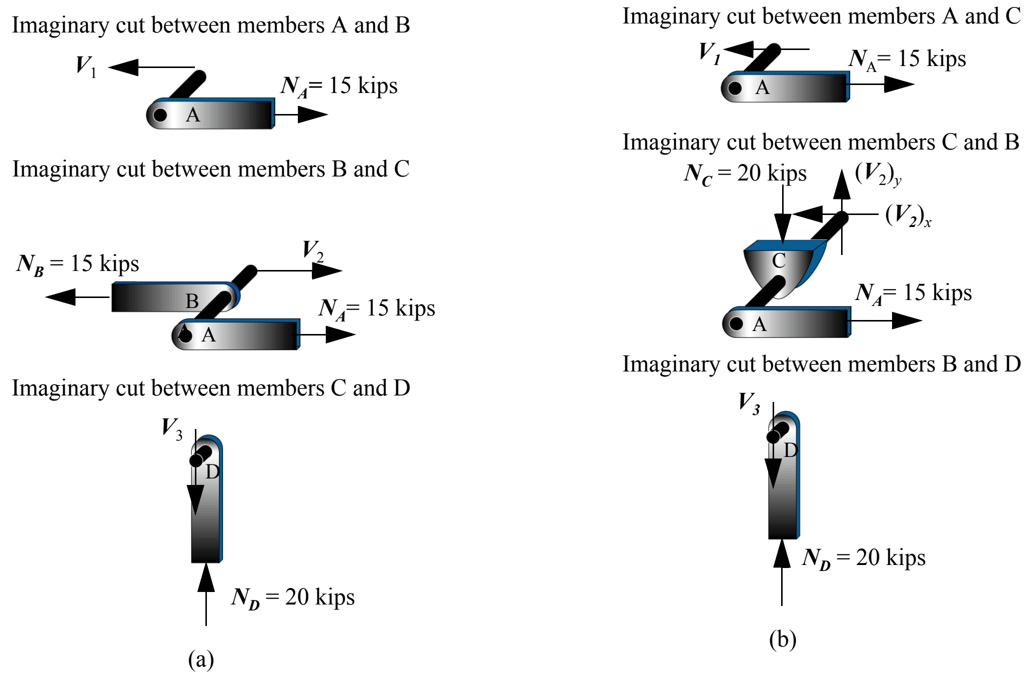


Figure 1.10 Free-body diagrams. (a) Configuration 1. (b) Configuration 2.

Configuration 1: From the free-body diagrams in Figure 1.10*a*

$$V_1 = 15 \text{ kips} \qquad V_2 = 0 \qquad V_3 = 20 \text{ kips} \tag{E1}$$

We see that the maximum shear force exists between members *C* and *D*. Thus the maximum shear stress is

$$\tau_{max} = V_3/A = 25.46 \text{ ksi.} \tag{E2}$$

Configuration 2: From the free-body diagrams in Figure 1.10*b*

$$V_1 = 15 \text{ kips} \qquad (V_2)_x = 15 \text{ kips} \qquad (V_2)_y = 20 \text{ kips} \qquad V_2 = \sqrt{15^2 + 20^2} = 25 \text{ kips.} \qquad V_3 = 20 \text{ kips} \tag{E3}$$

The maximum shear force exists the between *C* and *B*. Thus the maximum shear stress is

$$\tau_{max} = V_2/A = 31.8 \text{ ksi.} \tag{E4}$$

Comparing Equations (E2) and (E3) we conclude

ANS. The configuration 1 is preferred, as it will result in smaller shear stres

COMMENTS

1. Once more note the two steps: we first calculated the internal shear force by equilibrium and then calculated the shear stress from it.
2. The problem emphasizes the importance of visualizing the imaginary cut surface in the calculation of stresses.
3. A simple change in an assembly sequence can cause a joint to fail. This observation is true any time more than two members are joined together. Gusset plates are often used at the joints such as in bridge shown in Figure 1.11 to eliminate the problems associated with an assembly sequence.

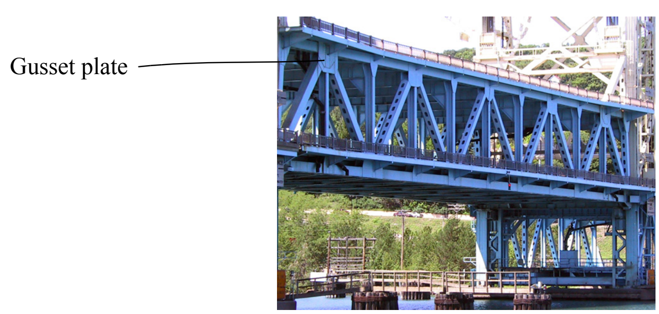


Figure 1.11 Use of gusset plates at joints in a bridge truss.

EXAMPLE 1.3

All members of the truss shown in Figure 1.12 have a cross-sectional area of 500 mm^2 and all pins have a diameter of 20 mm. Determine: (a) The axial stresses in members *BC* and *DE*, (b) The shear stress in the pin at *A*, assuming the pin is in double shear.

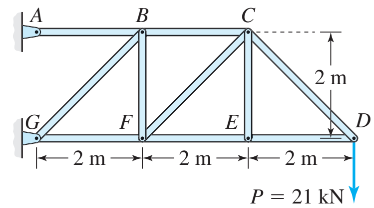


Figure 1.12 Truss.

PLAN

(a) The free-body diagram of joint D can be used to find the internal axial force in member DE. The free body diagram drawn after an imaginary cut through BC, CF, and EF can be used to find the internal force in member BC. (b) The free-body diagram of the entire truss can be used to find the support reaction at A, from which the shear stress in the pin at A can be found.

SOLUTION

The cross-sectional areas of pins and members can be calculated as in Equation (E1)

$$A_p = \frac{\pi(0.02\text{ m})^2}{4} = 314.2(10^{-6})\text{m}^2 \qquad A_m = 500(10^{-6})\text{m}^2 \tag{E1}$$

(a) Figure 1.13*a* shows the free-body diagram of joint *D*. The internal axial force N_{DE} can be found using equilibrium equations as shown in Equation (E3).

$$N_{DC}\sin 45^o - 21\text{ kN} = 0 \qquad \text{or} \qquad N_{DC} = 29.7\text{ kN} \tag{E2}$$

$$-N_{DE} - N_{DC}\cos 45^o = 0 \qquad \text{or} \qquad N_{DE} = -21\text{ kN} \tag{E3}$$

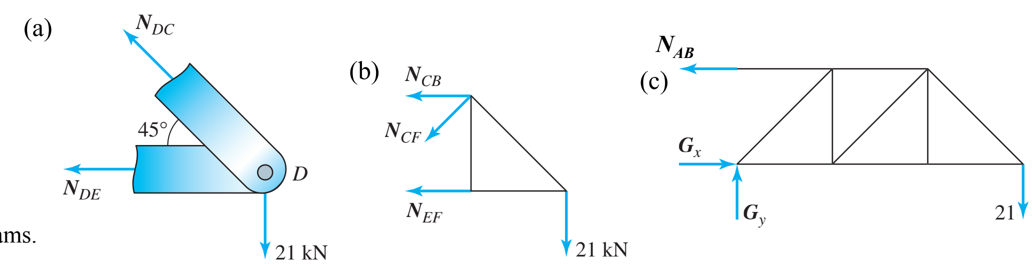


Figure 1.13 Free-body diagrams.

The axial stress in member *DE* can be found as shown in Equation (E4).

$$\sigma_{DE} = \frac{N_{DE}}{A_m} = \frac{[-21(10^3)\text{ N}]}{[500(10^{-6})\text{ m}^2]} = -42(10^6)\text{ N/m}^2 \tag{E4}$$

ANS. $\sigma_{DE} = 42\text{ MPa (C)}$

Figure 1.13*b* shows the free-body diagram after an imaginary cut is made through members *CB*, *CF*, and *EF*. By taking the moment about point *F* we can find the internal axial force in member *CB* as shown in Equation (E5).

$$N_{CB}(2\text{ m}) - (21\text{ kN})(4\text{ m}) = 0 \qquad \text{or} \qquad N_{CB} = 42\text{ kN} \tag{E5}$$

The axial stress in member *CB* can be found as shown in Equation (E6).

$$\sigma_{CB} = \frac{N_{CB}}{A_m} = 84(10^6)\text{ N/m}^2 \tag{E6}$$

ANS. $\sigma_{CD} = 84\text{ MPa (T)}$

(b) Figure 1.13*c* shows the free-body diagram of the entire truss.
By moment equilibrium about point *G* we obtain

$$N_{AB}(2\text{ m}) - 21\text{ kN}(6\text{ m}) = 0 \qquad \text{or} \qquad N_{AB} = 63\text{ kN} \tag{E7}$$

The shear force in the pin will be half the force of N_{AB} as it is in double shear. We obtain the shear stress in the pin as

$$\tau_A = \frac{N_{AB}/2}{A_p} = \frac{31.5(10^3)\text{ N}}{314.2(10^{-6})\text{m}^2} = 100(10^6)\text{ N/m}^2 \tag{E8}$$

ANS. $\tau_A = 100\text{ MPa}$

COMMENTS

1. We calculated the internal forces in each member before calculating the axial stresses, emphasizing the two steps Figure 1.2 of relating stresses to external forces.
2. In part (a) we could have solved for the force in *BC* by noting that *EC* is a zero force member and by drawing the free-body diagram of joint *C*.

PROBLEM SET 1.1

Tensile stress

1.1 In a tug of war, each person shown in Figure P1.1 exerts a force of 200 lb. If the effective diameter of the rope is $\frac{1}{2}$ in., determine the axial stress in the rope.

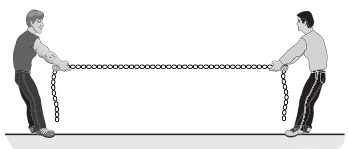


Figure P1.1

1.2 A weight is being raised using a cable and a pulley, as shown in Figure P1.2. If the weight $W = 200$ lb, determine the axial stress assuming: (a) the cable diameter is $\frac{1}{8}$ in. (b) the cable diameter is $\frac{1}{4}$ in.

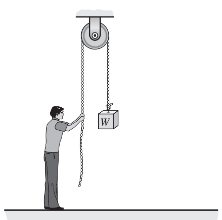


Figure P1.2

1.3 The cable in Figure P1.2 has a diameter of $\frac{1}{5}$ in. If the maximum stress in the cable must be limited to 4 ksi (T), what is the maximum weight that can be lifted?

1.4 The weight $W = 250$ lb in Figure P1.2. If the maximum stress in the cable must be limited to 5 ksi (T), determine the minimum diameter of the cable to the nearest $\frac{1}{16}$ in.

1.5 A 6-kg light shown in Figure P1.5 is hanging from the ceiling by wires of 0.75-mm diameter. Determine the tensile stress in wires *AB* and *BC*.

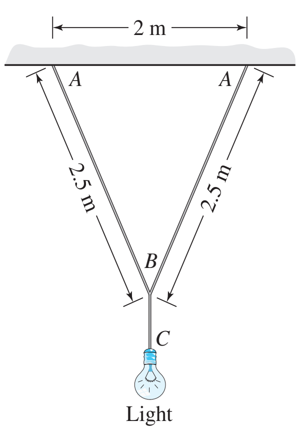


Figure P1.5

1.6 An 8-kg light shown in Figure P1.5 is hanging from the ceiling by wires. If the tensile stress in the wires cannot exceed 50 MPa, determine the minimum diameter of the wire, to the nearest tenth of a millimeter.

1.7 Wires of 0.5-mm diameter are to be used for hanging lights such as the one shown in Figure P1.5. If the tensile stress in the wires cannot exceed 80 MPa, determine the maximum mass of the light that can be hung using these wires.

1.8 A 3 kg picture is hung using a wire of 3 mm diameter, as shown in Figure P1.8. What is the average normal stress in the wires?

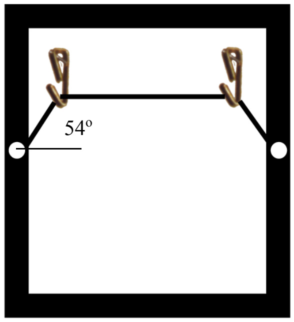


Figure P1.8

1.9 A 5 kg picture is hung using a wire, as shown in Figure P1.8. If the tensile stress in the wires cannot exceed 10MPa, determine the minimum required diameter of the wire to the nearest millimeter.

1.10 Wires of 16-mil diameter are used for hanging a picture, as shown in Figure P1.8. If the tensile stress in the wire cannot exceed 750 psi, determine the maximum weight of the picture that can be hung using these wires. 1 mil = $\frac{1}{1000}$ in.

1.11 A board is raised to lean against the left wall using a cable and pulley, as shown in Figure P1.11. Determine the axial stress in the cable in terms of the length L of the board, the specific weight γ per unit length of the board, the cable diameter d, and the angles θ and α, shown in Figure P1.11.

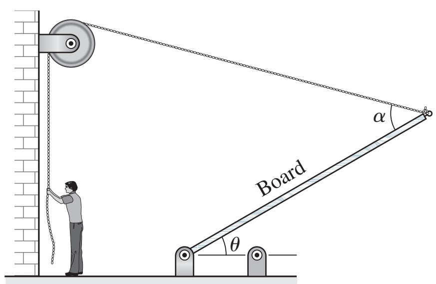


Figure P1.11

Compressive and bearing stresses

1.12 A hollow circular column supporting a building is attached to a metal plate and bolted into the concrete foundation, as shown in Figure P1.12. The column outside diameter is 100 mm and an inside diameter is 75 mm. The metal plate dimensions are 200 mm × 200 mm × 10 mm. The load P is estimated at 800 kN. Determine: (a) the compressive stress in the column; (b) the average bearing stress between the metal plate and the concrete.

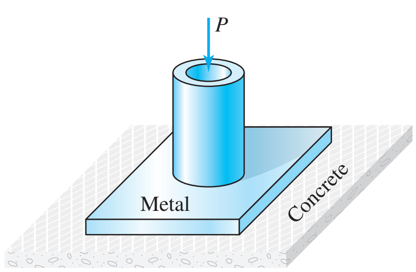


Figure P1.12

1.13 A hollow circular column supporting a building is attached to a metal plate and bolted into the concrete foundation, as shown in Figure P1.12. The column outside diameter is 4 in. and an inside diameter is 3.5 in. The metal plate dimensions are 10 in. × 10 in. × 0.75 in. If the allowable average compressive stress in the column is 30 ksi and the allowable average bearing stress in concrete is 2 ksi, determine the maximum load P that can be applied to the column.

1.14 A hollow square column supporting a building is attached to a metal plate and bolted into the concrete foundation, as shown in Figure P1.14. The column has outside dimensions of 120 mm × 120 mm and a thickness of 10 mm. The load P is estimated at 600 kN. The metal plate dimensions are 250 mm × 250 mm × 15 mm. Determine: (a) the compressive stress in the column; (b) the average bearing stress between the metal plate and the concrete.

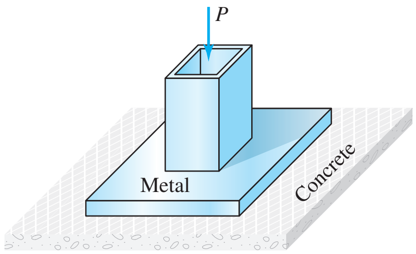
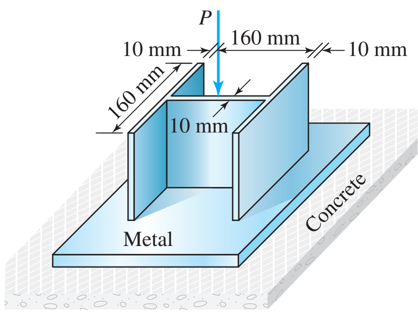


Figure P1.14

1.15 A column with the cross section shown in Figure P1.15 supports a building. The column is attached to a metal plate and bolted into the concrete foundation. The load P is estimated at 750 kN. The metal plate dimensions are 300 mm × 300 mm × 20 mm. Determine: (a) the compressive stress in the column; (b) the average bearing stress between the metal plate and the concrete



FigureP1.15

1.16 A 70-kg person is standing on a bathroom scale that has dimensions of 150 mm × 100 mm × 40 mm (Figures P1.16). Determine the bearing stress between the scale and the floor. Assume the weight of the scale is negligible.

Figure P1.16

1.17 A 30-ft-tall brick chimney has an outside diameter of 3 ft and a wall thickness of 4 in. (Figure P1.17). If the specific weight of the bricks is 80 lb/ft^3, determine the average bearing stress at the base of the chimney.

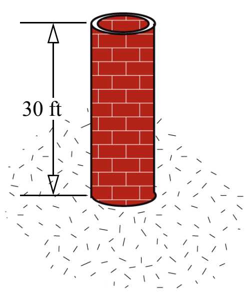


Figure P1.17

1.18 Determine the average bearing stress at the bottom of the block shown in Figure P1.18 in terms of the specific weight γ and the length dimensions a and h.

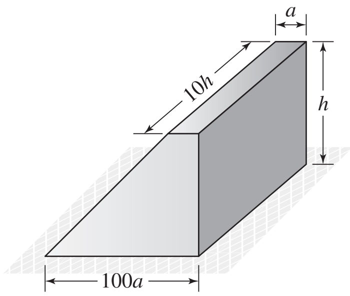


Figure P1.18

1.19 The Washington Monument is an obelisk with a hollow rectangular cross section that tapers along its length. An approximation of the monument geometry is shown in Figure P1.19. The thickness at the base is 4.5 m and at top it is 2.5 m. The monument is constructed from marble and granite. Using a specific weight of 28 kN/m^3 for these materials, determine the average bearing stress at the base of the monument.

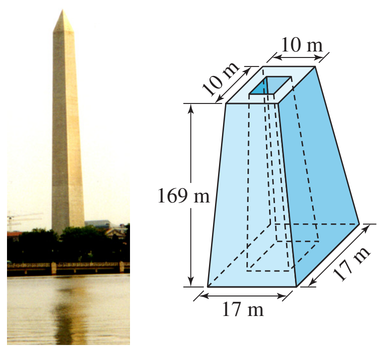


Figure P1.19

1.20 Show that the average compressive stress due to weight on a cross section at a distance x from the top of the wall in Figure P1.20*b* is half that of wall in Figure P1.20*a*, thus confirming the wisdom of ancient Egyptians in building inward-sloping walls for the pyramids. (Hint: Using γ the specific weight of wall material, H the height of the wall, t the thickness of the wall, and L the length of the wall, calculate the average compressive stress at any cross section at a distance x from the top for the two walls.).

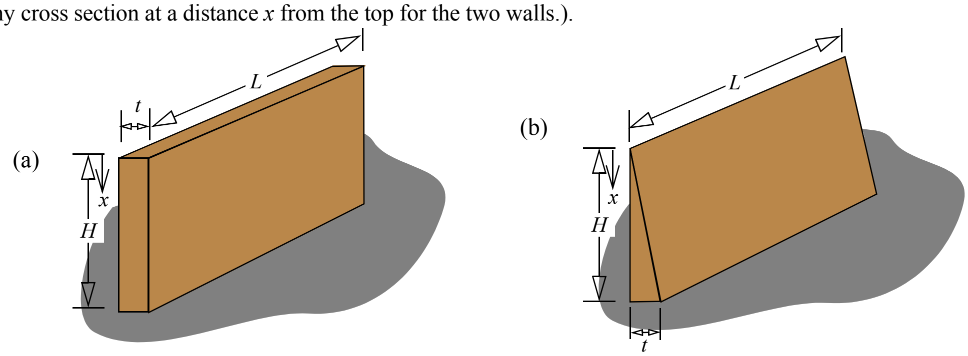


Figure P1.20 (a) Straight wall (b) Inward sloping tapered wall.

1.21 The Great pyramid of Giza shown in Figure 1.14*d* has a base of 757.7 ft x 757.7 ft and a height of 480.9 ft. Assume an average specific weight of γ = 75 lb/ft^3. Determine (a) the bearing stress at the base of the pyramid. (b) the average compressive stress at mid height.

1.22 The Bent pyramid shown in Figure 1.14*c* has a base of 188 m x 188 m. The initial slopes of the sides is 54°27'44". After a certain height the slope is 43°22'. The total height of the pyramid is 105 m. Assume an average mass density of 1200 kg/ m^3. Determine the bearing stress at the base of the pyramid.

1.23 A steel bolt of 25 mm diameter passes through an aluminum sleeve of thickness 4 mm and outside diameter of 48 mm as shown in Figure

P1.23. Determine the average normal stress in the sleeve if in the assembled position the bolt has an average normal stress of 100 MPa (T).

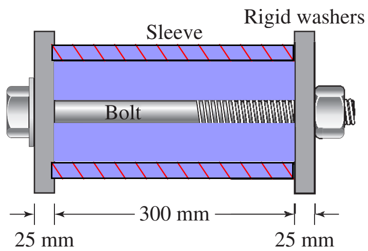


Figure P1.23

Shear stress

1.24 The device shown in Figure P1.24 is used for determining the shear strength of the wood. The dimensions of the wood block are 6 in. × 8 in. × 2 in. If the force required to break the wood block is 15 kips, determine the average shear strength of the wood.

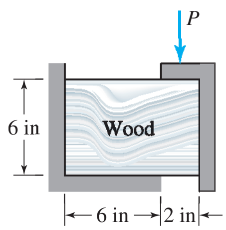


Figure P1.24

1.25 The dimensions of the wood block in Figure P1.24 are 6 in. × 8 in. × 1.5 in. Estimate the force P that should be applied to break the block if the average shear strength of the wood is 1.2 ksi.

1.26 The punch and die arrangement shown schematically in Figure P1.26 is used to punch out thin plate objects of different shapes. The cross section of the punch and die shown in Figure P1.26 is a circle of 1-in. diameter. A force $P = 6$ kips is applied to the punch. If the plate thickness $t = \frac{1}{8}$ in., determine the average shear stress in the plate along the path of the punch.

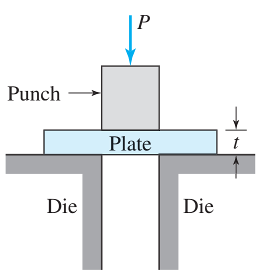


Figure P1.26

1.27 The cross section of the punch and die shown in Figure P1.26 is a square of 10 mm × 10 mm. The plate shown has a thickness $t = 3$ mm and an average shear strength of 200 MPa. Determine the average force P needed to drive the punch through the plate.

1.28 The schematic of a punch and die for punching washers is shown in Figure P1.28. Determine the force P needed to punch out washers, in terms of the plate thickness t, the average plate shear strength τ, and the inner and outer diameters of the washers d_i and d_o

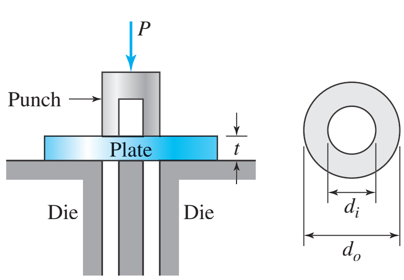


Figure P1.28

1.29 The magnified view of a pin joint in a truss are shown in Figure P1.29. The diameter of the pin is 25 mm. Determine the maximum transverse shear stress in the pin.

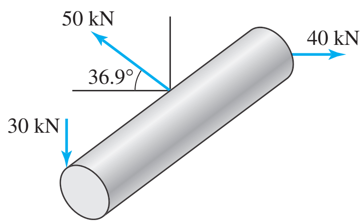


Figure P1.29

Normal and shear stresses

1.30 A weight W = 200 lb. is being raised using a cable and a pulley, as shown in Figure P1.30. The cable effective diameter is $\frac{1}{4}$ in. and the pin in the pulley has a diameter of $\frac{3}{8}$ in. Determine the axial stress in the cable and the shear stress in the pin, assuming the pin is in double shear.

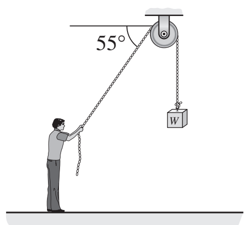


Figure P1.30

1.31 The cable in Figure P1.30 has a diameter of $\frac{1}{5}$ in. and the pin in the pulley has a diameter of $\frac{3}{8}$ in. If the maximum normal stress in the cable must be limited to 4 ksi (T) and the maximum shear stress in the pin is to be limited to 2 ksi, determine the maximum weight that can be lifted to the nearest lb. The pin is in double shear.

1.32 The manufacturer of the plastic carrier for drywall panels shown in Figure P1.32 prescribes a maximum load P of 200 lb. If the cross-sectional areas at sections AA and BB are 1.3 in.^{2} and 0.3 in.^{2}, respectively, determine the average shear stress at section AA and the average normal stress at section BB at the maximum load P.

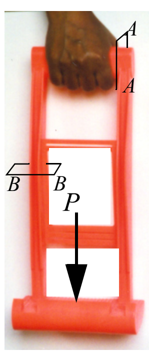


Figure P1.32

1.33 A bolt passing through a piece of wood is shown in Figure P1.33. Determine: (a) the axial stress in the bolt; (b) the average shear stress in the bolt head; (c) the average bearing stress between the bolt head and the wood; (d) the average shear stress in the wood.

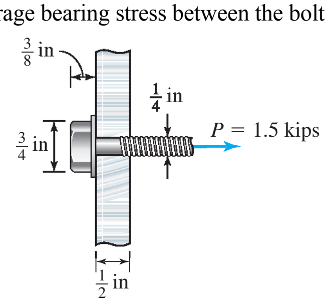


Figure P1.33

1.34 A load of P = 10 kips is transferred by the riveted joint shown in Figure P1.34. Determine (a) the average shear stress in the rivet. (b) the largest average normal stress in the members attached (c) the largest average bearing stress between the pins and members.

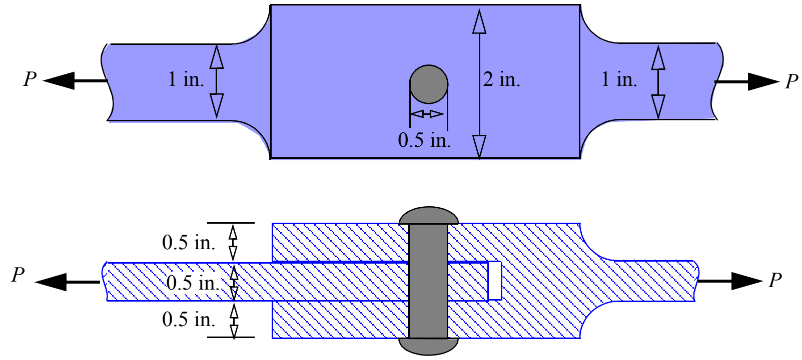


Figure P1.34

1.35 A joint in a wooden structure is shown in Figure P1.35. The dimension $h = 4\frac{3}{8}$ in. and $d = 1\frac{1}{8}$ in. Determine the average normal stress on plane BEF and average shear stress on plane BCD. Assume plane BEF and the horizontal plane at AB are a smooth surfaces.

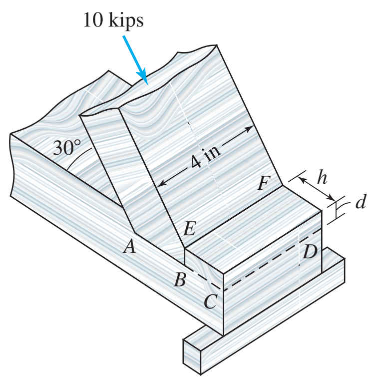


Figure P1.35

1.36 A metal plate welded to an I-beam is securely fastened to the foundation wall using four bolts of 1/2 in. diameter as shown in Figure P1.36. If P = 12 kips determine the normal and shear stress in each bolt. Assume the load is equally distributed among the four bolts.

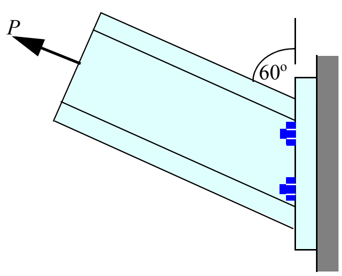


Figure P1.36

1.37 A metal plate welded to an I-beam is securely fastened to the foundation wall using four bolts as shown Figure P1.36. The allowable normal stress in the bolts is 100 MPa and the allowable shear stress is 70 MPa. Assume the load is equally distributed among the four bolts. If the beam load P= 50 kN, determine the minimum diameter to the nearest millimeter of the bolts.

1.38 A metal plate welded to an I-beam is securely fastened to the foundation wall using four bolts of 1/2 in. diameter as shown Figure P1.36. The allowable normal stress in the bolts is 15 ksi and the allowable shear stress is 12 ksi. Assume the load is equally distributed among the four bolts. Determine the maximum load P to the nearest pound the beam can support.

1.39 An adhesively bonded joint in wood is fabricated as shown in Figure P1.39. The length of the overlap is L= 4 in. and the thickness of the wood is 3/8 in. Determine the average shear stress in the adhesive.

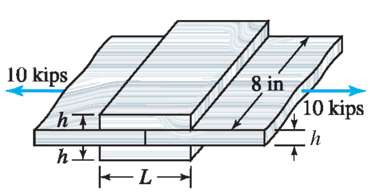


Figure P1.39

1.40 A double lap joint adhesively bonds three pieces of wood as shown in Figure P1.40. The joints transmits a force of P= 20 kips and has the following dimensions: L = 3 in., a = 8 in. and h = 2 in. Determine the maximum average normal stress in the wood and the average shear stress in the adhesive.

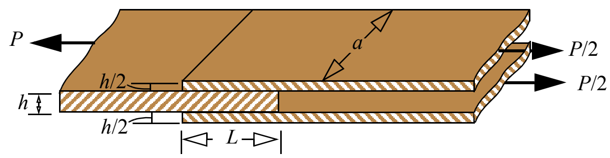


Figure P1.40

1.41 The wood in the double lap joint of Figure P1.40 has a strength of 15 MPa in tension and the strength of the adhesive in shear is 2 MPa. The joint has the following dimensions: L = 75 mm, a =200 mm, and h = 50 mm. Determine the maximum force P the joint can transfer.

1.42 A wooden dowel of diameter d = 20 mm is used for constructing the double lap joint in Figure P1.42. The wooden members have a strength of 10 MPa in tension, the bearing stress between the dowel and the members is to be limited to 18 MPa, the shear strength of the dowel is 25 MPa. The joint has the following dimensions: L = 75 mm, a =200 mm, and h = 50 mm. Determine the maximum force P the joint can transfer.

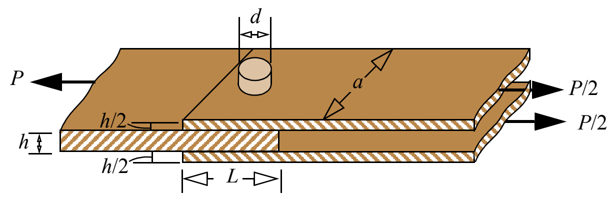


Figure P1.42

1.43 A couple is using the chair lift shown in Figure P1.43 to see the Fall colors in Michigan's Upper Peninsula. The pipes of the chair frame are 1/16 in. thick. Assuming each person weighs 180 lb, determine the average normal stress at section AA and BB and average shear stress at section CC assuming the chair is moving at a constant speed.

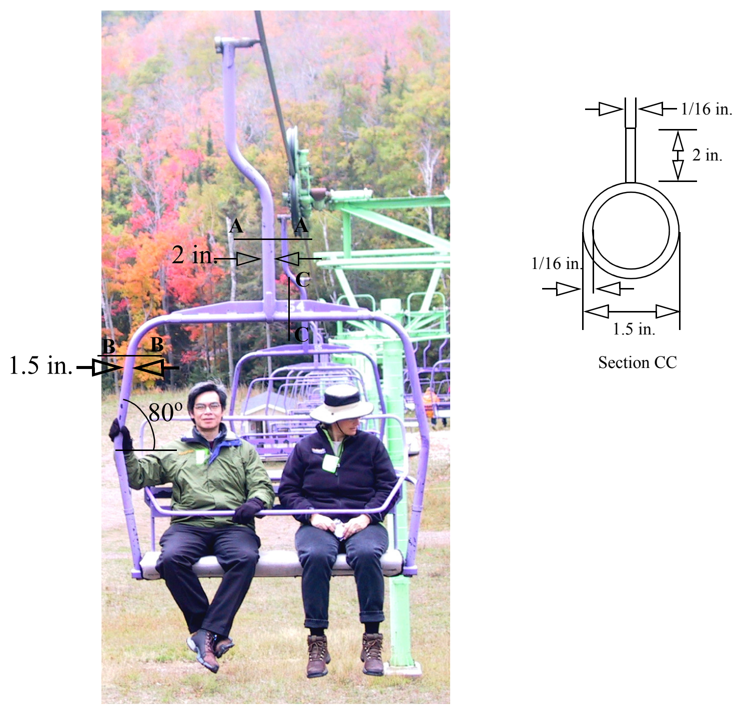


Figure P1.43

1.44 The axial force $P = 12$ kips acts on a rectangular member, as shown in Figure P1.44. Determine the average normal and shear stresses on the inclined plane AA.

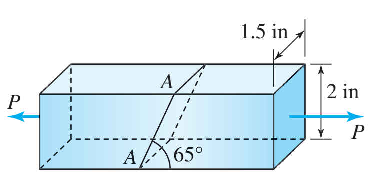


Figure P1.44

1.45 A wooden axial member has a cross section of 2 in.× 4 in. The member was glued along line AA and transmits a force of P = 80 kips as shown in Figure P1.45. Determine the average normal and shear stress on plane AA.

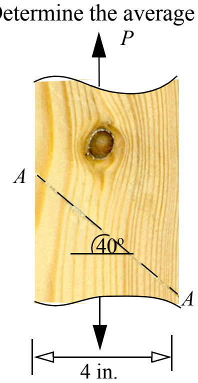


Figure P1.45

1.46 Two rectangular bars of 10-mm thickness are loaded as shown in Figure P1.46. If the normal stress on plane *AA* is 180 MPa (C), determine the force F_1 and the normal and shear stresses on plane *BB*.

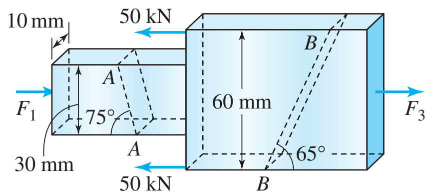


Figure P1.46

1.47 A butt joint is created by welding two plates to transmits a force of P = 250 kN as shown in Figure P1.47. Determine the average normal and shear stress on the plane AA of the weld.

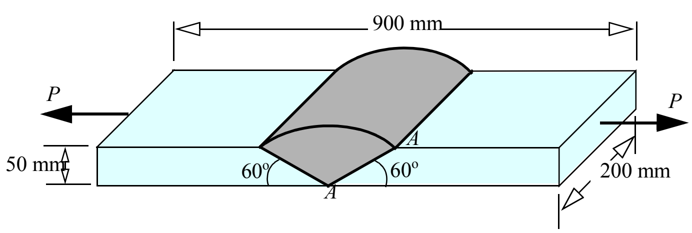


Figure P1.47

1.48 A square tube of 1/4 in thickness is welded along the seam and used for transmitting a force of P = 20 kips as shown in Figure P1.48. Determine average normal and shear stress on the plane AA of the weld.

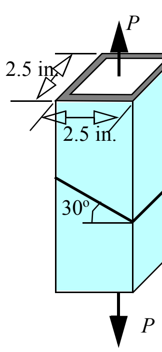


Figure P1.48

1.49 (a) In terms of *P*, *a*, *b*, and θ determine the average normal and shear stresses on the inclined plane *AA* shown in Figure P1.49. (b) Plot the normal and shear stresses as a function of θ and determine the maximum values of the normal and shear stresses. (c) At what angles of the inclined plane do the maximum normal and maximum shear stresses occurs.

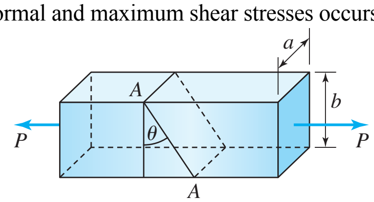
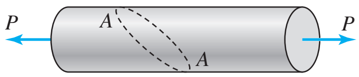


Figure P1.49

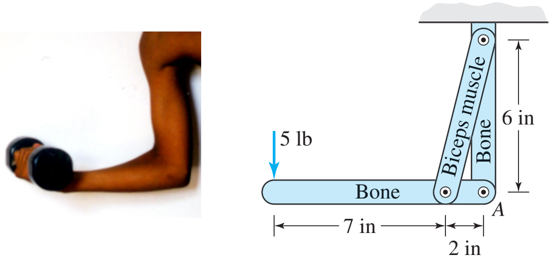
1.50 An axial load is applied to a 1-in-diameter circular rod (Figure P1.50). The shear stress on section *AA* was found to be 20 ksi. The section *AA* is at 45° to the axis of the rod. Determine the applied force *P* and the average normal stress acting on section *AA*.

Figure P1.50



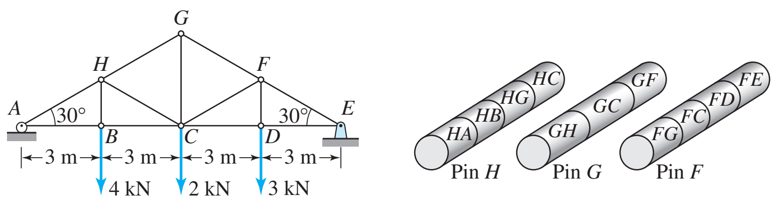
1.51 A simplified model of a child's arm lifting a weight is shown in Figure P1.51. The cross-sectional area of the biceps muscle is estimated as 2 in^2. Determine the average normal stress in the muscle and the average shear force at the elbow joint *A*.

Figure P1.51



1.52 Figure P1.52 shows a truss and the sequence of assembly of members at pins H, G, and F. All members of the truss have cross-sectional areas of 250 mm^2 and all pins have diameters of 15 mm. Determine (a) the axial stresses in members *HA*, *HB*, *HG*, and *HC*. (b) the maximum shear stress in pin *H*.

Figure P1.52

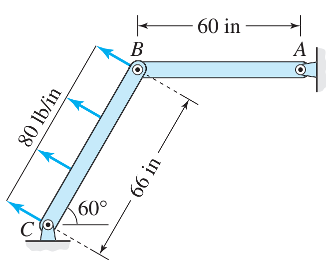


1.53 Figure P1.52 shows a truss and the sequence of assembly of members at pins H, G, and F. All members of the truss have cross-sectional areas of 250 mm^2 and all pins have diameters of 15 mm. Determine (a) the axial stresses in members *FG*, *FC*, *FD*, and *FE*. (b) the maximum shear stress in pin *F*.

1.54 Figure P1.52 shows a truss and the sequence of assembly of members at pins H, G, and F. All members of the truss have cross-sectional areas of 200 mm^2 and all pins have diameters of 10 mm. Determine (a) the axial stresses in members *GH*, *GC*, and *GF* of the truss shown in Figure P1.52. (b) the maximum shear stress in pin *G*

1.55 The pin at *C* in Figure P1.55 is has a diameter of $\frac{1}{2}$ in. and is in double shear. The cross-sectional areas of members *AB* and *BC* are 2 in.^{2} and 2.5 in.^{2}, respectively. Determine the axial stress in member *AB* and the shear stress in pin *C*.

Figure P1.55



1.56 All pins shown in Figure P1.56 are in single shear and have diameters of 40 mm. All members have square cross sections and the surface at E is smooth. Determine the maximum shear stresses in the pins and the axial stress in member *BD*.

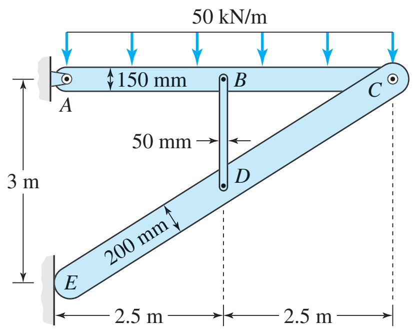


Figure P1.56

1.57 A student athlete is lifting weight W = 36 lbs as shown in Figure P1.57a. The weight of the athlete is W_A = 140 lb. A model of the student pelvis and legs is shown in Figure P1.57b. The weight of legs and pelvis W_L = 32 lb acts at the center of gravity G. Determine the normal stress in the erector spinae muscle that supports the trunk if the average muscle area at the time of lifting the weight is 1.75 in^2.

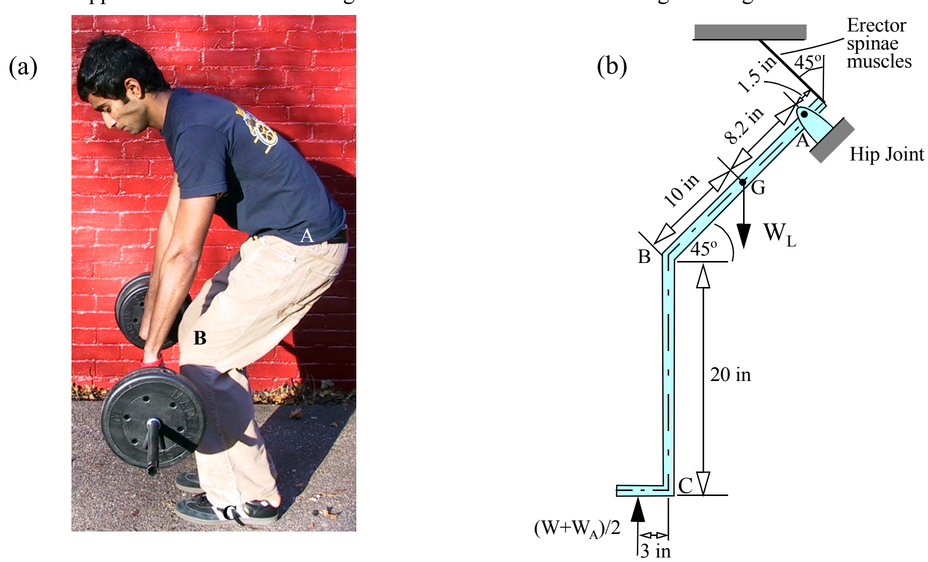


Figure P1.57

1.58 A student is exercising his shoulder muscles using a W = 15 lb dumbbell as shown in Figure P1.58a. The model of the student arm is shown in Figure P1.58b. The weight of the arm of W_A = 9 lb acts at the center of gravity G. Determine the average normal stress in the deltoid muscle if the average area of the muscle is 0.75 in^2 at the time the weight is in the horizontal position.

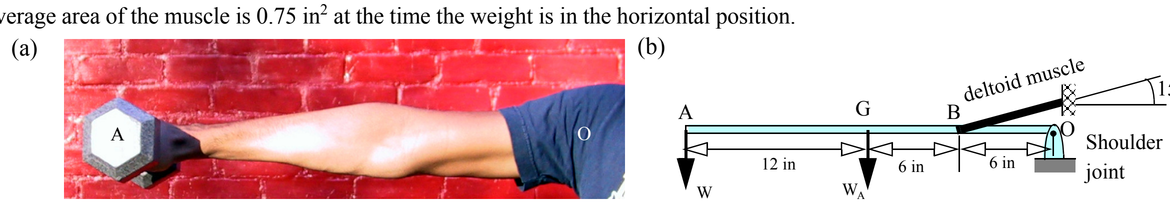


Figure P1.58

Design problems

1.59 The bottom screw in the hook shown in Figure P1.59 supports 60% of the load *P* while the remaining 40% of *P* is carried by the top screw. The shear strength of the screws is 50 MPa. Develop a table for the maximum load *P* that the hook can support for screw diameters that vary from 1 mm to 5 mm in steps of 1 mm.

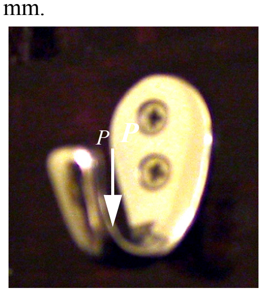


Figure P1.59

1.60 Determine the maximum force P that can be transferred by the riveted joint shown in Figure P1.34 if the limits apply: maximum normal stress in the attached members can be 30 ksi., maximum bearing stress between the pins and members can be 15 ksi, and the maximum shear stress in the rivet can be 20 ksi.

1.61 A tire swing is suspended using three chains, as shown in Figure P1.61. Each chain makes an angle of 12º with the vertical. The chain is made from links as shown. For design purposes assume that more than one person may use the swing, and hence the swing is to be designed to carry a weight of 500 lb. If the maximum average normal stress in the links is not to exceed 10 ksi, determine to the nearest $\frac{1}{16}$ in. the diameter of the wire that should be used for constructing the links.

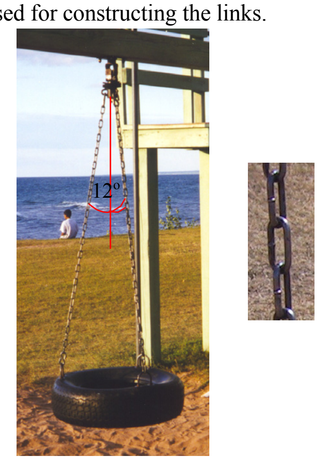


Figure P1.61

1.62 Two cast-iron pipes are held together by a bolt, as shown in Figure P1.62. The outer diameters of the two pipes are 50 mm and 70 mm and the wall thickness of each pipe is 10 mm. The diameter of the bolt is 15 mm. What is the maximum force P this assembly can transmit if the maximum permissible stresses in the bolt and the cast iron are 200 MPa in shear and 150 MPa in tension, respectively.

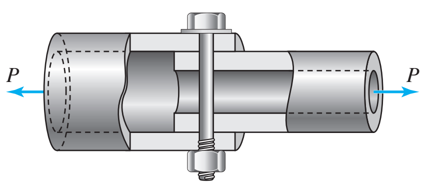


Figure P1.62

1.63 A normal stress of 20 ksi is to be transferred from one plate to another by riveting a plate on top, as shown in Figure P1.63. The shear strength of the $\frac{1}{2}$ in. rivets used is 40 ksi. Assuming all rivets carry equal shear stress, determine the minimum even number of rivets that must be used.

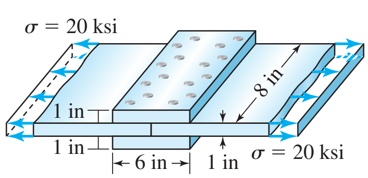


Figure P1.63

1.64 Two possible joining configurations are to be evaluated. The forces on joint in a truss were calculated and a magnified view is shown Figure P1.64. The pin diameter is 20 mm. Determine which joint assembly is better by calculating the maximum shear stress in the pin for each case.

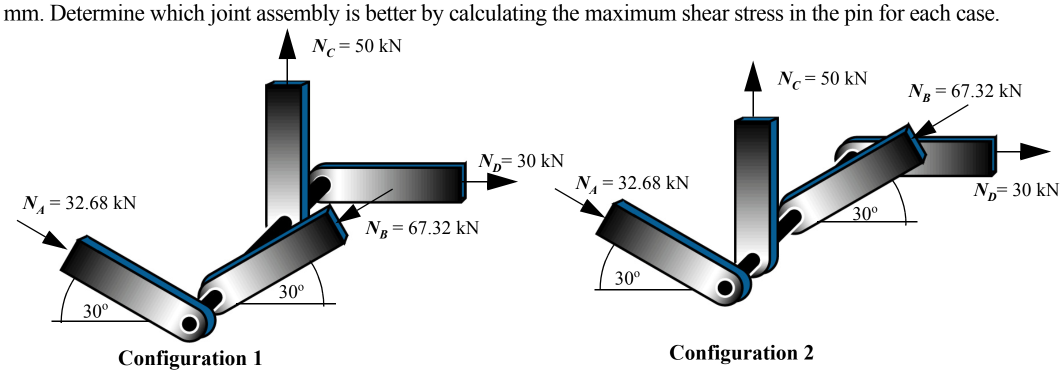


Figure P1.64

1.65 Truss analysis showed the forces at joint *A* given in Figure P1.65. Determine the sequence in which the three members at joint *A* should be assembled so that the shear stress in the pin is minimum.

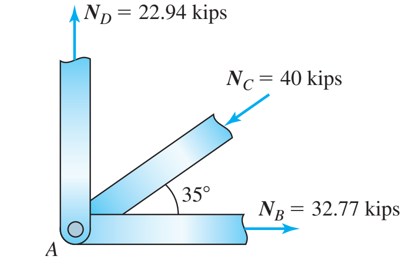


Figure P1.65

1.66 An 8 in × 8 in reinforced concrete bar needs to be designed to carry a compressive axial force of 235 kips. The reinforcement is done using $\frac{1}{2}$-in. round steel bars. Assuming the normal stress in concrete to be a uniform maximum value of 3 ksi and in steel bars to be a uniform value of 20 ksi, determine the minimum number of iron bars that are needed.

1.67 A wooden axial member has a cross section of 2 in. × 4 in. The member was glued along line *AA*, as shown in Figure P1.45. Determine the maximum force *P* that can be applied to the repaired axial member if the maximum normal stress in the glue cannot exceed 800 psi and the maximum shear stress in the glue cannot exceed 350 psi.

1.68 An adhesively bonded joint in wood is fabricated as shown in Figure P1.68. The length of the bonded region $L = 5$ in. Determine the maximum force *P* the joint can support if the shear strength of the adhesive is 300 psi and the wood strength is 6 ksi in tension.

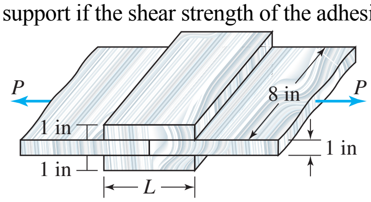


Figure P1.68

1.69 The joint in Figure P1.68 is to support a force $P = 25$ kips. What should be the length *L* of the bonded region if the adhesive strength in shear is 300 psi?

1.70 The normal stress in the members of the truss shown in Figure P1.70 is to be limited to 160 MPa in tension or compression. All members have circular cross sections. The shear stress in the pins is to be limited to 250 MPa. Determine (a) the minimum diameters to the nearest millimeter of members *ED*, *EG*, and *EF*. (b) the minimum diameter of pin *E* to the nearest millimeter and the sequence of assembly of members *ED*, *EG*, and *EF*.

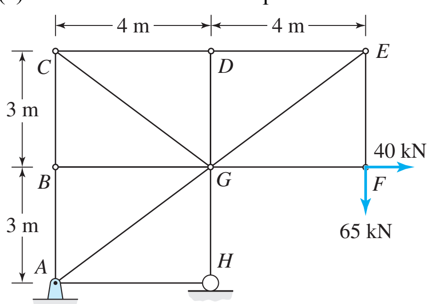


Figure P1.70

1.71 The normal stress in the members of the truss shown in Figure P1.70 is to be limited to 160 MPa in tension or compression. All members have circular cross sections. The shear stress in the pins is to be limited to 250 MPa. Determine (a) the minimum diameters to the nearest millimeter of members *CG*, *CD*, and *CB*. (b) the minimum diameter of pin *C* to the nearest millimeter and the sequence of assembly of members *CG*, *CD*, and *CB*.

Stretch yourself

1.72 Truss analysis showed the forces at joint *A* given in Figure P1.72. Determine the sequence in which the four members at joint *A* should be assembled to minimize the shear stress in the pin.

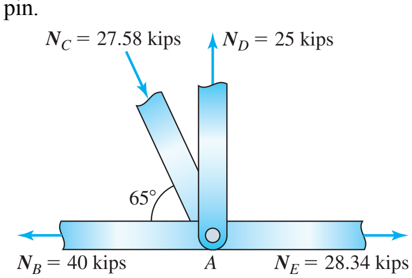


Figure P1.72

MoM in Action: Pyramids

The pyramids of Egypt are a remarkable engineering feat. The size, grandeur, and age of the pyramids excites the human imagination. Science fiction authors create stories about aliens building them. Pyramid design, however, is a story about human engineering in a design process that incorporates an intuitive understanding of material strength.

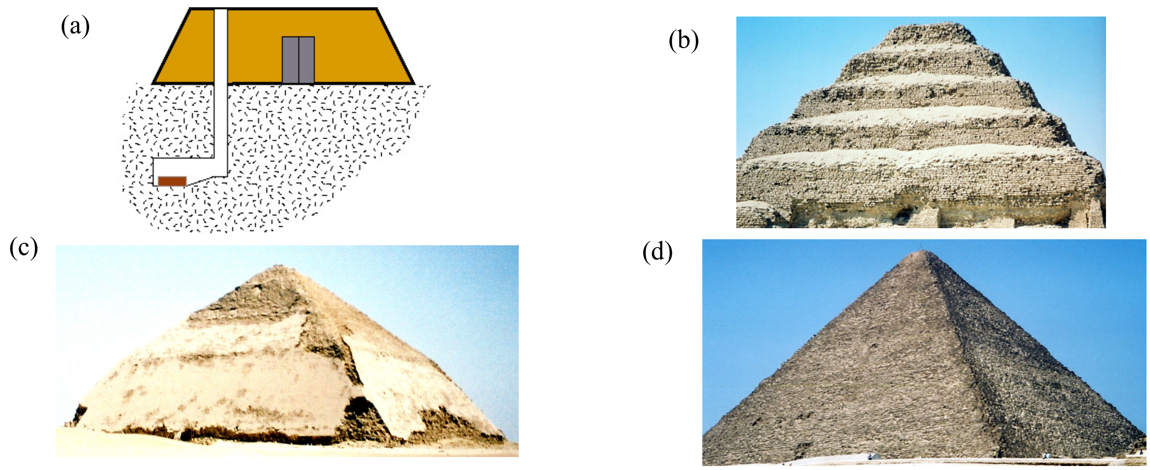


Figure 1.14 Pyramids of Egypt (a) Mastaba (b) Step pyramid (c) Bent pyramid (d) Great pyramid of Giza.

Before pyramids were built, Egyptians kings and nobles were buried in tombs called *mastaba (*Figure 1.14*a)*. Mastaba have underground chambers that are blocked off by dropping heavy stones down vertical shafts. On top of these underground burial chambers are rectangular structures with inward-sloping, tapered brick mud walls. The ancient Egyptians had learned by experience that inward-sloping walls that taper towards the top do not crumble as quickly as straight walls of uniform thickness (see problem 1.20).

Imhotep, the world's first renowned engineer-architect, took many of the design elements of *mastaba* to a very large scale in building the world's first Step pyramid (Figure 1.14*b*) for his pharaoh Djozer (2667-2648 BCE). By building it on a bedrock, Imhotep intuitively understood the importance of bearing stresses which were not properly accounted for in building of the leaning tower of Pisa 4000 years later. The Step pyramid rose in six steps to a height of 197 ft with a base of 397 ft x 358 ft. A 92-ft deep central shaft was dug beneath the base for the granite burial chamber. The slopes of the faces of the Step pyramid varied from 72^o to 78^o. Several pharaohs after Djozer tried to build their own step pyramids but were unsuccessful.

The next development in pyramid design took place in the reign of pharaoh Sneferu (2613-2589 BCE). Sneferu architects started by building a step pyramid but abandoned it because he wanted a pyramid with smooth sides. The pyramid shown in Figure 1.14*c* was started with a base of 617 ft x 617 ft and an initial slope of approximately 54^o. Signs of structural problem convinced the builders to change the slope to 43^o resulting in the unique bent shape seen in Figure 1.14*c* (see problem 1.22). Sneferu then ordered a third pyramid built. This pyramid had an initial slope of 43^o, stood on a base of 722 ft x722 ft, rose to a height of 345 ft, and had smooth sides. This experience was used by architects in the reign of Khufu (2589-2566 BCE) to build the largest pyramid in the world called the Great pyramid of Giza (Figure 1.14*d*). The Great Pyramid (see problem 1.21) stands on a base of 756.7 ft x 756.7 ft and has a height of 481 ft.

The ancient Egyptians did not have a formal definition of stress, but they had an intuitive understanding of the concept of strength. As happens often in engineering design they were able to design and construct pyramids through trial and error. Modern engineering can reduce this costly and time consuming process by developing rigorous methodologies and formulas. In this book we will be developing formulas for strength and stiffness design of structures and machine elements.

1.1.4 Internally Distributed Force Systems

In Sections 1.1.1 and 1.1.2 the normal stress and the shear stress were introduced as the average intensity of an internal normal and shear force distribution, respectively. But what if there are internal moments at a cross section? Would there be normal and shear stresses at such sections? How are the normal and shear stresses related to internal moments? To answer these questions and to get a better understanding of the character of normal stress and shear stress, we now consider an alternative and more fundamental view.

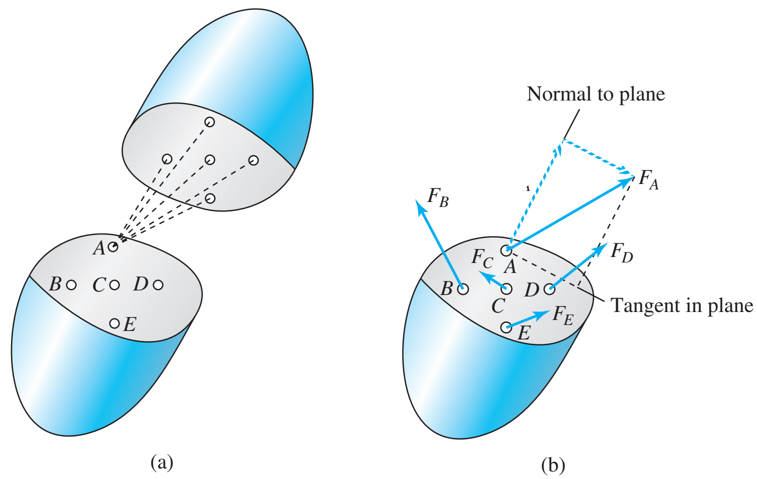


Figure 1.15 Internal forces between particles on two sides of an imaginary cut. (*a*) Forces between particles in a body, shown on particle *A*. (*b*) Resultant force on each particle.

The forces of attraction and repulsion between two particles (atoms or molecules) in a body are assumed to act along the line that joins the two particles.[1] The forces vary inversely as an exponent of the radial distance separating the two particles. Thus every particle exerts a force on every other particle, as shown symbolically in Figure 1.15*a* on an imaginary surface of a body. These forces between particles hold the body together and are referred to as *internal forces*. The shape of the body changes when we apply external forces thus changing distance between particles and hence changing the forces between the particles (internal forces). The body breaks when the change in the internal forces exceeds some characteristic material value. Thus the strength of the material can be characterized by the measure of *change in the intensity* of internal forces. This measure of change in the intensity of internal forces is what we call stress.

In Figure 1.15*b* we replace all forces that are exerted on any single particle by the resultants of these forces on that particle. The magnitude and direction of these resultant forces will vary with the location of the particle (point) implying that this is an internal distributed force system. The intensity of internally distributed forces on an imaginary cut surface of a body is called the **stress on the surface**. The internally distributed forces (stress on a surface) can be resolved into normal (perpendicular to the surface) and tangential (parallel to the surface) distribution. The intensity of an internally distributed force that is normal to the surface of an imaginary cut is called the **normal** stress on the surface. The intensity of an internally distributed force that is parallel to the surface of an imaginary cut surface is called the **shear** stress on the surface.

Normal stress on a surface may be viewed as the internal forces that develop due to the material resistance to the pulling apart or pushing together of two adjoining planes of an imaginary cut. Like pressure, normal stress is always perpendicular to the surface of the imaginary cut. But unlike pressure, which can only be compressive, normal stress can be tensile.

[1]Forces that act along the line joining two particles are called *central* forces. The concept of central forces started with Newton's universal gravitation law, which states: "the force between two particles is inversely proportional to the square of the radial distance between two particles and acts along the line joining the two particles." At atomic levels the central forces do not vary with the square of the radial distance but with an exponent, which is a power of 8 or 10.

Shear stress on a surface may be viewed as the internal forces that develop due to the material resistance to the sliding of two adjoining planes along the imaginary cut. Like friction, shear stresses act tangent to the plane in the direction opposite to the impending motion of the surface. But unlike friction, shear stress is not related to the normal forces (stresses).

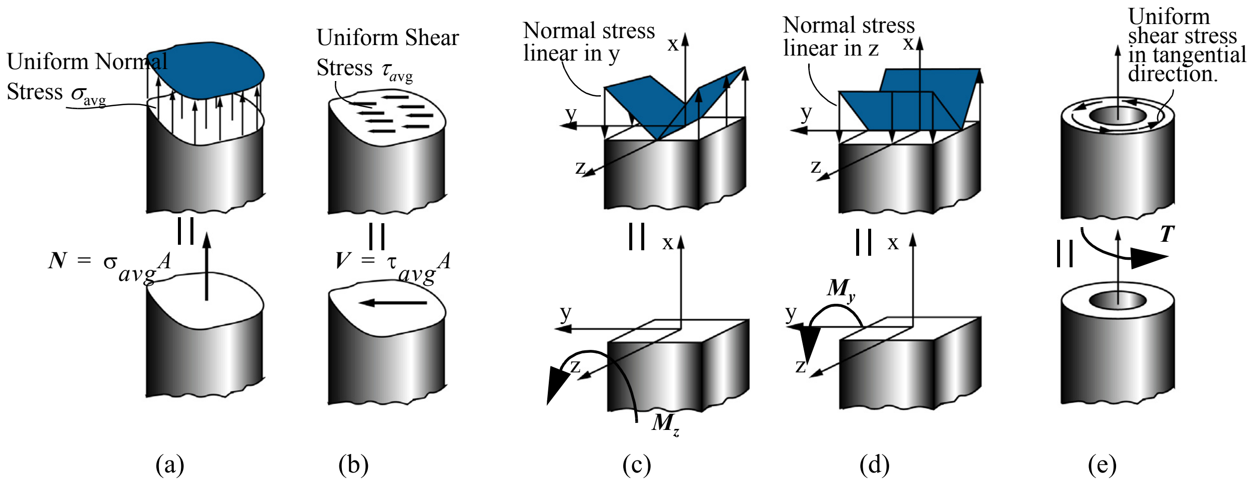


Figure 1.16 Static equivalency.

Now that we have established that the stress on a surface is an internally distributed force system, we are in a position to answer the questions raised at the beginning of the section. If the normal and shear stresses are constant in magnitude and direction across the cross section, as shown in Figure 1.16*a* and *b*, then these can be replaced by statically equivalent normal and shear forces. [We obtain the equivalent forms of Equations (1.1) and (1.2).] But if either the magnitude or the direction of the normal and shear stresses changes across the cross section, then internal bending moments M_y, M_z and the internal torque T may be necessary for static equivalency, as shown in Figure 1.16*c*, *d*, and *e*. Figure 1.16 shows some of the stress distributions we will see in this book. But how do we deduce the variation of stress on a surface when stress is an internal quantity that cannot be measured directly? The theories in this book that answer this question were developed over a long period of time using experimental observations, intuition, and logical deduction in an iterative manner. Assumptions have to be made regarding loading, geometry, and material properties of the structural member in the development of the theory. If the theoretical predictions do not conform to experimental observations, then assumptions have to be modified to include added complexities until the theoretical predictions are consistent with experimental observations. In Section 3.2, we will see the logic whose two links are shown in Figure 1.2. This logic with assumptions regarding loading, geometry, and material properties will be used to develop the simplified theories in Chapters 4 through 6.

EXAMPLE 1.4

Figure 1.17 shows a fiber pull-out test that is conducted to determine the shear strength of the interface between the fiber and the resin matrix in a composite material (see Section 3.12.3). Assuming a uniform shear stress τ at the interface, derive a formula for the shear stress in terms of the applied force P, the length of fiber L, and the fiber diameter D.

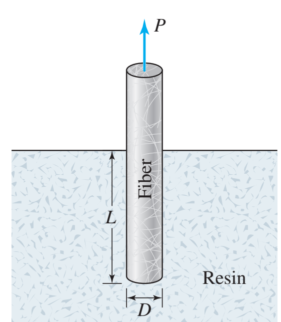


Figure 1.17 Fiber pull-out test.

PLAN

The shear stress is acting on the cylindrical surface area of the embedded fiber. The shear stress is uniform and hence can be replaced by an equivalent shear force V, which we can equate to P.

SOLUTION

Figure 1.18*a* shows the cylindrical surface of the fiber with the uniform shear stress on the surface. The surface area A is equal to the circumference multiplied by the fiber length L as shown by the Equation (E1).

$$A = \pi DL \tag{E1}$$

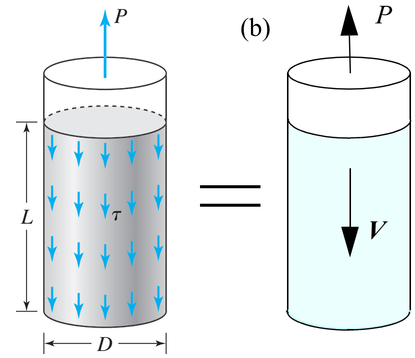


Figure 1.18 Free body diagrams of the fiber in Example 1.4 (a) with shear stresses, (b) with equivalent internal shear force.

The shear force is the shear stress multiplied by the surface area,

$$V = \tau A = (\pi DL)\tau \tag{E2}$$

By equilibrium of forces in Figure 1.18*b*

$$V = P \qquad \text{or} \qquad (\pi DL)\tau = P \tag{E3}$$

ANS. $\tau = P/(\pi DL)$

COMMENTS

1. First, we replaced an internal distributed force system (shear stress) by an equivalent shear force. Second, we related the internal shear force to external force by equilibrium.
2. In the preceding test it is implicitly assumed that the strength of the fiber is greater than the interface strength. Otherwise the fiber would break before it gets pulled out.
3. In a test the force P is increased slowly until the fiber is pulled out. The pull-out force is recorded, and the shear strength can be calculated.
4. Suppose we have determined the shear strength from our formula for specific dimensions D and L of the fiber. Now we should be able to predict the force P that a fiber with different dimensions would support. If on conducting the test the experimental value of P is significantly different from the value predicted, then our assumption of uniform shear stress in the interface is most likely incorrect.

EXAMPLE 1.5

Figure 1.19 shows a test to determine the shear strength of an adhesive. A torque (a moment along the axis) is applied to two thin cylinders joined together with the adhesive. Assuming a uniform shear stress τ in the adhesive, develop a formula for the adhesive shear stress τ in terms of the applied torque T_{ext}, the cylinder radius R, and the cylinder thickness t.

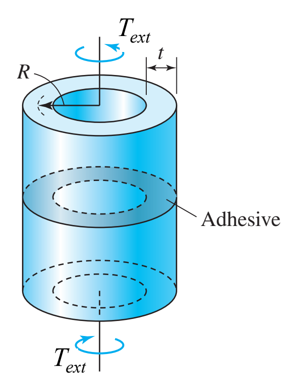


Figure 1.19 Adhesive shear strength test.

PLAN

A free body diagram can be constructed after making an imaginary cut through the adhesive layer. On a differential area the internal shear force can be found and the moment from the internal shear force on the differential area obtained. By integrating we can find the total internal moment acting on the adhesive, which we can equate to the applied external moment T_{ext}.

SOLUTION

We make an imaginary cut through the adhesive and draw the shear stress in the tangential direction, as shown in Figure 1.20*a*.

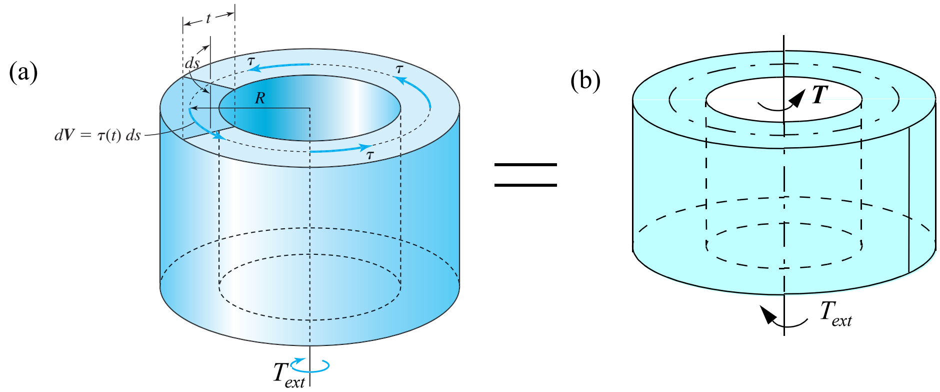


Figure 1.20 Free-body diagrams in Example 1.5 (a) with shear stress, (b) with equivalent internal torque.

The differential area is the differential arc length ds multiplied by the thickness t.
The differential tangential shear force dV is the shear stress multiplied by the differential area.
The differential internal torque (moment) is the moment arm R multiplied by dV, that is, $d\boldsymbol{T} = RdV = R\tau t ds$
Noting that $[ds = R\,d\theta]$, we obtain the total internal torque by integrating over the entire circumference.

$$\boldsymbol{T} = \int R\,dV = \int R(\tau t)R\,d\theta = \int_0^{2\pi} \tau t R^2\,d\theta = \tau t R^2(2\pi) \tag{E1}$$

By equilibrium of moment in Figure 1.20*b*

$$\boldsymbol{T} = T_{ext} \qquad \text{or} \qquad 2\pi R^2 t\tau = T_{ext} \tag{E2}$$

ANS. $\tau = \dfrac{T_{ext}}{2\pi R^2 t}$

COMMENTS

1. By recording the value of the torque at which the top half of the cylinder separates from the bottom half, we can calculate the shear strength of the adhesive.
2. The assumption of uniform shear stress can only be justified for thin cylinders. In Chapter 5 we will see that shear stress for thicker cylinders varies linearly in the radial direction.
3. First, we replaced an internal distributed force system (shear stress) by an equivalent internal torque. Second, we related the internal torque to external torque by equilibrium.

EXAMPLE 1.6

Figure 1.21 shows a drill being used to make a L = 12-in.-deep hole for placing explosive charges in a granite rock. The shear strength of the granite is τ = 5 ksi. Determine the minimum torque T that must be applied to a drill of radius R = 1-in., assuming a uniform shear stress along the length of the drill. Neglect the taper at the end.

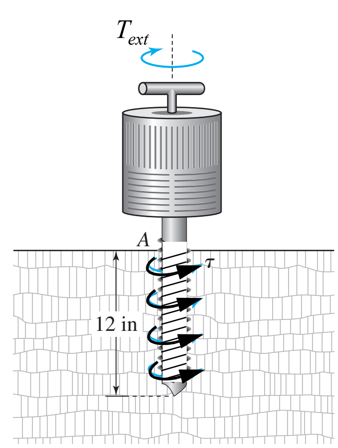


Figure 1.21 Torque on a drill.

PLAN

The imaginary cut surface is the surface of the hole in the granite. The shear stress on the surface of the hole would act like a distributed frictional force on the cylindrical surface of the drill bit. We can find the moment from this frictional force and relate it to the applied torque.

SOLUTION

The shear stress acts tangential to the cylindrical surface of the drill bit, as shown in Figure 1.22*a*.

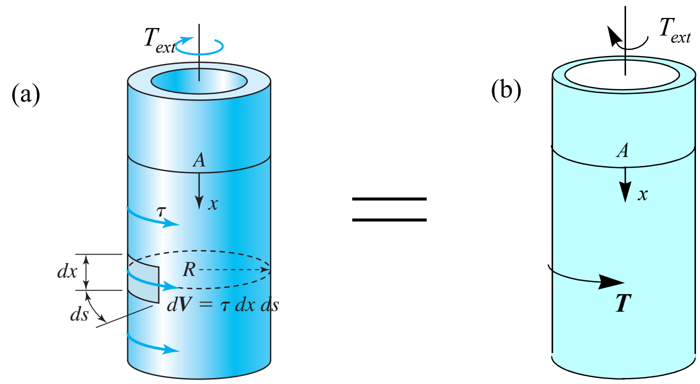


Figure 1.22 Free body diagram of drill bit in Example 1.6 (a) with shear stress, (b) with equivalent internal torque.

Multiplying the shear stress by the differential surface area *ds dx* we obtain the differential tangential shear force dV. Multiplying dV by the moment arm *R*, we obtain the internal torque $d\boldsymbol{T} = R dV = R\tau ds dx$, which is due to the shear stress over the differential surface area.
Integrating over the circumference $ds = R d\theta$ and the length of the drill, we obtain the total internal torque.

$$\boldsymbol{T} = \int R\, dV = \int_0^L \int_0^{2\pi} R(\tau) R\, d\theta\, dx = \tau R^2 \int_0^L \int_0^{2\pi} d\theta\, dx = \tau R^2 \int_0^L 2\pi\, dx = 2\pi = 2\pi\tau R^2 L \quad \text{or}$$

$$\boldsymbol{T} = 2\pi(5\text{ ksi})(1\text{ in.})^2(12\text{ in.}) = 120\pi\ \text{in.}\cdot\text{kips} \tag{E1}$$

By equilibrium of moment in Figure 1.22*b*

$$T_{ext} = \boldsymbol{T} \tag{E2}$$

ANS. $T_{ext} = 377\ \text{in.}\cdot\text{kips}$

COMMENTS

1. In this example and in Example 1.4 shear stress acted on the outside cylindrical surface. In Example 1.4 we replaced the shear stresses by just an internal shear force, whereas in this example we replaced the shear stresses by an internal torque. The difference comes from the direction of the shear stress.
2. In Example 1.5 and in this example the surfaces on which the shear stresses are acting are different. Yet in both examples we replaced the shear stresses by the equivalent internal torque.
3. The two preceding comments emphasize that before we can define which internal force or which internal moment is statically equivalent to the internal stress distribution, we must specify the direction of stress and the orientation of the surface on which the stress is acting. We shall develop this concept further in Section 1.2.

Consolidate your knowledge

1. In your own words describe stress on a surface.

QUICK TEST 1.1

Time: 15 minutes Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E. Give yourself one point for each correct answer (true or false) and one point for every correct explanation.

1. You can measure stress directly with an instrument the way you measure temperature with a thermometer.
2. There can be only one normal stress component acting on the surface of an imaginary cut.
3. If a shear stress component on the left surface of an imaginary cut is upward, then on the right surface it will be downward.
4. If a normal stress component puts the left surface of an imaginary cut in tension, then the right surface will be in compression.
5. The correct way of reporting shear stress is $\tau = 70$ kips.
6. The correct way of reporting positive axial stress is $\sigma = +15$ MPa.
7. 1 GPa equals 10^6 Pa.
8. 1 psi is approximately equal to 7 Pa.
9. A common failure stress value for metals is 10,000 Pa.
10. Stress on a surface is the same as pressure on a surface as both quantities have the same units.

PROBLEM SET 1.2

Internally Distributed Force Systems

1.73 The post shown in Figure P1.73 has a rectangular cross section of 2 in. × 4 in. The length L of the post buried in the ground is 12 in. and the average shear strength of the soil is 2 psi. Determine the force P needed to pull the post out of the ground.

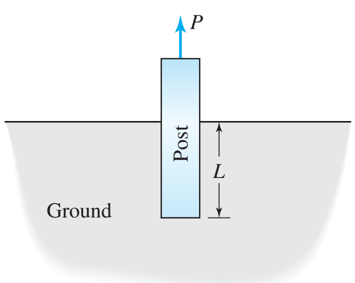


Figure P1.73

1.74 The post shown in Figure P1.73 has a circular cross section of 100-mm diameter. The length L of the post buried in the ground is 400 mm. It took a force of 1250 N to pull the post out of the ground. What was the average shear strength of the soil?

1.75 The cross section of the post shown in Figure P1.73 is an equilateral triangle with each side of dimension a. If the average shear strength of the soil is τ, determine the force P needed to pull the post out of the ground in terms of τ, L, and a.

1.76 A force $P = 10$ lb is applied to the handle of a hammer in an effort to pull a nail out of the wood, as shown in Figure P1.76. The nail has a diameter of $\frac{1}{8}$ in. and is buried in wood to a depth of 2 in. Determine the average shear stress acting on the nail.

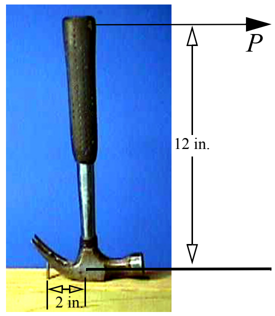


Figure P1.76

1.77 Two cast-iron pipes are adhesively bonded together over a length of 200 mm as shown in Figure P1.77. The outer diameters of the two pipes are 50 mm and 70 mm, and the wall thickness of each pipe is 10 mm. The two pipes separated while transmitting a force of 100 kN. What was the average shear stress in the adhesive just before the two pipes separated?

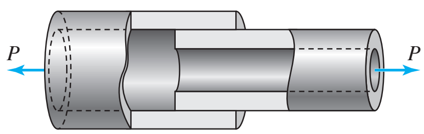


Figure P1.77

1.78 Two cast-iron pipes are adhesively bonded together over a length of 200 mm (Figure P1.78). The outer diameters of the two pipes are 50 mm and 70 mm, and the wall thickness of each pipe is 10 mm. The two pipes separated while transmitting a torque of 2 $kN \cdot m$. What was the average shear stress in the adhesive just before the two pipes separated?

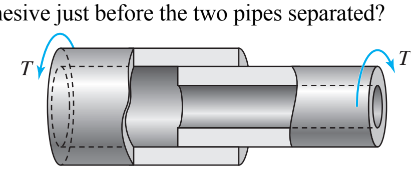


Figure P1.78

1.79 Two cast-iron pipes are held together by a bolt, as shown in Figure P1.79. The outer diameters of the two pipes are 50 mm and 70 mm, and the wall thickness of each pipe is 10 mm. The diameter of the bolt is 15 mm. The bolt broke while transmitting a torque of 2 $kN \cdot m$. On what surface(s) did the bolt break? What was the average shear stress in the bolt on the surface where it broke?

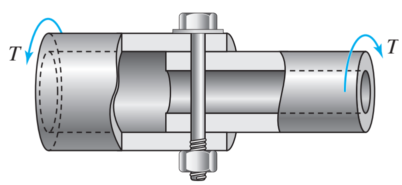


Figure P1.79

1.80 The can lid in Figure P1.80a gets punched on two sides AB and AC of an equilateral triangle ABC. Figure P1.80b is the top view showing relative location of the points. The thickness of the lid is t = 1/64 in. and the lid material can at most support a shear stress of 1800 psi. Assume a uniform shear stress during punching and point D acts like a pin joint. Use a= 1/2 in., b = 3 in. and c =1/4 in. Determine the minimum force F that must be applied to the can opener.

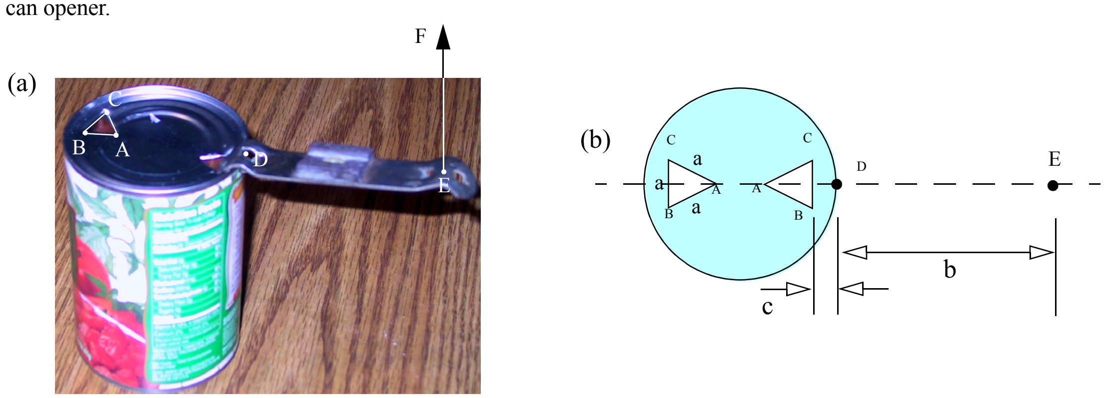


Figure P1.80

1.81 It is proposed to use $\frac{1}{2}$-in. diameter bolts in a 10-in.-diameter coupling for transferring a torque of 100 in. · kips from one 4-in.-diameter shaft onto another (Figure P1.81). The maximum average shear stress in the bolts is to be limited to 20 ksi. How many bolts are needed, and at what radius should the bolts be placed on the coupling? (Note there are multiple answers.)

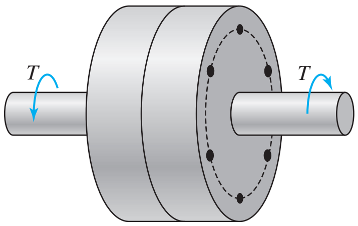


Figure P1.81

1.82 A human hand can comfortably apply a torsional moment of 15 in.·lb (Figure P1.82). (a) What should be the breaking shear strength of a seal between the lid and the bottle, assuming the lid has a diameter of $1\frac{1}{2}$ in. and a height of $\frac{1}{2}$ in.? (b) If the same sealing strength as in part (a) is used on a lid that is 1 in. in diameter and $\frac{1}{2}$ in. in height, what would be the torque needed to open the bottle?

Figure P1.82

1.83 The torsional moment on the lid is applied by hand exerting a force F on the handle of bottle opener as shown in Figure P1.83. Assume the average shear strength of the bond between the lid and the bottle is 10 psi. Determine the minimum force F needed to open the bottle. Use t = 3/8 in. d = 2 1/2 in. and a = 4 in.

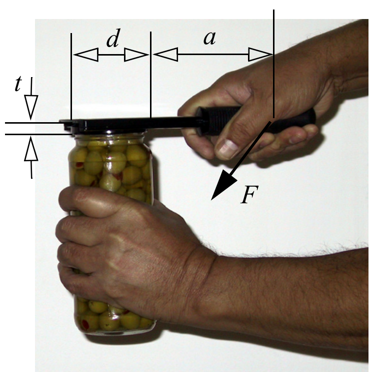


Figure P1.83

1.2 STRESS AT A POINT

The breaking of a structure starts at the point where the internal force intensity—that is, where stress exceeds some material characteristic value. This implies that we need to refine our definition of stress on a surface to that of *Stress at a Point*. But an infinite number of planes (surfaces) can pass through a point. Which imaginary surface do we shrink to zero? And when we shrink the surface area to zero, which equation should we use, (1.1) or (1.2)? Both difficulties can be addressed by assigning an orientation to the imaginary surface and to the internal force on this surface. We label these directions with subscripts for the stress components, in the same way that subscripts x, y, and z describe the components of vectors.

Figure 1.23 shows a body cut by an imaginary plane that has an outward normal in the i direction. On this surface we have a differential area ΔA_i on which a resultant force acts. $\Delta \boldsymbol{F}_j$ is the component of the force in the j direction. A component of average stress is $\Delta F_j / \Delta A_i$. If we shrink ΔA_i to zero we get the definition of a stress component at a point as shown by the Equation (1.3).

$$\sigma_{ij} = \lim_{\Delta A_i \to 0} \left(\frac{\Delta F_j}{\Delta A_i} \right) \tag{1.3}$$

direction of outward normal to imaginary cut surface

direction of internal force component

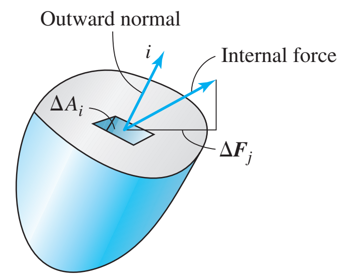


Figure 1.23 Stress at a point.

Now when we look at a stress component, the first subscript tells us the orientation of the imaginary surface and the second the direction of the internal force.

In three dimensions each subscript i and j can refer to an x, y, or z direction. In other words, there are nine possible combinations of the two subscripts. This is shown in the stress matrix in Equation (1.4). The diagonal elements in the stress matrix are the normal stresses and all off-diagonal elements represent the shear stresses.

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.4}$$

To specify the stress at a point, we need a magnitude and two directions. In three dimensions we need nine components of stress, and in two dimensions we need four components of stress to completely define stress at a point. Table 1.3 shows the number of components needed to specify a scalar, a vector, and stress. Now force, moment, velocity, and acceleration are all different quantities, but they all are called vectors. In a similar manner, stress belongs to a category called *tensors*. More specifically, stress is a **second-order tensor**,[2] where 'second order' refers to the exponent in the last row. In this terminology, a vector is a tensor of order 1, and a scalar is a tensor of order 0.

TABLE 1.3 **Comparison of number of components**

Quantity	One Dimension	Two Dimensions	Three Dimensions
Scalar	$1 = 1^0$	$1 = 2^0$	$1 = 3^0$
Vector	$1 = 1^1$	$2 = 2^1$	$3 = 3^1$
Stress	$1 = 1^2$	$4 = 2^2$	$9 = 3^2$

1.2.1 Sign convention

To obtain the sign of a stress component in Equation (1.3) we establish the following sign convention.

> **Sign Convention:** Differential area ΔA_i will be considered positive if the outward normal to the surface is in the positive i direction. If the outward normal is in the negative i direction, then ΔA_i will be considered negative.

We can now deduce the sign for stress. A stress component can be positive in two ways. Both the numerator and the denominator are positive or both the numerator and the denominator are negative in Equation (1.3). Alternatively, if numerator and the

[2]To be labeled as tensor, a quantity must also satisfy certain coordinate transformation properties, which will be discussed briefly in Chapter 8.

denominator in Equation (1.3) have: the same sign the stress component is *positive*; if they have opposite signs the stress component is *negative*.

We conclude this section with the following points to remember.

- Stress is an internal quantity that has units of force per unit area.
- A stress component at a *point* is specified by a magnitude and two directions. Stress at a point is a second-order tensor.
- Stress on a *surface* is specified by a magnitude and only one direction. Stress on a surface thus is a vector.
- The first subscript on stress gives the direction of the outward normal of the imaginary cut surface. The second subscript gives the direction of the internal force.
- The sign of a stress component is determined from the direction of the internal force and the direction of the outward normal to the imaginary cut surface.

1.3 STRESS ELEMENTS

The previous section showed that stress at a point is an abstract quantity. Stress on a surface, however, is easier to visualize as the intensity of a distributed force on a surface. A **stress element** is an imaginary object that helps us visualize stress at a point by constructing surfaces that have outward normals in the coordinate directions. In Cartesian coordinates the stress element is a cube; in cylindrical or a spherical coordinates the stress element is a fragment of a cylinder or a sphere, respectively. We start our discussion with the construction of a stress element in Cartesian coordinates to emphasize the basic construction principles. We can use a similar process to draw stress elements in cylindrical and spherical coordinate systems as demonstrated in Example 1.9.

1.3.1 Construction of a Stress Element for Axial Stress

Suppose we wish to visualize a positive stress component σ_{xx} at a point that may be generated in an bar under axial forces shown in Figure 1.24*a*. Around this point imagine an object that has sides with outward normals in the coordinate direction. The cube has six surfaces with outward normals that are either in the positive or in the negative coordinate direction, as shown in Figure 1.24. The first subscript of σ_{xx} tells us it must be on the surface that has an outward normal in the *x* direction. Thus, the two surfaces on which σ_{xx} will be shown are at *A* and *B*.

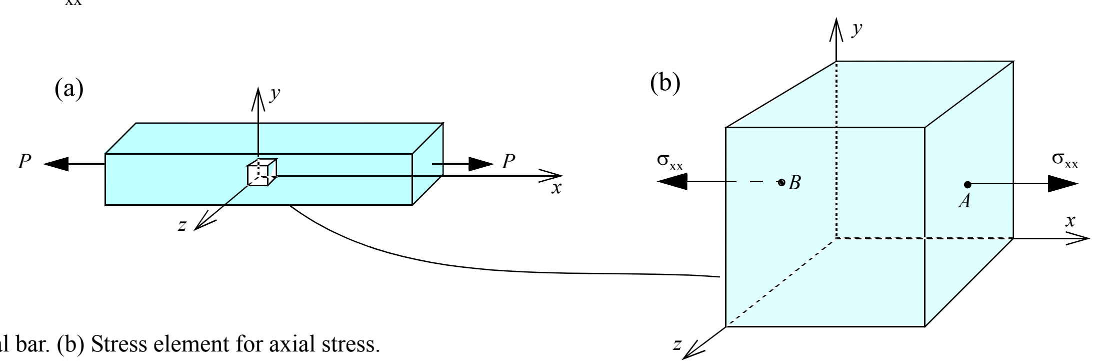


Figure 1.24 (a) Axial bar. (b) Stress element for axial stress.

The direction of the outward normal on surface *A* is in the positive *x* direction [the denominator is positive in Equation (1.3)]. For the stress component to be positive on surface *A*, the force must be in the positive *x* direction [the numerator must be positive in Equation (1.3)], as shown in Figure 1.24*b*.

The direction of the outward normal on surface *B* is in the negative *x* direction [the denominator is negative in Equation (1.3)]. For the stress component to be positive on surface *B*, the force must be in the negative *x* direction [the numerator must be negative in Equation (1.3)], as shown in Figure 1.24*b*.

The positive stress component σ_{xx} are pulling the cube in opposite directions; that is, the cube is in tension due to a positive normal stress component. We can use this information to draw normal stresses in place of subscripts. A tensile normal stress will pull the surface away from the interior of the element and a compressive normal stress will push the surface into the element. As mentioned earlier, normal stresses are usually reported as tension or compression and not as positive or negative.

It should be emphasized that the single arrow used to show the stress component does not imply that the stress component is a force. Showing the stress components as distributed forces on surfaces *A* and *B* as in Figure 1.25 is visually more accurate but very tedious to draw every time we need to visualize stress. We will show stress components using single arrows as in Figure 1.24, but visualize them as shown in Figure 1.25.

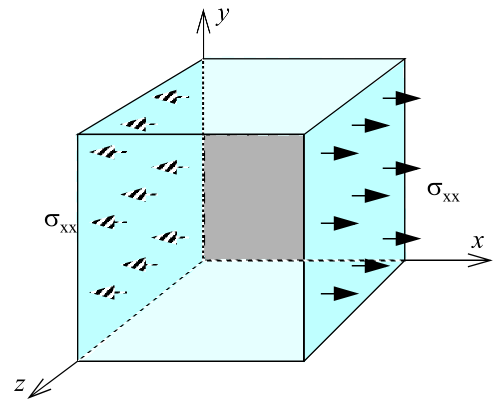


Figure 1.25 Stress components are distributed forces on a surface.

1.3.2 Construction of a Stress Element for Plane Stress

Plane stress is one of the two types of two-dimensional simplifications used in mechanics of materials. In Chapter 2 we will study the other type, plane strain. In Chapter 3 we will study the difference between the two types. By two dimensional we imply that one of the coordinates does not play a role in the description of the problem. If we choose z to be the coordinate, we set all stresses with subscript z to zero to get

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

We assume that the stress components in Equation (1.5) are positive. Let us consider the first row. The first subscript gives us the direction of the outward normal, which is the x direction. Surfaces A and B in Figure 1.26*a* have outward normals in the x direction, and it is on these surfaces that the stress component of the first row will be shown.

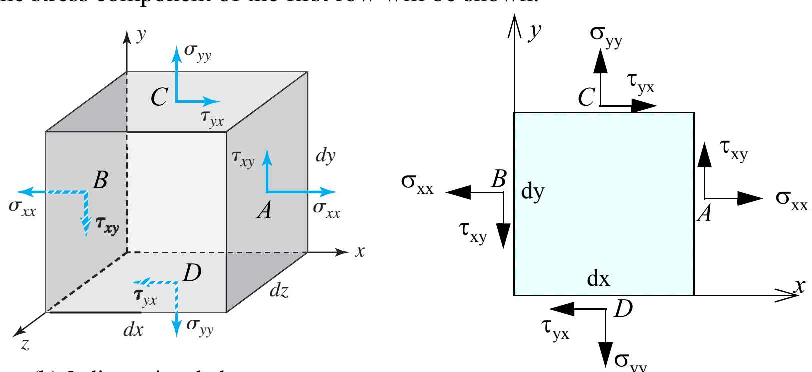


Figure 1.26 Plane stress: (a) 3-dimensional element (b) 2-dimensional element.

The direction of the outward normal on surface A is in the positive x direction [the denominator is positive in Equation (1.3)]. For the stress component to be positive on surface A, the force must be in the positive direction [the numerator must be positive in Equation (1.3)], as shown in Figure 1.26*a*.

The direction of the outward normal on surface B is in the negative x direction [the denominator is negative in Equation (1.3)]. For the stress component to be positive on surface B, the force must be in the negative direction [the numerator must be negative in Equation (1.3)], as shown in Figure 1.26*a*.

Now consider row 2 in the stress matrix in Equation (1.5). From the first subscript we know that the normal to the surface is in the y direction. Surface C has an outward normal in the positive y direction, therefore all forces on surface C are in the positive direction of the second subscript, as shown in Figure 1.26*a*. Surface D has an outward normal in the negative y direction, therefore all forces on surface D are in the negative direction of the second subscript, as shown in Figure 1.26*a*.

We note that the plane with outward normal in the z direction is stress-free. Stress-free surfaces are also called **free surfaces**, and these surfaces play an important role in stress analysis.

Figure 1.26b shows the two-dimensional representation of the stress element that will be seen looking down the z axis.

1.4 SYMMETRIC SHEAR STRESSES

If a body is in equilibrium, then all points on the body are in equilibrium. Is the stress element that represents a point on the body in equilibrium? To answer this question we need to convert the stresses into forces by multiplying by the surface area. We take a simple problem of plane stress and assume that the cube in Figure 1.26 has lengths of *dx*, *dy*, and *dz* in the coordinate directions. We draw a two-dimensional picture of the stress cube after multiplying each stress component by the surface area and get the force diagram of Figure 1.27.[3]

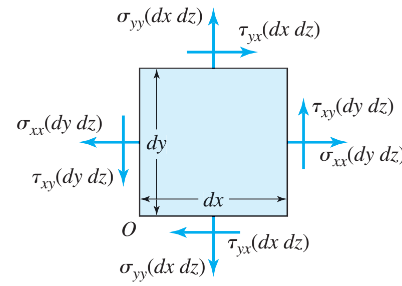


Figure 1.27 Force diagram for plane stress.

In Figure 1.27 we note that the equations of force equilibrium are satisfied by the assumed state of stress at a point. We consider the moment about point *O* and obtain

$$(\tau_{xy}\,dy\,dz)dx = (\tau_{yx}\,dx\,dz)dy \tag{1.6}$$

We cancel the differential volume (dx dy dz) on both sides to obtain

$$\boxed{\tau_{xy} = \tau_{yx}} \tag{1.7a}$$

In a similar manner we can show that

$$\boxed{\tau_{yz} = \tau_{zy}} \tag{1.7b}$$

$$\boxed{\tau_{zx} = \tau_{xz}} \tag{1.7c}$$

Equations (1.7a) through (1.7c) emphasize that shear stress is symmetric. The symmetry of shear stress implies that in *three dimensions* there are only *six independent* stress components out of the nine components necessary to specify stress at a point. In *two dimensions* there are only *three independent* stress components out of the four components necessary to specify stress at a point. In Figure 1.26 notice that the shear stress components τ_{xy} and τ_{yx} point either toward the corners or away from the corners. This observation can be used in drawing the symmetric pair of shear stresses after drawing the shear stress on one of the surfaces of the stress cube.

EXAMPLE 1.7

Show the non-zero stress components on the surfaces of the two cubes shown in different coordinate systems in Figure 1.28.

$$\begin{bmatrix} \sigma_{xx} = 80\text{MPa(T)} & \tau_{xy} = 30\text{MPa} & 0 \\ \tau_{yx} = 30\text{MPa} & \sigma_{yy} = 40\text{MPa(C)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

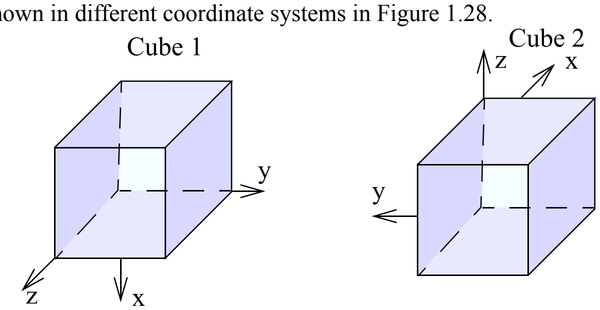


Figure 1.28 Cubes in different coordinate systems in Example 1.7.

PLAN

We can identify the surface with the outward normal in the direction of the first subscript. Using the sign convention and Equation (1.3) we draw the force in the direction of the second subscript.

[3]Figure 1.27 is only valid if we assume that the stresses are varying very slowly with the *x* and *y* coordinates. If this were not true, we would have to account for the increase in stresses over a differential element. But a more rigorous analysis will also reveal that shear stresses are symmetric, see Problem 1.105.

SOLUTION

Cube 1: The first subscript of σ_{xx} and τ_{xy} shows that the outward normal is in the x direction; hence these components will be shown on surfaces C and D in Figure 1.29*a*.

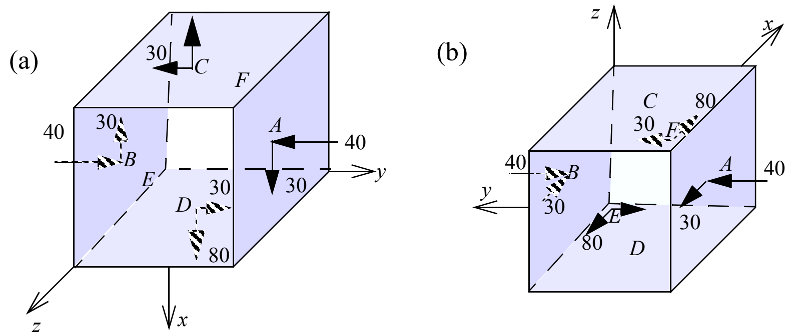


Figure 1.29 Solution of Example 1.7.(a) Cube 1. (b) Cube 2.

The outward normal on surface C is in the negative x direction; hence the denominator in Equation (1.3) is negative. Therefore on Figure 1.29*a:*

- The internal force has to be in the negative x direction to produce a positive (tensile) σ_{xx}.
- The internal force has to be in the negative y direction to produce a positive τ_{xy}.

The outward normal on surface D is in the positive x direction; hence the denominator in Equation (1.3) is positive. Therefore on Figure 1.29*a:*

- The internal force has to be in the positive x direction to produce a positive (tensile) σ_{xx}.
- The internal force has to be in the positive y direction to produce a positive τ_{xy}.

The first subscript of τ_{yx} and σ_{yy} shows that the outward normal is in the y direction; hence this component will be shown on surfaces A and B.

The outward normal on surface A is in the positive y direction; hence the denominator in Equation (1.3) is positive. Therefore on Figure 1.29*a:*

- The internal force has to be in the positive x direction to produce a positive τ_{yx}.
- The internal force has to be in the negative y direction to produce a negative (compressive) σ_{yy}.

The outward normal on surface B is in the negative y direction; hence the denominator in Equation (1.3) is negative. Therefore on Figure 1.29*a:*

- The internal force has to be in the negative x direction to produce a positive τ_{yx}.
- The internal force has to be in the positive y direction to produce a negative (compressive) σ_{yy}.

Cube 2: The first subscript of σ_{xx} and τ_{xy} shows that the outward normal is in the x direction; hence these components will be shown on surfaces E and F.

The outward normal on surface E is in the negative x direction; hence the denominator in Equation (1.3) is negative. Therefore on Figure 1.29*a:*

- The internal force has to be in the negative x direction to produce a positive (tensile) σ_{xx}.
- The internal force has to be in the negative y direction to produce a positive τ_{xy}.

The outward normal on surface F is in the positive x direction; hence the denominator in Equation (1.3) is positive. Therefore on Figure 1.29*a:*

- The internal force has to be in the positive x direction to produce a positive (tensile) σ_{xx}.
- The internal force has to be in the positive y direction to produce a positive τ_{xy}.

The first subscript of τ_{yx} and σ_{yy} shows that the outward normal is in the y direction; hence this component will be shown on surfaces A and B.

The outward normal on surface A is in the negative y direction; hence the denominator in Equation (1.3) is negative. Therefore on Figure 1.29*a:*

- The internal force has to be in the negative x direction to produce a positive τ_{yx}.
- The internal force has to be in the positive y direction to produce a negative (compressive) σ_{yy}.

The outward normal on surface B is in the positive y direction; hence the denominator in Equation (1.3) is positive. Therefore on Figure 1.29*a:*

- The internal force has to be in the positive x direction to produce a positive τ_{yx}.
- The internal force has to be in the negative y direction to produce a negative (compressive) σ_{yy}.

COMMENTS

1. Figure 1.30 shows the two-dimensional representations of stress cubes shown in Figure 1.29. These two-dimensional representations are easier to draw but it must be kept in mind that the point is in three-dimensional space with surfaces with outward normals in the z direction being stress free.

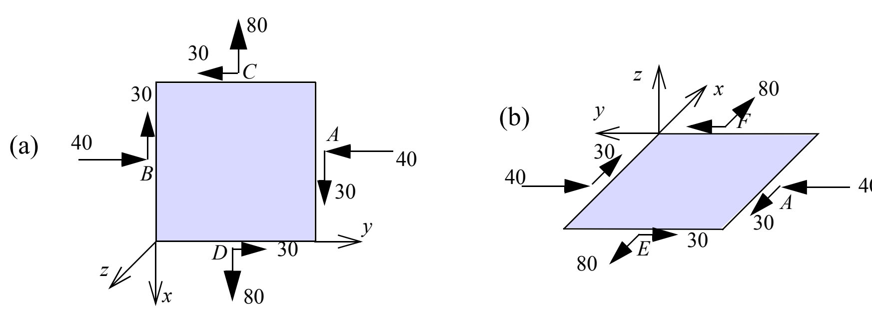


Figure 1.30 Two-dimensional depiction of the solution of Example 1.7.(a) Cube 1. (b) Cube 2.

2. We note that σ_{xx} pulls the surfaces outwards and σ_{yy} pushes the surfaces inwards in Figures 1.29 and 1.30 as these are tensile and compressive stresses, respectively. Hence, we can use this information to draw these stress components without using the subscripts.
3. The shear stress τ_{xx} and τ_{yx} either point towards the corner or away from the corner as seen in Figures 1.29 and 1.30. Using this information we can draw the shear stress on the appropriate surfaces after obtaining the direction on one surface using subscripts.

1.5* CONSTRUCTION OF A STRESS ELEMENT IN 3-DIMENSION

We once more visualize a cube with outward normals in the coordinate direction around the point we wish to show our stress components. The cube has six surfaces with outward normals that are either in the positive or in the negative coordinate direction, as shown in Figure 1.31. In other words, we have now accounted for the first subscript in our stress definition. We know that force is in the positive or negative direction of the second subscript. We use our sign convention to show the stress in the direction of the force on each of the six surfaces.

(a) $$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}$$

(b)

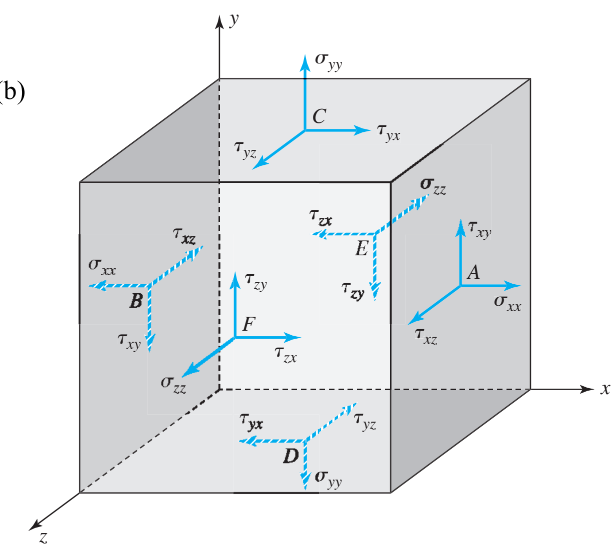


Figure 1.31 Stress cube showing all positive stress components in three dimensions.

To demonstrate the construction of the stress element we will assume that all nine stress components in the stress matrix shown in Figure 1.31*a* are positive. Let us consider the first row. The first subscript gives us the direction of the outward normal, which is the *x* direction. Surfaces *A* and *B* in Figure 1.31*b* have outward normals in the *x* direction, and it is on these surfaces that the stress component of the first row will be shown.

The direction of the outward normal on surface *A* is in the positive *x* direction [the denominator is positive in Equation (1.3)]. For the stress component to be positive on surface *A*, the force must be in the positive direction [the numerator must be positive in Equation (1.3)], as shown in Figure 1.31*b*.

The direction of the outward normal on surface *B* is in the negative *x* direction [the denominator is negative in Equation (1.3)]. For the stress component to be positive on surface *B*, the force must be in the negative direction [the numerator must be negative in Equation (1.3)], as shown in Figure 1.31*b*.

Now consider row 2 in the stress matrix in Figure 1.31*a*. From the first subscript we know that the normal to the surface is in the *y* direction. Surface *C* has an outward normal in the positive *y* direction, therefore all forces on surface *C* are in the positive direction of the second subscript, as shown in Figure 1.31*b*. Surface *D* has an outward normal in the negative *y* direction, therefore all forces on surface *D* are in the negative direction of the second subscript, as shown in Figure 1.31*b*.

By the same logic, the components of row 3 in the stress matrix are shown on surfaces *E* and *F* in Figure 1.31*b*.

EXAMPLE 1.8

Show the nonzero stress components on surfaces A, B, and C of the two cubes shown in different coordinate systems in Figure 1.32.

$$\begin{bmatrix} \sigma_{xx} = 80 \text{ MPa (T)} & \tau_{xy} = 30 \text{ MPa} & \tau_{xz} = -70 \text{ MPa} \\ \tau_{yx} = 30 \text{ MPa} & \sigma_{yy} = 0 & \tau_{yz} = 0 \\ \tau_{zx} = -70 \text{ MPa} & \tau_{zy} = 0 & \sigma_{zz} = 40 \text{ MPa (C)} \end{bmatrix}$$

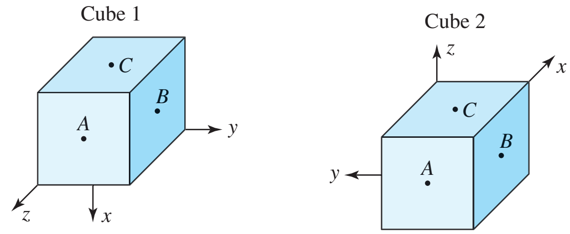


Figure 1.32 Cubes in different coordinate systems.

PLAN

We can identify the surface with the outward normal in the direction of the first subscript. Using the sign convention and Equation (1.3) we draw the force in the direction of the second subscript.

SOLUTION

Cube 1: The first subscript of σ_{xx}, τ_{xy}, and τ_{xz} shows that the outward normal is in the x direction; hence these components will be shown on surface C. The outward normal on surface C is in the negative x direction; hence the denominator in Equation (1.3) is negative. Therefore on Figure 1.33*a:*

- The internal force has to be in the negative x direction to produce a positive (tensile) σ_{xx}.
- The internal force has to be in the negative y direction to produce a positive τ_{xy}.
- The internal force has to be in the positive z direction to produce a negative τ_{xz}.

The first subscript of τ_{yx} shows that the outward normal is in the y direction; hence this component will be shown on surface B. The outward normal on surface B is in the positive y direction; hence the denominator in Equation (1.3) is positive. Therefore on Figure 1.33*a:*

- The internal force has to be in the positive x direction to produce a positive τ_{yx}.

The first subscript of τ_{zx}, σ_{zz} shows that the outward normal is in the z direction; hence these components will be shown on surface A. The outward normal on surface A is in the positive z direction; hence the denominator in Equation (1.3) is positive. Therefore on Figure 1.33*a:*

- The internal force has to be in the negative x direction to produce a negative τ_{zx}.
- The internal force has to be in the negative z direction to produce a negative (compressive) σ_{zz}.

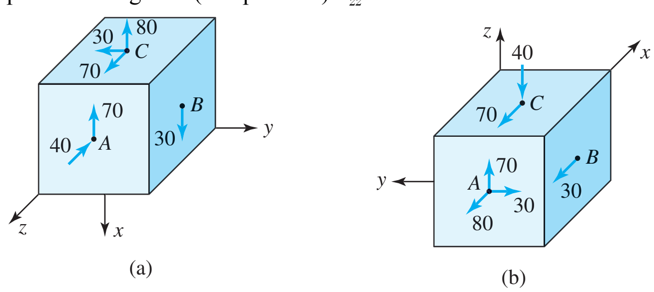


Figure 1.33 Solution of Example 1.8.

Cube 2: The first subscript of σ_{xx}, τ_{xy}, and τ_{xz} shows that the outward normal is in the x direction; hence these components will be shown on surface A. The outward normal on surface A is in the negative x direction; hence the denominator in Equation (1.3) is negative. Therefore in Figure 1.33*b*:

- The internal force has to be in the negative x direction to produce a positive (tensile) σ_{xx}.
- The internal force has to be in the negative y direction to produce a positive τ_{xy}.
- The internal force has to be in the positive z direction to produce a negative τ_{xz}.

The first subscript of τ_{yx} shows that the outward normal is in the y direction; hence this component will be shown on surface B. The outward normal on surface B is in the negative y direction; hence the denominator in Equation (1.3) is negative. Therefore in Figure 1.33*b*:

- The internal force has to be in the negative y direction to produce a positive τ_{yx}.

The first subscript of τ_{zx}, σ_{zz} shows that the outward normal is in the z direction; hence these components will be shown on surface C. The outward normal on surface C is in the positive z direction; hence the denominator in Equation (1.3) is positive. Therefore in Figure 1.33*b*:

- The internal force has to be in the negative x direction to produce a negative τ_{zx}.
- The internal force has to be in the negative z direction to produce a negative (compressive) σ_{zz}.

COMMENTS

1. In drawing the normal stresses we could have made use of the fact that σ_{xx} is tensile and hence pulls the surface outward. σ_{zz} is compressive and hence pushes the surface inward. This is a quicker way of getting the directions of these stress components than the arguments based on signs and subscripts.

2. Once we have drawn τ_{xy} and τ_{xz} using the subscripts, we could draw τ_{yx} and τ_{zx} using the observation that the pair of symmetric shear stresses point towards or away from the corner formed by the two adjoining surfaces, thus saving some effort in the construction of the stress element.

EXAMPLE 1.9

Show the following positive stress components on a stress element drawn in the spherical coordinate system shown in Figure 1.34.

$$\begin{bmatrix} \sigma_{rr} & \tau_{r\theta} & \tau_{r\phi} \\ \tau_{\theta r} & \sigma_{\theta\theta} & \tau_{\theta\phi} \\ \tau_{\phi r} & \tau_{\phi\theta} & \sigma_{\phi\phi} \end{bmatrix}$$

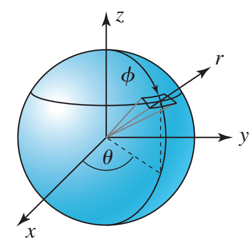


Figure 1.34 Stresses in spherical coordinates.

PLAN

We construct a stress element with surfaces that have outward normals in the r, θ, and ϕ directions. The first subscript will identify the surface on which the row of stress components is to be shown. The second subscript then will show the direction of the stress component on the surface.

SOLUTION

We draw a stress element with lines in the directions of r, θ, and ϕ, as shown in Figure 1.35.

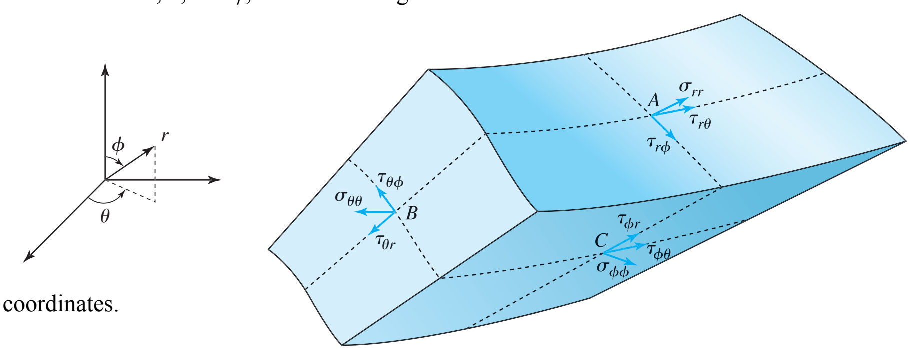


Figure 1.35 Stress element in spherical coordinates.

The stresses σ_{rr}, $\tau_{r\theta}$, and $\tau_{r\phi}$ will be on surface A in Figure 1.35. The outward normal on surface A is in the positive r direction. Thus the forces have to be in the positive r, θ, and ϕ directions to result in positive σ_{rr}, $\tau_{r\theta}$, and $\tau_{r\phi}$.

The stresses $\tau_{\theta r}$, $\sigma_{\theta\theta}$, and $\tau_{\theta\phi}$ will be on surface B in Figure 1.35. The outward normal on surface B is in the negative θ direction. Thus the forces have to be in the negative r, θ, and ϕ directions to result in positive $\tau_{\theta r}$, $\sigma_{\theta\theta}$, and $\tau_{\theta\phi}$.

The stresses $\tau_{\phi r}$, $\tau_{\phi\theta}$, and $\sigma_{\phi\phi}$ will be on surface C in Figure 1.35. The outward normal on surface C is in the positive ϕ direction. Thus the forces have to be in the positive r, θ, and ϕ directions to result in positive $\tau_{\phi r}$, $\tau_{\phi\theta}$, and $\sigma_{\phi\phi}$.

COMMENT

1. This example demonstrates that use of subscripts in determining the direction of stress components follows the same procedure as in cartesian coordinates even though the stress element is a fragment of a sphere.

Consolidate your knowledge

1. In your own words describe stress at a point and how it differs from stress on a surface

QUICK TEST 1.2

Time: 15 minutes/Total: 20 points

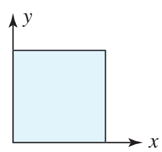
Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E. Give yourself one point for every correct answer (true or false) and one point for every correct explanation.

1. Stress at a point is a vector like stress on a surface.
2. In three dimensions stress has nine components.
3. In three dimensions stress has six independent components.
4. At a point in plane stress there are three independent stress components.
5. At a point in plane stress there are always six zero stress components.
6. If the shear stress on the left surface of an imaginary cut is upward and defined as positive, then on the right surface of the imaginary cut it is downward and negative.
7. A stress element can be drawn to any scale.
8. A stress element can be drawn at any orientation.
9. Stress components are opposite in direction on the two surfaces of an imaginary cut.
10. Stress components have opposite signs on the two surfaces of an imaginary cut.

PROBLEM SET 1.3

Plane Stress: Cartesian Coordinates

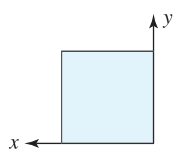
1.84 Show the stress components of a point in plane stress on the square in Figure P1.84.



$$\begin{bmatrix} \sigma_{xx} = 100\text{ MPa (T)} & \tau_{xy} = -75\text{ MPa} \\ \tau_{yx} = -75\text{ MPa} & \sigma_{yy} = 85\text{ MPa (T)} \end{bmatrix}$$

Figure P1.84

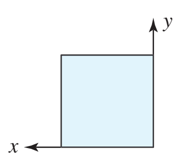
1.85 Show the stress components of a point in plane stress on the square in Figure P1.85.



$$\begin{bmatrix} \sigma_{xx} = 85\text{ MPa (C)} & \tau_{xy} = 75\text{ MPa} \\ \tau_{yx} = 75\text{ MPa} & \sigma_{yy} = 100\text{ MPa (T)} \end{bmatrix}$$

Figure P1.85

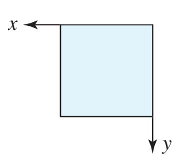
1.86 Show the stress components of a point in plane stress on the square in Figure P1.86.



$$\begin{bmatrix} \sigma_{xx} = 27\text{ ksi (C)} & \tau_{xy} = 18\text{ ksi} \\ \tau_{yx} = 18\text{ ksi} & \sigma_{yy} = 85\text{ ksi (T)} \end{bmatrix}$$

Figure P1.86

1.87 Show the stress components of a point in plane stress on the square in Figure P1.87.



$$\begin{bmatrix} \sigma_{xx} = 27\text{ ksi (C)} & \tau_{xy} = 18\text{ ksi} \\ \tau_{yx} = 18\text{ ksi} & \sigma_{yy} = 85\text{ ksi (T)} \end{bmatrix}$$

Figure P1.87

1.88 Show the nonzero stress components on the A, B, and C faces of the cube in Figure P1.88.

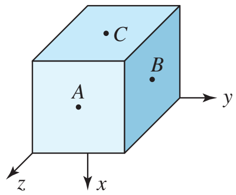


Figure P1.88

$$\begin{bmatrix} \sigma_{xx} = 70 \text{ MPa (T)} & \tau_{xy} = -40 \text{ MPa} & \tau_{xz} = 0 \\ \tau_{yx} = -40 \text{ MPa} & \sigma_{yy} = 85 \text{ MPa (C)} & \tau_{yz} = 0 \\ \tau_{zx} = 0 & \tau_{zy} = 0 & \sigma_{zz} = 0 \end{bmatrix}$$

1.89 Show the nonzero stress components on the A, B, and C faces of the cube in Figure P1.89.

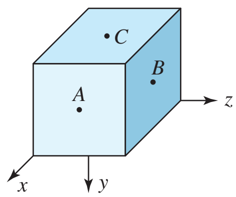


Figure P1.89

$$\begin{bmatrix} \sigma_{xx} = 70 \text{ MPa (T)} & \tau_{xy} = -40 \text{ MPa} & \tau_{xz} = 0 \\ \tau_{yx} = -40 \text{ MPa} & \sigma_{yy} = 85 \text{ MPa (C)} & \tau_{yz} = 0 \\ \tau_{zx} = 0 & \tau_{zy} = 0 & \sigma_{zz} = 0 \end{bmatrix}$$

Plane Stress: Polar Coordinates

1.90 Show the stress components of a point in plane stress on the stress element in polar coordinates in Figure P1.90.

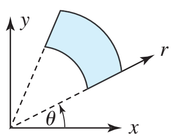


Figure P1.90

$$\begin{bmatrix} \sigma_{rr} = 125 \text{ MPa (T)} & \tau_{r\theta} = -65 \text{ MPa} \\ \tau_{\theta r} = -65 \text{ MPa} & \sigma_{\theta\theta} = 90 \text{ MPa (C)} \end{bmatrix}$$

1.91 Show the stress components of a point in plane stress on the stress element in polar coordinates in Figure P1.91.

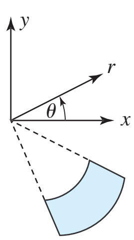


Figure P1.91

$$\begin{bmatrix} \sigma_{rr} = 125 \text{ MPa (T)} & \tau_{r\theta} = -65 \text{ MPa} \\ \tau_{\theta r} = -65 \text{ MPa} & \sigma_{\theta\theta} = 90 \text{ MPa (C)} \end{bmatrix}$$

1.92 Show the stress components of a point in plane stress on the stress element in polar coordinates in Figure P1.92.

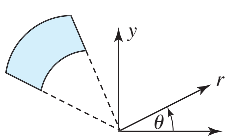


Figure P1.92

$$\begin{bmatrix} \sigma_{rr} = 18 \text{ ksi (T)} & \tau_{r\theta} = -12 \text{ ksi} \\ \tau_{\theta r} = -12 \text{ ksi} & \sigma_{\theta\theta} = 25 \text{ ksi (C)} \end{bmatrix}$$

1.93 Show the stress components of a point in plane stress on the stress element in polar coordinates in Figure P1.93.

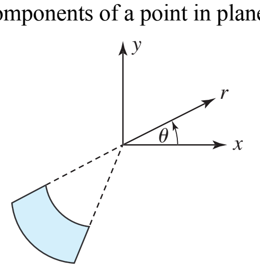
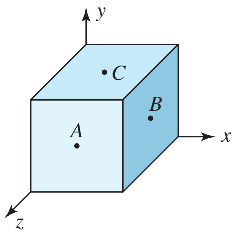


Figure P1.93

$$\begin{bmatrix} \sigma_{rr} = 25 \text{ ksi (C)} & \tau_{r\theta} = 12 \text{ ksi} \\ \tau_{\theta r} = 12 \text{ ksi} & \sigma_{\theta\theta} = 18 \text{ ksi (T)} \end{bmatrix}$$

Stress Element in 3-dimensions

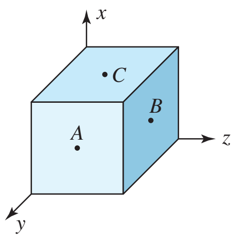
1.94 Show the nonzero stress components on the *A*, *B*, and *C* faces of the cube in Figure P1.94.



$$\begin{bmatrix} \sigma_{xx} = 100 \text{ MPa (T)} & \tau_{xy} = 200 \text{ MPa} & \tau_{xz} = -125 \text{ MPa} \\ \tau_{yx} = 200 \text{ MPa} & \sigma_{yy} = 175 \text{ MPa (C)} & \tau_{yz} = 225 \text{ MPa} \\ \tau_{zx} = -125 \text{ MPa} & \tau_{zy} = 225 \text{ MPa} & \sigma_{zz} = 150 \text{ MPa (T)} \end{bmatrix}$$

Figure P1.94

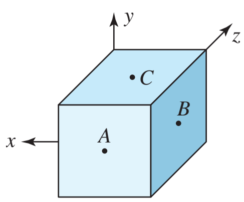
1.95 Show the nonzero stress components on the *A*, *B*, and *C* faces of the cube in Figure P1.95.



$$\begin{bmatrix} \sigma_{xx} = 90 \text{ MPa (T)} & \tau_{xy} = 200 \text{ MPa} & \tau_{xz} = 0 \\ \tau_{yx} = 200 \text{ MPa} & \sigma_{yy} = 175 \text{ MPa (T)} & \tau_{yz} = -225 \text{ MPa} \\ \tau_{zx} = 0 & \tau_{zy} = -225 \text{ MPa} & \sigma_{zz} = 150 \text{ MPa (C)} \end{bmatrix}$$

Figure P1.95

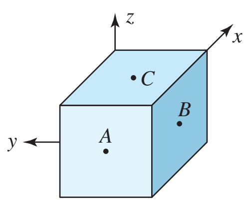
1.96 Show the nonzero stress components on the *A*, *B*, and *C* faces of the cube in Figure P1.96.



$$\begin{bmatrix} \sigma_{xx} = 0 & \tau_{xy} = 15 \text{ksi} & \tau_{xz} = 0 \\ \tau_{yx} = 15 \text{ ksi} & \sigma_{yy} = 10 \text{ ksi (T)} & \tau_{yz} = -25 \text{ ksi} \\ \tau_{zx} = 0 & \tau_{zy} = -25 \text{ ksi} & \sigma_{zz} = 20 \text{ ksi (C)} \end{bmatrix}$$

Figure P1.96

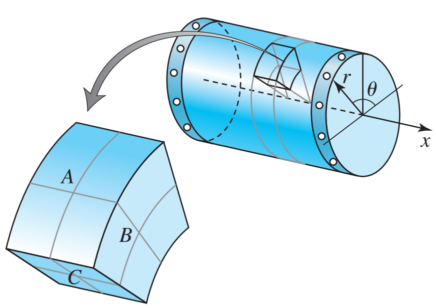
1.97 Show the nonzero stress components on the *A*, *B*, and *C* faces of the cube in Figure P1.97.



$$\begin{bmatrix} \sigma_{xx} = 0 & \tau_{xy} = -15 \text{ ksi} & \tau_{xz} = 0 \\ \tau_{yx} = -15 \text{ ksi} & \sigma_{yy} = 10 \text{ ksi (C)} & \tau_{yz} = 25 \text{ ksi} \\ \tau_{zx} = 0 & \tau_{zy} = 25 \text{ ksi} & \sigma_{zz} = 20 \text{ ksi (T)} \end{bmatrix}$$

Figure P1.97

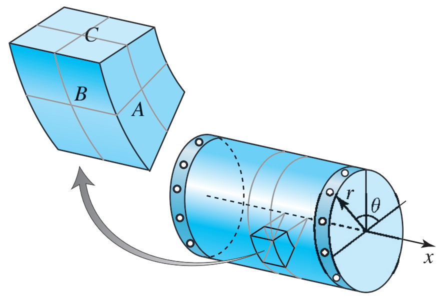
1.98 Show the nonzero stress components in the r, θ, and x cylindrical coordinate system on the A, B, and C faces of the stress elements shown in Figures P1.98.



$$\begin{bmatrix} \sigma_{rr} = 150 \text{ MPa (T)} & \tau_{r\theta} = -100 \text{ MPa} & \tau_{rx} = 125 \text{ MPa} \\ \tau_{\theta r} = -100 \text{ MPa} & \sigma_{\theta\theta} = 160 \text{ MPa (C)} & \tau_{\theta x} = 165 \text{ MPa} \\ \tau_{xr} = 125 \text{ MPa} & \tau_{x\theta} = 165 \text{ MPa} & \sigma_{xx} = 145 \text{ MPa (C)} \end{bmatrix}$$

Figure P1.98

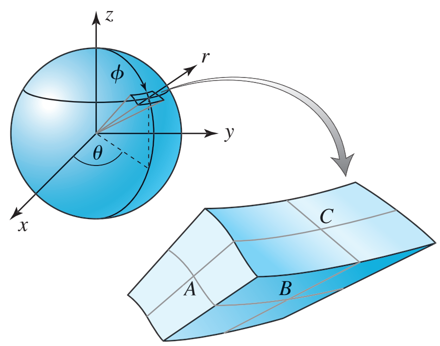
1.99 Show the nonzero stress components in the r, θ, and x cylindrical coordinate system on the A, B, and C faces of the stress elements shown in P1.99.



$$\begin{bmatrix} \sigma_{rr} = 10\text{ ksi (C)} & \tau_{r\theta} = 22\text{ ksi} & \tau_{rx} = 32\text{ ksi} \\ \tau_{\theta r} = 22\text{ ksi} & \sigma_{\theta\theta} = 0 & \tau_{\theta x} = 25\text{ ksi} \\ \tau_{xr} = 32\text{ ksi} & \tau_{x\theta} = 25\text{ ksi} & \sigma_{xx} = 20\text{ ksi (T)} \end{bmatrix}$$

Figure P1.99

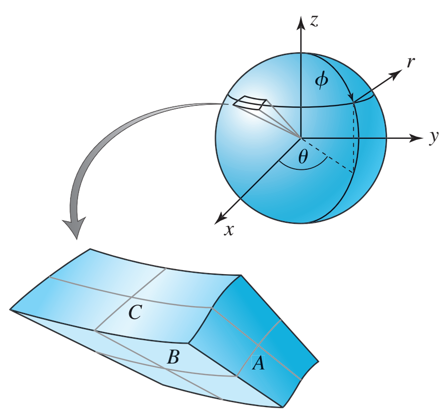
1.100 Show the nonzero stress components in the r, θ, and ϕ spherical coordinate system on the A, B, and C faces of the stress elements shown in Figure P1.100.



$$\begin{bmatrix} \sigma_{rr} = 150\text{ MPa (T)} & \tau_{r\theta} = 100\text{ MPa} & \tau_{r\phi} = 125\text{ MPa} \\ \tau_{\theta r} = 100\text{ MPa} & \sigma_{\theta\theta} = 160\text{ MPa (C)} & \tau_{\theta\phi} = -175\text{ MPa} \\ \tau_{\phi r} = 125\text{ MPa} & \tau_{\phi\theta} = -175\text{ MPa} & \sigma_{\phi\phi} = 135\text{ MPa (C)} \end{bmatrix}$$

Figure P1.100

1.101 Show the nonzero stress components in the r, θ, and ϕ spherical coordinate system on the A, B, and C faces of the stress elements shown in P1.101.



$$\begin{bmatrix} \sigma_{rr} = 0 & \tau_{r\theta} = -18\text{ ksi} & \tau_{r\phi} = 0 \\ \tau_{\theta r} = -18\text{ ksi} & \sigma_{\theta\theta} = 10\text{ ksi (C)} & \tau_{\theta\phi} = 25\text{ ksi} \\ \tau_{\phi r} = 0 & \tau_{\phi\theta} = 25\text{ ksi} & \sigma_{\phi\phi} = 20\text{ ksi (T)} \end{bmatrix}$$

Figure P1.101

Stretch yourself

1.102 Show that the normal stress σ_{xx} on a surface can be replaced by the equivalent internal normal force $\boldsymbol{N}$ and internal bending moments $\boldsymbol{M}_y$ and $\boldsymbol{M}_z$ as shown in Figure P1.102 and given by the equations (1.8a) through (1.8c).

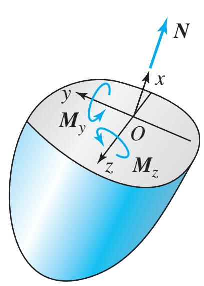


Figure P1.102

$$\boldsymbol{N} = \int_A \sigma_{xx}\, dA \tag{1.8a}$$

$$\boldsymbol{M}_y = -\int_A z\sigma_{xx}\, dA \tag{1.8b}$$

$$\boldsymbol{M}_z = -\int_A y\sigma_{xx}\, dA \tag{1.8c}$$

1.103 The normal stress on a cross section is given by $\sigma_{xx} = a + by$, where y is measured from the centroid of the cross section. If A is the cross-sectional area, I_{zz} is the area moment of inertia about the z axis, and $\boldsymbol{N}$ and $\boldsymbol{M}_z$ are the internal axial force and the internal bending moment given by Equations (1.8a) and (1.8c), respectively, prove the result in Equation (1.8).

$$\sigma_{xx} = \frac{\boldsymbol{N}}{A} - \left(\frac{\boldsymbol{M}_z}{I_{zz}}\right)y \tag{1.8}$$

We will encounter Equation (1.8) in combined axial and symmetric bending problems in later Chapter 10.

1.104 The normal stress on a cross section is given by $\sigma_{xx} = a + by + cz$, where y and z are measured from the centroid of the cross section. Using Equations (1.8a), (1.8b), and (1.8c) prove the result of Equation (1.9).

$$\sigma_{xx} = \frac{\boldsymbol{N}}{A} - \left(\frac{\boldsymbol{M}_z I_{yy} - \boldsymbol{M}_y I_{yz}}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{\boldsymbol{M}_y I_{zz} - \boldsymbol{M}_z I_{yz}}{I_{yy}I_{zz} - I_{yz}^2}\right)z \tag{1.9}$$

where I_{yy}, I_{zz}, and I_{yz} are the area moment of inertias. Equation (1.9) is used in the unsymmetrical bending of beams. Note that if either y or z is an axis of symmetry, then $I_{yz} = 0$. In such a case Equation (1.9) simplifies considerably.

1.105 An infinitesimal element in plane stress is shown in Figure P1.105. F_x and F_y are the body forces acting at the point and have the dimensions of force per unit volume. By converting stresses into forces and writing equilibrium equations obtain the results in Equations (1.10a) through (1.10c).

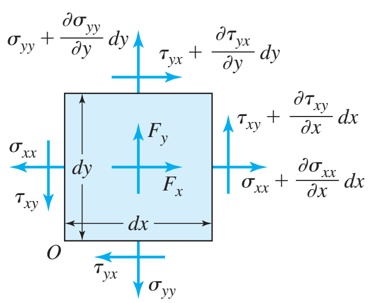


Figure P1.105

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0 \tag{1.10a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0 \tag{1.10b}$$

$$\tau_{xy} = \tau_{yx} \tag{1.10c}$$

1.6* CONCEPT CONNECTOR

Formulating the concept of stress took 500 years of struggle, briefly described in Section 1.6.1. In hindsight, the long evolution of quantifier of the strength is not surprising, because stress is not a single idea. It is a package of ideas that may be repackaged in many ways, depending on the needs of the analysis. Our chapter dealt with only one such package, called *Cauchy's stress,* which is used most in engineering design and analysis.

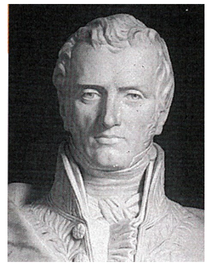
1.6.1 History: The Concept of Stress

The first formal treatment of strength is seen in the notes of the inventor and artist Leonardo da Vinci (1452–1519). Leonardo conducted several experiments on the strength of structural materials. His notes on "testing the strength of iron wires of various lengths" includes a sketch of how to measure the strength of wire experimentally. We now recognize that the dependence of the strength of a material on its length is due to the variations in manufacturing defects along the length.

The first indication of a concept of stress is found in Galileo Galilei (1564–1642). Galileo was born in Pisa and became a professor of mathematics at the age of twenty-five. For his belief in the Copernican theory on the motion of planets, which contradicted the interpretation of scriptures at that time, Galileo was put under house arrest for the last eight years of his life. During that period he wrote *Two New Sciences,* which lays out his contributions to the field of mechanics. Here he discusses the strength of a cantilever beam bending under the action of its own weight. Galileo viewed strength as the resistance to fracture, concluding that the strength of a bar depends on its cross-sectional area but is independent of its length. We will discuss Galileo's work on beam bending in Section 6.7.

The first person to differentiate between normal stress and shear stress was Charles-Augustin Coulomb (1736–1806) born in Angoulême. He was honored by the French Academy of Sciences in 1781 for his memoir *Theorie des machines simples,* in which he discussed friction between bodies. The theory of dry friction is named after him. Given the similarities between shear stress and friction, it seems only natural that Coulomb would be the first to differentiate between normal and shear stress. We will see other works of Coulomb when we come to failure theory and on the torsion of circular shafts.

Claude Louis Navier (1785–1857) initiated the mathematical development of the concept of stress starting with Newton's concept of a central force—one that acts along a line between two particles. His approach led to a controversy that took eighty years to resolve, as we shall see in Section 3.12.



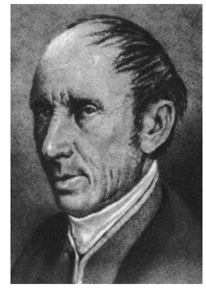


Figure 1.36 Pioneers of stress concept.

Claude Louis Navier

Augustin Cauchy.

Augustin Cauchy (1789–1857) brought the concept of stress to the form we studied it in this chapter (Figure 1.36). Forced to leave Paris, his birthplace, during the French Revolution, he took refuge in the village of Arcueil along with many other mathematicians and scientists of the period. At the age of twenty-one Cauchy worked engineering at the port of Cherbourg, which must have enhanced his understanding of the hydrodynamic concept of pressure. Pressure acts always normal to a surface, but Cauchy assumed that on an internal surface it acts at an angle hence he reasoned it can be resolved into components, the normal stress and shear stress. Combining this idea with his natural mathematical abilities, Cauchy developed what is now called Cauchy's stress. We shall see Cauchy's genius again in chapters on strain, material properties, and stress and strain transformation.

We have seen that unlike force, which is indivisible into more elementary ideas, stress is a package of ideas. Other packages will contain related but different elementary ideas. If instead of the cross-sectional area of an undeformed body, we use the cross-sectional area of a deformed body, then we get *true stress*. If we use the cross-sectional area of a deformed body and take the component of this area in the undeformed configuration, then we get *Kirchhoff's stress*. Still other stress measures are used in nonlinear analysis.

The English physicist James Clerk Maxwell (1831–1879) recognized the fact that the symmetry of shear stress given by Equations (1.7a) through (1.7c) is a consequence of there being no body moments. If a body moment is present, as in electromagnetic fields, then shear stresses will not be symmetric.

In Figure 1.15 we replaced the internal forces on a particle by a resultant force but no moment, because we assumed a central force between two particles. Woldemar Voigt (1850–1919), a German scientist who worked extensively with crystals is credited with introducing the stress tensor. Voigt recognized that in some cases a couple vector should be included when representing the interaction between particles by equivalent internal loads. If stress analysis is conducted at a very small scale, as the frontier research in nanostructures, then the moment transmitted by bonds between molecules may need to be included. The term *couple stress* is sometimes used to indicate the presence of a couple vector.

As history makes clear, stress has many definition. We choose the definition depending on the problem at hand and the information we are seeking. Most engineering analysis is linear and deals with large bodies, for which Cauchy's stress gives very good results. Cauchy's stress is thus sometimes referred to as engineering stress. Unless stated otherwise, *stress* always means Cauchy's stress in mechanics of materials and in this book.

1.7 CHAPTER CONNECTOR

In this chapter we have established the linkage between stresses, internal forces and moments, and external forces and moments. We have seen that to replace stresses by internal forces and internal moments requires knowledge of how the stress varies at each point on the surface. Although we can deduce simple stress behavior on a cross section, we would like to have other alternatives, in particular ones in which the danger of assuming physically impossible deformations is eliminated. This can be achieved if we can establish a relationship between stresses and deformations. Before we can discuss this relationship we need to understand the measure of deformation, which is the subject of Chapter 2. We will relate stresses and strains in

Chapter 3. In Section 3.2 we will synthesize the links introduced in Chapters 1, 2, and 3 into a logic that is used in mechanics of materials. We will use the logic to obtain simplified theories of one-dimensional structure members in Chapters 4, 5, 6, and 7.

All analyses in mechanics are conducted in a coordinate system, which is chosen for simplification whenever possible. Thus the stresses we obtain are in a given coordinate system. Now, our motivation for learning about stress is to define a measure of strength. Thus we can conclude that a material will fail when the stress at a point reaches some critical maximum value. There is no reason to expect that the stresses will be maximum in the arbitrarily chosen coordinate system. To determine the maximum stress at a point thus implies that we establish a relationship between stresses in different coordinate systems, as we shall do in Chapter 8.

We have seen that the concept of stress is a difficult one. If this concept is to be internalized so that an intuitive understanding is developed, then it is imperative that a discipline be developed to visualize the imaginary surface on which the stress is being considered.

POINTS AND FORMULAS TO REMEMBER

- Stress is an internal quantity.
- The internally distributed force on an imaginary cut surface of a body is called stress on a surface.
- Stress has units of force per unit area.
- 1 psi is equal to 6.95 kPa, or approximately 7 kPa. 1 kPa is equal to 0.145 psi, or approximately 0.15 psi.
- The internally distributed force that is normal (perpendicular) to the surface of an imaginary cut is called normal stress on a surface.
- Normal stress is usually reported as tensile or compressive and not as positive or negative.
- Average stress on a surface:
- $$\sigma_{av} = \boldsymbol{N}/A \quad (1.1) \qquad \tau_{av} = \boldsymbol{V}/A \quad (1.2)$$
- where σ_{av} is the average normal stress, τ_{av} is the average shear stress, $\boldsymbol{N}$ is the internal normal force, $\boldsymbol{V}$ is the internal shear force, and A is the cross-sectional area of the imaginary cut on which $\boldsymbol{N}$ and $\boldsymbol{V}$ act.
- The relationship of external forces (and moments) to internal forces and the relationship of internal forces to stress distributions are two distinct ideas.
- Stress at a point:
- $$\sigma_{ij} = \lim_{\Delta A_i \to 0}\left(\frac{\Delta \boldsymbol{F}_j}{\Delta A_i}\right) \quad (1.3)$$
- where i is the direction of the outward normal to the imaginary cut surface, and j is the outward normal to the direction of the internal force.
- Stress at a point needs a magnitude and two directions to specify it, i.e., stress at a point is a second-order tensor.
- The first subscript on stress denotes the direction of the outward normal of the imaginary cut surface. The second subscript denotes the direction of the internal force.
- The sign of a stress component is determined from the direction of the internal force and the direction of the outward normal to the imaginary cut surface.
- Stress element is an imaginary object that helps us visualize stress at a point by constructing surfaces that have outward normals in the coordinate directions.
- $$\tau_{xy} = \tau_{yx} \quad (1.7a) \qquad \tau_{yz} = \tau_{zy} \quad (1.7b) \qquad \tau_{zx} = \tau_{xz} \quad (1.7c)$$
- Shear stress is symmetric.
- In three dimensions there are nine stress components, but only six are independent.
- In two dimensions there are four stress components, but only three are independent.
- The pair of symmetric shear stress components point either toward the corner or away from the corner on a stress element.
- A point on a free surface is said to be in plane stress.

CHAPTER TWO
STRAIN

Learning objectives

1. Understand the concept of strain.
2. Understand the use of approximate deformed shapes for calculating strains from displacements.

How much should the drive belts (Figure 2.1*a*) stretch when installed? How much should the nuts in the turnbuckles (Figure 2.1*b*) be tightened when wires are attached to a traffic gate? Intuitively, the belts and the wires must stretch to produce the required tension. As we see in this chapter *strain* is a measure of the intensity of deformation used in the design against deformation failures.

(a)

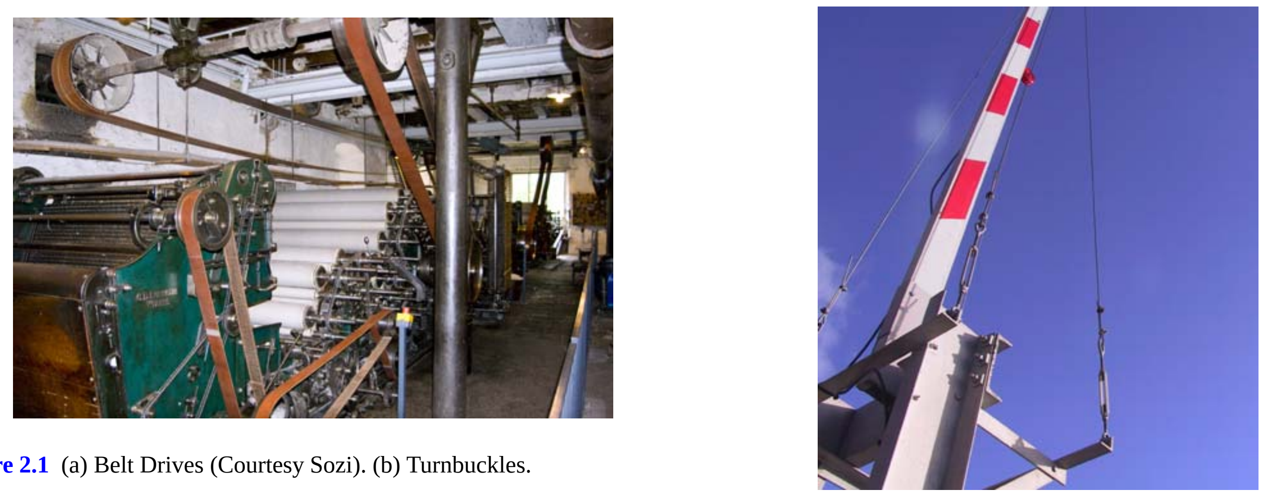


Figure 2.1 (a) Belt Drives (Courtesy Sozi). (b) Turnbuckles.

A change in shape can be described by the displacements of points on the structure. The relationship of strain to displacement depicted in Figure 2.2 is thus a problem in geometry—or, since displacements involve motion, a problem in kinematics. This relationship shown in Figure 2.2 is a link in the logical chain by which we shall relate displacements to external forces as discussed in Section 3.2. The primary tool for relating displacements and strains is drawing the body's approximate deformed shape. This is analogous to drawing a free-body diagram to obtain forces.

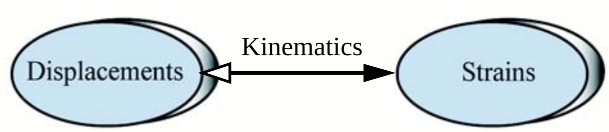


Figure 2.2 Strains and displacements.

2.1 DISPLACEMENT AND DEFORMATION

Motion due to applied forces is of two types. (i) In rigid-body motion, the body as a whole moves without changing shape. (ii) In motion due to *deformation*, the body shape change. But, how do we decide if a moving body is undergoing deformation?

In rigid body, by definition, the distance between any two points does not change. In translation, for example, any two points on a rigid body will trace parallel trajectories. If the distance between the trajectories of two points changes, then the body is deforming. In addition to translation, a body can also rotate. On rigid bodies all lines rotate by equal amounts. If the angle between two lines on the body changes, then the body is deforming.

Whether it is the distance between two points or the angle between two lines that is changing, deformation is described in terms of the relative movements of points on the body. **Displacement** is the absolute movement of a point with respect to a fixed reference frame. **Deformation** is the relative movement with respect to another point on the same body. Several examples and problems in this chapter will emphasize the distinction between deformation and displacement.

2.2 LAGRANGIAN AND EULERIAN STRAIN

A handbook cost $L_0 = \$100$ a year ago. Today it costs $L_f = \$125$. What is the percentage change in the price of the handbook? Either of the two answers is correct. (i) The book costs 25% *more* than what it cost a year ago. (ii) The book cost 20% *less* a year ago than what it costs today. The first answer treats the original value as a reference: $\text{change} = [(L_f - L_0)/L_0] \times 100$. The second answer uses the final value as the reference: $\text{change} = [(L_0 - L_f)/L_f] \times 100$. The two arguments emphasize the necessity to specify the reference value from which change is calculated.

In the contexts of deformation and strain, this leads to the following definition: **Lagrangian strain** is computed by using the original undeformed geometry as a reference. **Eulerian strain** is computed using the final deformed geometry as a reference. The Lagrangian description is usually used in solid mechanics. The Eulerian description is usually used in fluid mechanics. When a material undergoes very large deformations, such as in soft rubber or projectile penetration of metals, then either description may be used, depending on the need of the analysis. We will use Lagrangian strain in this book, except in a few “stretch yourself” problems.

2.3 AVERAGE STRAIN

In Section 2.1 we saw that to differentiate the motion of a point due to translation from deformation, we need to measure changes in length. To differentiate the motion of a point due to rotation from deformation, we need to measure changes in angle. In this section we discuss *normal* strain and *shear* strain, which are measures of changes in length and angle, respectively.

2.3.1 Normal Strain

Figure 2.3 shows a line on the surface of a balloon that grows from its original length L_0 to its final length L_f as the balloon expands. The change in length $L_f - L_0$ represents the deformation of the line. *Average* normal strain is the intensity of deformation defined as a ratio of deformation to original length.

$$\boxed{\varepsilon_{av} = \frac{L_f - L_0}{L_0}} \qquad (2.1)$$

where ε is the Greek symbol epsilon used to designate normal strain and the subscript av emphasizes that the normal strain is an average value. The following sign convention follows from Equation (2.1). Elongations ($L_f > L_0$) result in **positive normal strains**. Contractions ($L_f < L_0$) result in **negative normal strains**.

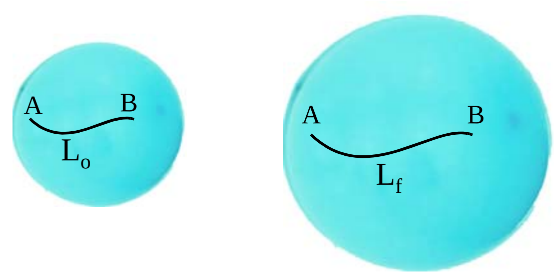
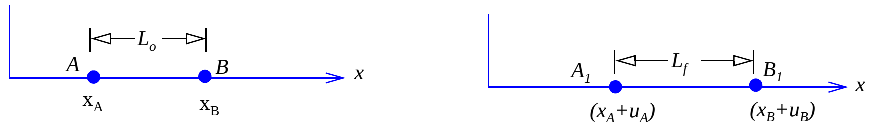


Figure 2.3 Normal strain and change in length.

An alternative form of Equation (2.1) is:

$$\boxed{\varepsilon_{av} = \frac{\delta}{L_0}} \qquad (2.2)$$

where the Greek letter delta (δ) designates deformation of the line and is equal to $L_f - L_0$.



$$L_0 = x_B - x_A$$

$$L_f = (x_B + u_B) - (x_A + u_A) = L_o + (u_B - u_A)$$

Figure 2.4 Normal strain and displacement.

We now consider a special case in which the displacements are in the direction of a straight line. Consider two points A and B on a line in the x direction, as shown in Figure 2.4. Points A and B move to A_1 and B_1, respectively. The coordinates of the point change from x_A and x_B to $x_A + u_A$ and $x_B + u_B$, respectively. From Figure 2.4 we see that $L_0 = x_B - x_A$ and $L_f - L_0 = u_B - u_A$. From Equation (2.1) we obtain

$$\boxed{\varepsilon_{av} = \frac{u_B - u_A}{x_B - x_A}} \tag{2.3}$$

where u_A and u_B are the displacements of points A and B, respectively. Hence $u_B - u_A$ is the relative displacement, that is, it is the deformation of the line.

2.3.2 Shear Strain

Figure 2.5 shows an elastic band with a grid attached to two wooden bars using masking tape. The top wooden bar is slid to the right, causing the grid to deform. As can be seen, the angle between lines ABC changes. The measure of this change of angle is defined by shear strain, usually designated by the Greek letter gamma (γ). The average Lagrangian shear strain is defined as the change of angle from a right angle:

$$\boxed{\gamma_{av} = \frac{\pi}{2} - \alpha} \tag{2.4}$$

where the Greek letter alpha (α) designates the final angle measured in radians (rad), and the Greek letter pi (π) equals 3.14159 rad. Decreases in angle ($\alpha < \pi/2$) result in **positive shear strains**. Increases in angle ($\alpha > \pi/2$) result in **negative shear strains**.

(a) (b)

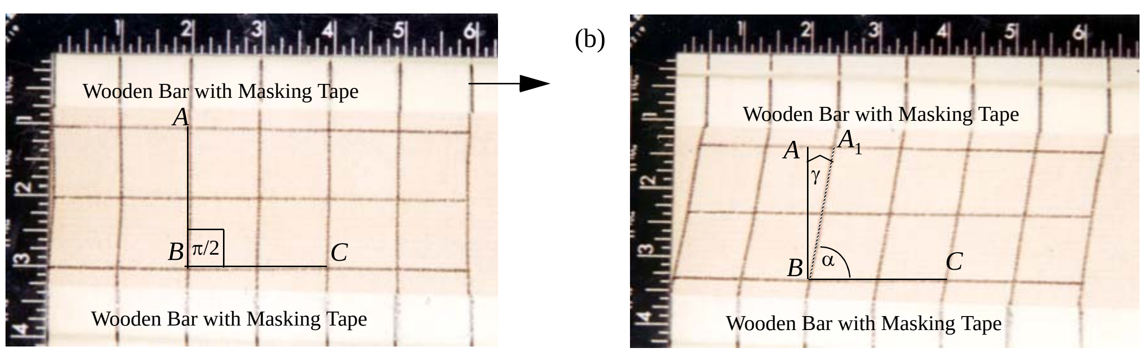


Figure 2.5 Shear strain and angle changes. (*a*) Undeformed grid. (*b*) Deformed grid.

2.3.3 Units of Average Strain

Equation (2.1) shows that normal strain is dimensionless, and hence should have no units. However, to differentiate average strain and strain at a point (discussed in Section 2.5), average normal strains are reported in units of length, such as in/in, cm/cm, or m/m. Radians are used in reporting average shear strains.

A percentage change is used for strains in reporting large deformations. Thus a normal strain of 0.5% is equal to a strain of 0.005. The Greek letter mu (μ) representing *micro* ($\mu = 10^{-6}$), is used in reporting small strains. Thus a strain of 1000 μ in/in is the same as a normal strain of 0.001 in/in.

EXAMPLE 2.1

The displacements in the x direction of the rigid plates in Figure 2.6 due to a set of axial forces were observed as given. Determine the axial strains in the rods in sections AB, BC, and CD.

$u_A = -0.0100$ in. $u_B = 0.0080$ in.

$u_C = -0.0045$ in. $u_D = 0.0075$ in.

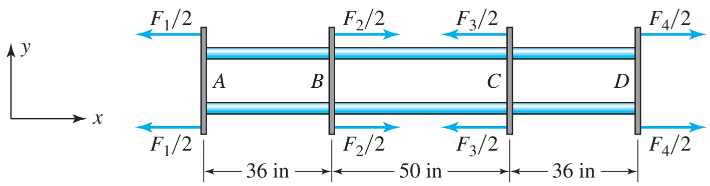


Figure 2.6 Axial displacements in Example 2.1.

PLAN

We first calculate the relative movement of rigid plates in each section. From this we can calculate the normal strains using Equation (2.3).

SOLUTION

The strains in each section can be found as shown in Equations (E1) through (E3).

$$\varepsilon_{AB} = \frac{u_B - u_A}{x_B - x_A} = \frac{0.018 \text{ in.}}{36 \text{ in.}} = 0.0005\frac{\text{in.}}{\text{in.}} \tag{E1}$$

ANS. $\varepsilon_{AB} = 500 \ \mu\text{in./in.}$

$$\varepsilon_{BC} = \frac{u_C - u_B}{x_C - x_B} = \frac{-0.0125 \text{ in.}}{50 \text{ in.}} = -0.00025\frac{\text{in.}}{\text{in.}} \tag{E2}$$

ANS. $\varepsilon_{BC} = -250 \ \mu\text{in./in.}$

$$\varepsilon_{CD} = \frac{u_D - u_C}{x_D - x_C} = \frac{0.012 \text{ in.}}{36 \text{ in.}} = 0.0003333\frac{\text{in.}}{\text{in.}} \tag{E3}$$

ANS. $\varepsilon_{CD} = 333.3 \ \mu\text{in./in.}$

COMMENT

1. This example brings out the difference between the displacements, which were given, and the deformations, which we calculated before finding the strains.

EXAMPLE 2.2

A bar of hard rubber is attached to a rigid bar, which is moved to the right relative to fixed base A as shown in Figure 2.7. Determine the average shear strain at point A.

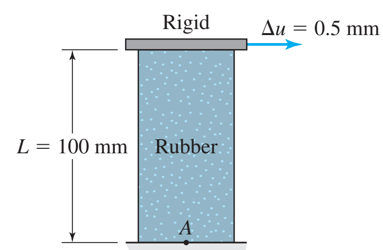


Figure 2.7 Geometry in Example 2.2.

PLAN

The rectangle will become a parallelogram as the rigid bar moves. We can draw an approximate deformed shape and calculate the change of angle to determine the shear strain.

SOLUTION

Point B moves to point B_1, as shown in Figure 2.8. The shear strain represented by the angle between BAB_1 is:

$$\gamma = \tan^{-1}\left(\frac{BB_1}{AB}\right) = \tan^{-1}\left(\frac{0.5 \text{ mm}}{100 \text{ mm}}\right) = 0.005 \text{ rad} \tag{E1}$$

ANS. $\gamma = 5000\mu\text{rad}$.

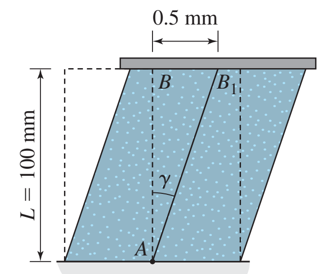


Figure 2.8 Exaggerated deformed shape.

COMMENTS

1. We assumed that line *AB* remained straight during the deformation in Figure 2.8. If this assumption were not valid, then the shear strain would vary in the vertical direction. To determine the varying shear strain, we would need additional information. Thus our assumption of line *AB* remaining straight is the simplest assumption that accounts for the given information.
2. The values of γ and tan γ are roughly the same when the argument of the tangent function is small. Thus for small shear strains the tangent function can be approximated by its argument.

EXAMPLE 2.3

A thin ruler, 12 in. long, is deformed into a circular arc with a radius of 30 in. that subtends an angle of 23° at the center. Determine the average normal strain in the ruler.

PLAN

The final length is the length of a circular arc and original length is given. The normal strain can be obtained using Equation (2.1).

SOLUTION

The original length $L_0 = 12$ in. The angle subtended by the circular arc shown in Figure 2.9 can be found in terms of radians:

$$\Delta\theta = \frac{(23^o)\pi}{180^o} = 0.4014 \text{ rads} \tag{E1}$$

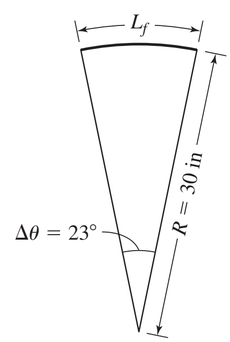


Figure 2.9 Deformed geometry in Example 2.3.

The length of the arc is:

$$L_f = R\Delta\theta = 12.04277 \text{ in.} \tag{E2}$$

and average normal strain is

$$\varepsilon_{av} = \frac{L_f - L_0}{L_0} = \frac{0.04277 \text{ in.}}{12 \text{ in.}} = 3.564(10^{-3})\frac{\text{in.}}{\text{in.}} \tag{E3}$$

ANS. $\varepsilon_{av} = 3564\ \mu\text{in./in.}$

COMMENTS

1. In Example 2.1 the normal strain was generated by the displacements in the axial direction. In this example the normal strain is being generated by bending.
2. In Chapter 6 on the symmetric bending of beams we shall consider a beam made up of lines that will bend like the ruler and calculate the normal strain due to bending as we calculated it in this example.

EXAMPLE 2.4

A belt and a pulley system in a VCR has the dimensions shown in Figure 2.10. To ensure adequate but not excessive tension in the belts, the average normal strain in the belt must be a minimum of 0.019 mm/mm and a maximum of 0.034 mm/mm. What should be the minimum and maximum undeformed lengths of the belt to the nearest 1/10th of a millimeter?

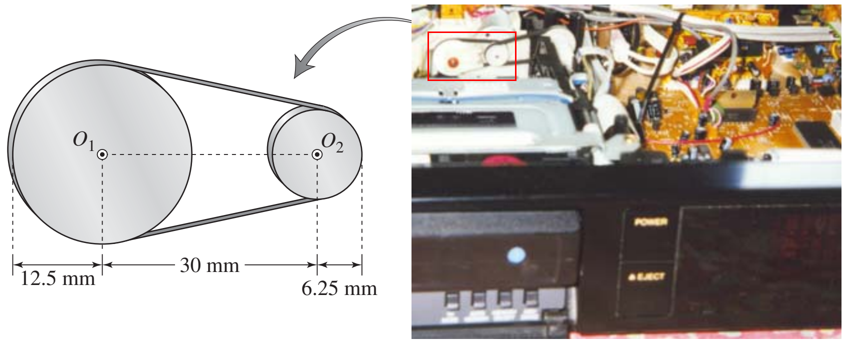


Figure 2.10 Belt and pulley in a VCR.

PLAN

The belt must be tangent at the point where it comes in contact with the pulley. The deformed length of the belt is the length of belt between the tangent points on the pulleys, plus the length of belt wrapped around the pulleys. Once we calculate the deformed length of the belt using geometry, we can find the original length using Equation (2.1) and the given limits on normal strain.

SOLUTION

We draw radial lines from the center to the tangent points A and B, as shown in Figure 2.11. The radial lines O_1A and O_2B must be perpendicular to the belt AB, hence both lines are parallel and at the same angle θ with the horizontal. We can draw a line parallel to AB through point O_2 to get line CO_2. Noting that CA is equal to O_2B, we can obtain CO_1 as the difference between the two radii.

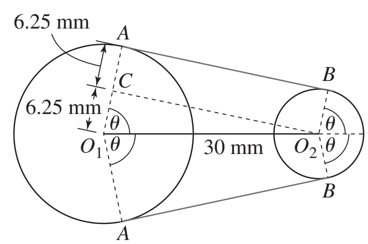


Figure 2.11 Analysis of geometry.

Triangle O_1CO_2 in Figure 2.11 is a right triangle, so we can find side CO_2 and the angle θ as:

$$AB = CO_2 = \sqrt{(30\text{ mm})^2 - (6.25\text{ mm})^2} = 29.342\text{ mm} \tag{E1}$$

$$\cos\theta = \frac{CO_1}{O_1O_2} = \frac{6.25\text{ mm}}{30\text{ mm}} \quad or \quad \theta = \cos^{-1}(0.2083) = 1.3609\text{ rad} \tag{E2}$$

The deformed length L_f of the belt is the sum of arcs AA and BB and twice the length AB:

$$AA = (12.5\text{ mm})(2\pi - 2\theta) = 44.517\text{ mm} \tag{E3}$$

$$BB = (6.25\text{ mm})(2\theta) = 17.011\text{ mm} \tag{E4}$$

$$L_f = 2(AB) + AA + BB = 120.212\text{ mm} \tag{E5}$$

We are given that $0.019 \le \varepsilon \le 0.034$. From Equation (2.1) we obtain the limits on the original length:

$$\varepsilon = \frac{L_f - L_0}{L_0} \le 0.034 \quad \text{or} \quad L_0 \ge \frac{120.212}{1 + 0.034}\text{ mm} \quad \text{or} \quad L_0 \ge 116.26\text{ mm} \tag{E6}$$

$$\varepsilon = \frac{L_f - L_0}{L_0} \ge 0.019 \quad or \quad L_0 \le \frac{120.212}{1 + 0.019}\text{ mm} \quad or \quad L_0 \le 117.97\text{ mm} \tag{E7}$$

To satisfy Equations (E6) and (E7) to the nearest millimeter, we obtain the following limits on the original length L_0:

ANS. $116.3\text{ mm} \le L_0 \le 117.9\text{ mm}$

COMMENTS

1. We rounded upward in Equation (E6) and downwards in Equation (E7) to ensure the inequalities.

2. Tolerances in dimensions must be specified for manufacturing. Here we have a tolerance range of 0.6 mm.
3. The difficulty in this example is in the analysis of the geometry rather than in the concept of strain. This again emphasizes that the analysis of deformation and strain is a problem in geometry. Drawing the approximate deformed shape is essential.

2.4 SMALL-STRAIN APPROXIMATION

In many engineering problems, a body undergoes only small deformations. A significant simplification can then be achieved by approximation of small strains, as demonstrated by the simple example shown in Figure 2.12. Due to a force acting on the bar, point P moves by an amount D at an angle θ to the direction of the bar. From the cosine rule in triangle APP_1, the length L_f can be found in terms of L_0, D, and θ:

$$L_f = \sqrt{L_0^2 + D^2 + 2L_0 D\cos\theta} = L_0\sqrt{1 + \left(\frac{D}{L_0}\right)^2 + 2\left(\frac{D}{L_0}\right)\cos\theta}$$

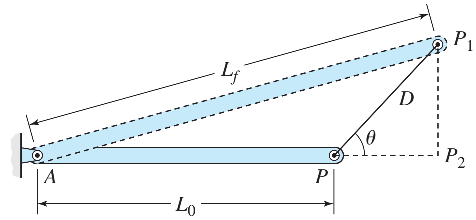


Figure 2.12 Small normal-strain calculations.

From Equation (2.1) we obtain the average normal strain in bar AP:

$$\varepsilon = \frac{L_f - L_0}{L_0} = \sqrt{1 + \left(\frac{D}{L_0}\right)^2 + 2\left(\frac{D}{L_0}\right)\cos\theta} - 1 \tag{2.5}$$

Equation (2.5) is valid regardless of the magnitude of the deformation D. Now suppose that D/L_0 is small. In such a case we can neglect the $(D/L_0)^2$ term and expand the radical by binomial[1] expansion:

$$\varepsilon \approx \left(1 + \frac{D}{L_0}\cos\theta + \dots + \dots\right) - 1$$

Neglecting the higher-order terms, we obtain an approximation for small strain in Equation (2.6).

$$\varepsilon_{\text{small}} = \frac{D\cos\theta}{L_0} \tag{2.6}$$

In Equation (2.6) the deformation D and strain are linearly related, whereas in Equation (2.5) deformation and strain are nonlinearly related. This implies that small-strain calculations require only a linear analysis, a significant simplification.

Equation (2.6) implies that in small-strain calculations only the component of deformation in the direction of the original line element is used. We will make significant use of this observation. Another way of looking at small-strain approximation is to say that the deformed length AP_1 is approximated by the length AP_2.

TABLE 2.1 **Small-strain approximation**

$\varepsilon_{\text{small}}$, **[Equation (2.6)]**	ε, **[Equation (2.5)]**	**% Error,** $\left(\frac{\varepsilon - \varepsilon_{\text{small}}}{\varepsilon}\right) \times 100$
1.000	1.23607	19.1
0.500	0.58114	14.0
0.100	0.10454	4.3
0.050	0.00512	2.32
0.010	0.01005	0.49
0.005	0.00501	0.25
0.001	0.00100	0.05

What is small strain? To answer this question we compare strains from Equation (2.6) to those from Equation (2.5). For different values of small strain and for $\theta = 45°$, the ratio of D/L is found from Equation (2.6), and the strain from Equation

[1]For small d, binomial expansion is $(1 + d)^{1/2} = 1 + d/2$ + terms of d^2 and higher order.

(2.5) is calculated as shown in Table 2.1. Equation (2.6) is an approximation of Equation (2.5), and the error in the approximation is shown in the third column of Table 2.1. It is seen from Table 2.1 that when the strain is less than 0.01, then the error is less than 1%, which is acceptable for most engineering analyses.

We conclude this section with summary of our observations.

1. Small-strain approximation may be used for strains less than 0.01.
2. Small-strain calculations result in linear deformation analysis.
3. Small normal strains are calculated by using the deformation component in the original direction of the line element, regardless of the orientation of the deformed line element.
4. In small shear strain (γ) calculations the following approximations may be used for the trigonometric functions: $\tan\gamma \approx \gamma$, $\sin\gamma \approx \gamma$, and $\cos\gamma \approx 1$.

EXAMPLE 2.5

Two bars are connected to a roller that slides in a slot, as shown in Figure 2.13. Determine the strains in bar *AP* by: (a) Finding the deformed length of *AP* without small-strain approximation. (b) Using Equation (2.6). (c) Using Equation (2.7).

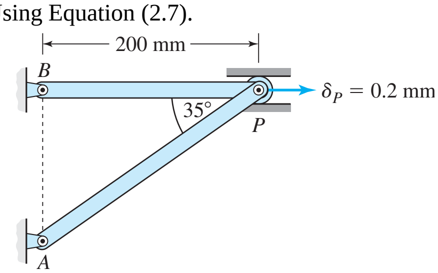


Figure 2.13 Small-strain calculations.

PLAN

(a) An exaggerated deformed shape of the two bars can be drawn and the deformed length of bar *AP* found using geometry. (b) The deformation of bar *AP* can be found by dropping a perpendicular from the final position of point *P* onto the original direction of bar *AP* and using geometry. (c) The deformation of bar *AP* can be found by taking the dot product of the unit vector in the direction of *AP* and the displacement vector of point *P*.

SOLUTION

The length *AP* used in all three methods can be found as $AP = (200\text{ mm})/\cos 35^\circ = 244.155$ mm.

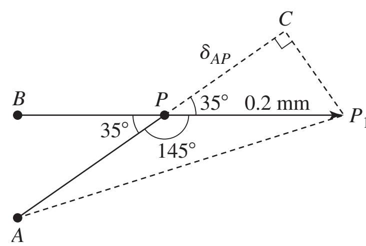


Figure 2.14 Exaggerated deformed shape.

(a) Let point *P* move to point P_1, as shown in Figure 2.14. The angle APP_1 is 145°. From the triangle APP_1 we can find the length AP_1 using the cosine formula and find the strain using Equation (2.1).

$$AP_1 = \sqrt{AP^2 + PP_1^2 - 2(AP)(PP_1)\cos 145^\circ} = 244.3188 \text{ mm} \tag{E1}$$

$$\varepsilon_{AP} = \frac{AP_1 - AP}{AP} = \frac{244.3188\text{ mm} - 244.155\text{ mm}}{244.155\text{ mm}} = 0.67112(10^{-3})\text{ mm/mm} \tag{E2}$$

ANS. $\varepsilon_{AP} = 671.12\ \mu\text{mm/mm}$

(b) We drop a perpendicular from P_1 onto the line in direction of *AP* as shown in Figure 2.14. By the small-strain approximation, the strain in AP is then

$$\delta_{AP} = 0.2\cos 35^\circ = 0.1638 \text{ mm} \tag{E3}$$

$$\varepsilon_{AP} = \frac{\delta_{AP}}{AP} = \frac{0.1638\text{ mm}}{244.155\text{ mm}} = 0.67101(10^{-3})\text{ mm/mm} \tag{E4}$$

ANS. $\varepsilon_{AP} = 671.01\,\mu\text{mm/mm}$

(c) Let the unit vectors in the x and y directions be given by $\bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$. The unit vector in direction of AP and the deformation vector $\bar{\mathbf{D}}$ can be written as

$$\bar{\mathbf{i}}_{AP} = \cos 35°\bar{\mathbf{i}} + \sin 35°\bar{\mathbf{j}}, \qquad \bar{\mathbf{D}} = 0.2\bar{\mathbf{i}}, \tag{E5}$$

The strain in AP can be found using Equation (2.7):

$$\delta_{AP} = \bar{\mathbf{D}} \cdot \bar{\mathbf{i}}_{AP} = (0.2\text{ mm})\cos 35 = 0.1638\text{ mm} \tag{E6}$$

$$\varepsilon_{AP} = \frac{\delta_{AP}}{AP} = \frac{0.1638\text{ mm}}{244.155\text{ mm}} = 0.67101(10^{-3})\ \text{mm/mm} \tag{E7}$$

ANS. $\varepsilon_{AP} = 671.01\ \mu\text{mm/mm}$

COMMENTS

1. The calculations for parts (b) and (c) are identical, since there is no difference in the approximation between the two approaches. The strain value for part (a) differs from that in parts (b) and (c) by 0.016%, which is insignificant in engineering calculations.
2. To a small-strain approximation the final length AP_1 is being approximated by length AC.
3. If we do not carry many significant figures in part (a) we may get a prediction of zero strain as the first three significant figures subtract out.

EXAMPLE 2.6

A gap of 0.18 mm exists between the rigid plate and bar B before the load P is applied on the system shown in Figure 2.15. After load P is applied, the axial strain in rod B is $-2500\ \mu$m/m. Determine the axial strain in rods A.

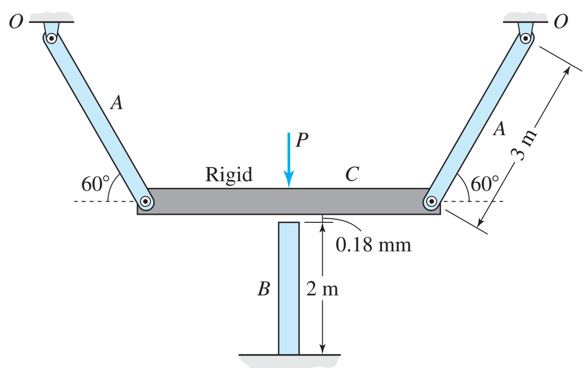


Figure 2.15 Undeformed geometry in Example 2.6.

PLAN

The deformation of bar B can be found from the given strain and related to the displacement of the rigid plate by drawing an approximate deformed shape. We can then relate the displacement of the rigid plate to the deformation of bar A using small-strain approximation.

SOLUTION

From the given strain of bar B we can find the deformation of bar B:

$$\delta_B = \varepsilon_B L_B = (2500)(10^{-6})(2\text{ m}) = 0.005\ \text{m contraction} \tag{E1}$$

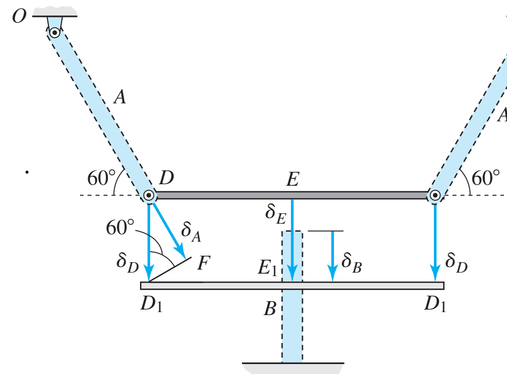


Figure 2.16 Deformed geometry.

Let points D and E be points on the rigid plate. Let the position of these points be D_1 and E_1 after the load P has been applied, as shown in Figure 2.16.

From Figure 2.16 the displacement of point E is

$$\delta_E = \delta_B + 0.00018\text{ m} = 0.00518\text{ m} \tag{E2}$$

As the rigid plate moves downward horizontally without rotation, the displacements of points D and E are the same:

$$\delta_D = \delta_E = 0.00518\text{ m} \tag{E3}$$

We can drop a perpendicular from D_1 to the line in the original direction OD and relate the deformation of bar A to the displacement of point D:

$$\delta_A = \delta_D \sin 60° = (0.00518\text{ m})\sin 60° = 0.004486\text{ m} \tag{E4}$$

The normal strain in A is then

$$\varepsilon_A = \frac{\delta_A}{L_A} = \frac{0.004486\text{ m}}{3\text{ m}} = 1.49539(10^{-3})\ \text{m/m} \tag{E5}$$

ANS. $\varepsilon_A = 1495\ \mu\ \text{m/m}$

COMMENTS

1. Equation (E3) is the relationship of points on the rigid bar, whereas Equations (E2) and (E4) are the relationship between the movement of points on the rigid bar and the deformation of the bar. This two-step process simplifies deformation analysis as it reduces the possibility of mistakes in the calculations.
2. We dropped the perpendicular from D_1 to OD and not from D to OD_1 because OD is the original direction, and not OD_1.

EXAMPLE 2.7

Two bars of hard rubber are attached to a rigid disk of radius 20 mm as shown in Figure 2.17. The rotation of the rigid disk by an angle $\Delta\phi$ causes a shear strain at point A of 2000 μrad. Determine the rotation $\Delta\phi$ and the shear strain at point C.

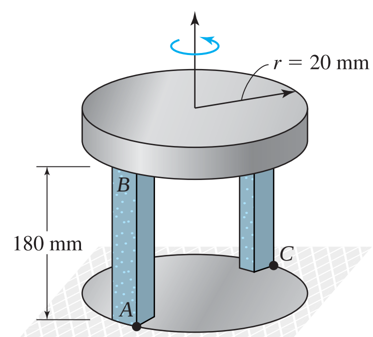


Figure 2.17 Geometry in Example 2.7.

PLAN

The displacement of point B can be related to shear strain at point A as in Example 2.2. All radial lines rotate by equal amounts of $\Delta\phi$ on the rigid disk. We can find $\Delta\phi$ by relating displacement of point B to $\Delta\phi$ assuming small strains. We repeat the calculation for the bar at C to find the strain at C.

SOLUTION

The shear strain at A is $\gamma_A = 2000\ \mu\text{rad} = 0.002\text{ rad}$. We draw the approximated deformed shape of the two bars as shown in Figure 2.18*a*. The displacement of point B is approximately equal to the arc length BB_1, which is related to the rotation of the disk, as shown in Figure 2.18*a* and *b* and given as

$$\Delta u_B = (20\text{ mm})(\Delta\phi) \tag{E1}$$

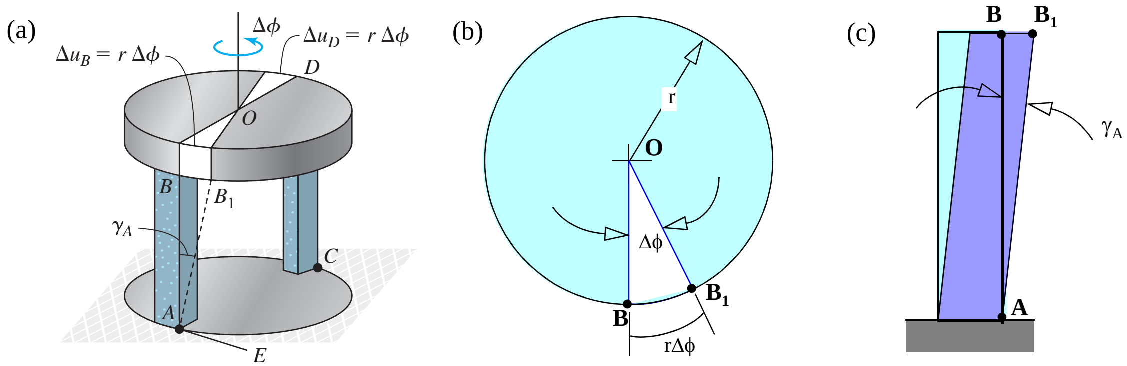


Figure 2.18 (a) Deformed geometry in Example 2.7. (b) Top view of disc. (c) Side view of bar.

The displacement of point B can also be related to the shear strain at A, and we can find $\Delta\phi$ as

$$\tan\gamma_A \approx \gamma_A = \frac{BB_1}{AB} = \frac{\Delta u_B}{AB} = \frac{(20\Delta\phi)\text{ mm}}{180\text{ mm}} = \frac{\Delta\phi}{9} \tag{E2}$$

$$\Delta\phi = 9\gamma_A = (9)(0.002) = 0.018\text{ rad} \tag{E3}$$

ANS. $\Delta\phi = 0.018\text{ rad}$

The displacement of point D can be found and the shear strain at C obtained from

$$\gamma_C = \frac{\Delta u_D}{CD} = \frac{20\Delta\phi}{180} = \frac{(20\text{ mm})(0.018\text{ rad})}{180\text{ mm}} = 0.002\text{ rad} \tag{E4}$$

ANS. $\gamma_C = 2000\ \mu\text{rad}$

COMMENTS

1. We approximated the arc BB_1 by a straight line, which is valid only if the deformations are small.
2. The shear strain was found from the change in angle formed by the tangent line AE and the axial line AB.
3. In Chapter 5, on the torsion of circular shafts, we will consider a shaft made up of bars and calculate the shear strain due to torsion as in this example.

2.4.1 Vector Approach to Small-Strain Approximation

To calculate strains from known displacements of the pins in truss problems is difficult using the small-strain approximation given by Equation (2.6). Similar algebraic difficulties are encountered in three-dimension. A vector approach helps address these difficulties.

The deformation of the bar in Equation (2.6) is given by $\delta = D\cos\theta$ and can be written in vector form using the dot product:

$$\boxed{\delta = \bar{\mathbf{D}}_{AP} \cdot \bar{\mathbf{i}}_{AP}} \tag{2.7}$$

where $\bar{\mathbf{D}}_{AP}$ is the deformation vector of the bar AP and $\bar{\mathbf{i}}_{AP}$ is the unit vector in the original direction of bar AP. With point A fixed in Figure 2.12 the vector $\bar{\mathbf{D}}_{AP}$ is also the displacement vector of point P. If point A is also displaced, then the deformation vector is obtained by taking the difference between the displacement vectors of point P and point A. If points A and P have coordinates (x_A, y_A, z_A) and (x_P, y_P, z_P), respectively, and are displaced by amounts (u_A, v_A, w_A) and (u_P, v_P, w_P) in the x, y, and z directions, respectively, then the deformation vector $\overline{\mathbf{D}}_{AP}$ and the unit vector $\overline{\mathbf{i}}_{AP}$ can be written as

$$\overline{\mathbf{D}}_{AP} = (u_P - u_A)\overline{\mathbf{i}} + (v_P - v_A)\overline{\mathbf{j}} + (w_P - w_A)\overline{\mathbf{k}}$$
$$\overline{\mathbf{i}}_{AP} = (x_P - x_A)\overline{\mathbf{i}} + (y_P - y_A)\overline{\mathbf{j}} + (z_P - z_A)\overline{\mathbf{k}} \tag{2.8}$$

where $\overline{\mathbf{i}}$, $\overline{\mathbf{j}}$, and $\overline{\mathbf{k}}$ are the unit vectors in the x, y, and z directions, respectively. The important point to remember about the calculation of $\mathbf{D}_{AP}$ and $\overline{\mathbf{i}}_{AP}$ is that the same reference point (A) must be used in calculating deformation vector and the unit vector.

EXAMPLE 2.8*

The displacements of pins of the truss shown Figure 2.19 were computed by the finite-element method (see Section 4.8) and are given below. u and v are the pin displacement x and y directions, respectively. Determine the axial strains in members BC, HB, HC, and HG.

$u_B = 2.700$ mm $v_B = -9.025$ mm

$u_C = 5.400$ mm $v_C = -14.000$ mm

$u_G = 8.000$ mm $v_G = -14.000$ mm

$u_H = 9.200$ mm $v_H = -9.025$ mm

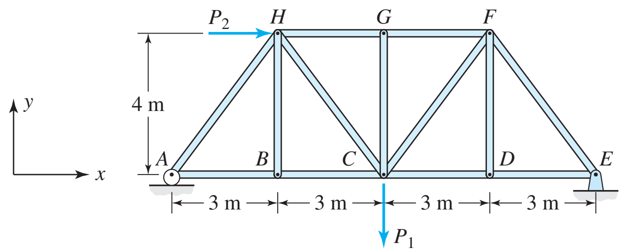


Figure 2.19 Truss in Example 2.8.

PLAN

The deformation vectors for each bar can be found from the given displacements. The unit vectors in directions of the bars BC, HB, HC, and HG can be determined. The deformation of each bar can be found using Equation (2.7) from which we can find the strains.

SOLUTION

Let the unit vectors in the x and y directions be given by $\overline{\mathbf{i}}$ and $\overline{\mathbf{j}}$, respectively. The deformation vectors for each bar can be found for the given displacement as

$$\overline{\mathbf{D}}_{BC} = (u_C - u_B)\overline{\mathbf{i}} + (v_C - v_B)\overline{\mathbf{j}} = (2.7\overline{\mathbf{i}} - 4.975\overline{\mathbf{j}})\text{ mm} \qquad \overline{\mathbf{D}}_{HB} = (u_B - u_H)\overline{\mathbf{i}} + (v_B - v_H)\overline{\mathbf{j}} = (-6.5\overline{\mathbf{i}})\text{ mm}$$
$$\overline{\mathbf{D}}_{HC} = (u_C - u_H)\overline{\mathbf{i}} + (v_C - v_H)\overline{\mathbf{j}} = (-3.8\overline{\mathbf{i}} - 4.975\overline{\mathbf{j}})\text{ mm} \qquad \overline{\mathbf{D}}_{HG} = (u_G - u_H)\overline{\mathbf{i}} + (v_G - v_H)\overline{\mathbf{j}} = (-1.2\overline{\mathbf{i}} - 4.975\overline{\mathbf{j}})\text{ mm} \tag{E1}$$

The unit vectors in the directions of bars BC, HB, and HG can be found by inspection as these bars are horizontal or vertical:

$$\overline{\mathbf{i}}_{BC} = \overline{\mathbf{i}} \qquad \overline{\mathbf{i}}_{HB} = -\overline{\mathbf{j}} \qquad \overline{\mathbf{i}}_{HG} = \overline{\mathbf{i}} \tag{E2}$$

The position vector from point H to C is $\overline{\mathbf{HC}} = 3\overline{\mathbf{i}} - 4\overline{\mathbf{j}}$. Dividing the position vector by its magnitude we obtain the unit vector in the direction of bar HC:

$$\overline{\mathbf{i}}_{HC} = \frac{\overline{\mathbf{HC}}}{|\overline{\mathbf{HC}}|} = \frac{(3\text{ mm})\overline{\mathbf{i}} - (4\text{ mm})\overline{\mathbf{j}}}{\sqrt{(3\text{ mm})^2 + (4\text{ mm})^2}} = 0.6\overline{\mathbf{i}} - 0.8\overline{\mathbf{j}} \tag{E3}$$

We can find the deformation of each bar from Equation (2.7):

$$\delta_{BC} = \overline{\mathbf{D}}_{BC} \cdot \overline{\mathbf{i}}_{BC} = 2.7\text{ mm} \qquad \delta_{HG} = \overline{\mathbf{D}}_{HG} \cdot \overline{\mathbf{i}}_{HG} = -1.2\text{ mm}$$
$$\delta_{HB} = \overline{\mathbf{D}}_{HB} \cdot \overline{\mathbf{i}}_{HB} = 0 \qquad \delta_{HC} = \overline{\mathbf{D}}_{HC} \cdot \overline{\mathbf{i}}_{HC} = (0.6\text{ mm})(-3.8) + (-4.975\text{ mm})(-0.8) = 1.7\text{ mm} \tag{E4}$$

Finally, Equation (2.2) gives the strains in each bar:

$$\varepsilon_{BC} = \frac{\delta_{BC}}{L_{BC}} = \frac{2.7\text{ mm}}{3 \times 10^3\text{ mm}} = 0.9 \times 10^{-3}\text{ mm/mm} \qquad \varepsilon_{HG} = \frac{\delta_{HG}}{L_{HG}} = \frac{-1.2\text{ mm}}{3 \times 10^3\text{ mm}} = -0.4 \times 10^{-3}\text{ mm/mm}$$
$$\varepsilon_{HB} = \frac{\delta_{HB}}{L_{HB}} = 0 \qquad \varepsilon_{HC} = \frac{\delta_{HC}}{L_{HC}} = \frac{1.7\text{ mm}}{3 \times 10^3\text{ mm}} = 0.340 \times 10^{-3}\text{ mm/mm} \tag{E5}$$

ANS. $\varepsilon_{BC} = 900\ \mu\text{mm/mm}$ $\varepsilon_{HG} = -400\ \mu\text{mm/mm}$ $\varepsilon_{HB} = 0$ $\varepsilon_{HC} = 340\ \mu\text{mm/mm}$

COMMENTS

1. The zero strain in *HB* is not surprising. By looking at joint *B*, we can see that *HB* is a zero-force member. Though we have yet to establish the relationship between internal forces and deformation, we know intuitively that internal forces will develop if a body deforms.
2. We took a very procedural approach in solving the problem and, as a consequence, did several additional computations. For horizontal bars *BC* and *HG* we could have found the deformation by simply subtracting the *u* components, and for the vertical bar *HB* we can find the deformation by subtracting the v component. But care must be exercised in determining whether the bar is in extension or in contraction, for otherwise an error in sign can occur.
3. In Figure 2.20 point *H* is held fixed (reference point), and an exaggerated relative movement of point *C* is shown by the vector $\overline{\mathbf{D}}_{HC}$. The calculation of the deformation of bar *HC* is shown graphically.

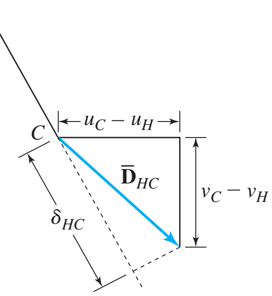


Figure 2.20 Visualization of the deformation vector for bar *HC*.

4. Suppose that instead of finding the relative movement of point *C* with respect to *H*, we had used point *C* as our reference point and found the relative movement of point *H*. The deformation vector would be $\overline{\mathbf{D}}_{CH}$, which is equal to $-\overline{\mathbf{D}}_{HC}$. But the unit vector direction would also reverse, that is, we would use $\bar{\mathbf{i}}_{CH}$, which is equal to $-\bar{\mathbf{i}}_{HC}$. Thus the dot product to find the deformation would yield the same number and the same sign. The result independent of the reference point is true only for small strains, which we have implicitly assumed.

PROBLEM SET 2.1

Average normal strains

2.1 An 80-cm stretch cord is used to tie the rear of a canoe to the car hook, as shown in Figure P2.1. In the stretched position the cord forms the side *AB* of the triangle shown. Determine the average normal strain in the stretch cord.

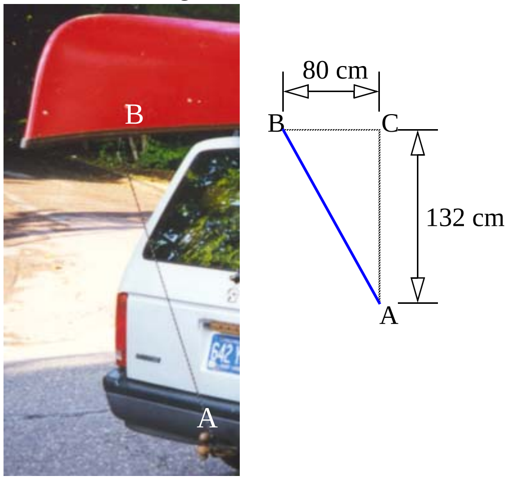


Figure P2.1

2.2 The diameter of a spherical balloon shown in Figure P2.2 changes from 250 mm to 252 mm. Determine the change in the average circumferential normal strain.

Figure P2.2

2.3 Two rubber bands are used for packing an air mattress for camping as shown in Figure P2.3. The undeformed length of a rubber band is 7 in. Determine the average normal strain in the rubber bands if the diameter of the mattress is 4.1 in. at the section where the rubber bands are on the mattress.

Figure P2.3

2.4 A canoe on top of a car is tied down using rubber stretch cords, as shown in Figure P2.4*a*. The undeformed length of the stretch cord is 40 in. Determine the average normal strain in the stretch cord assuming that the path of the stretch cord over the canoe can be approximated as shown in Figure P2.4*b*.

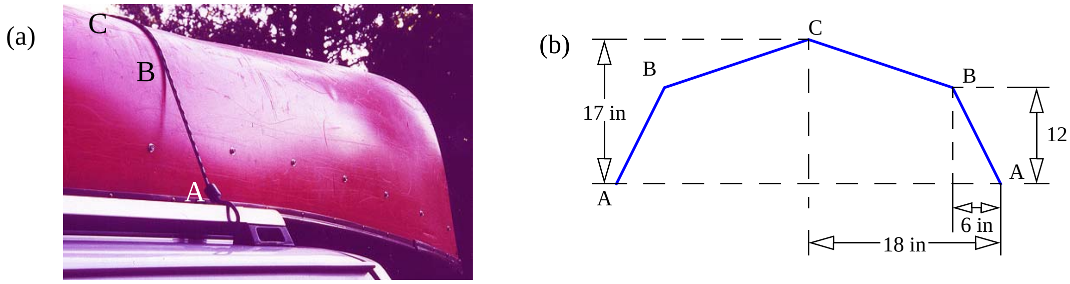


Figure P2.4

2.5 The cable between two poles shown in Figure P2.5 is taut before the two traffic lights are hung on it. The lights are placed symmetrically at 1/3 the distance between the poles. Due to the weight of the traffic lights the cable sags as shown. Determine the average normal strain in the cable.

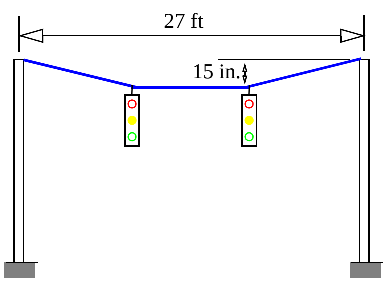


Figure P2.5

2.6 The displacements of the rigid plates in x direction due to the application of the forces in Figure P2.6 are $u_B = -1.8$ mm, $u_C = 0.7$ mm, and $u_D = 3.7$ mm. Determine the axial strains in the rods in sections *AB*, *BC*, and *CD*.

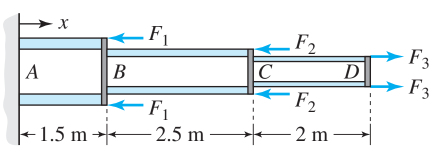


Figure P2.6

2.7 The average normal strains in the bars due to the application of the forces in Figure P2.6 are $\varepsilon_{AB} = -800\ \mu$, $\varepsilon_{BC} = 600\ \mu$, and $\varepsilon_{CD} = 1100\ \mu$. Determine the movement of point *D* with respect to the left wall.

2.8 Due to the application of the forces, the rigid plate in Figure P2.8 moves 0.0236 in to the right. Determine the average normal strains in bars *A* and *B*.

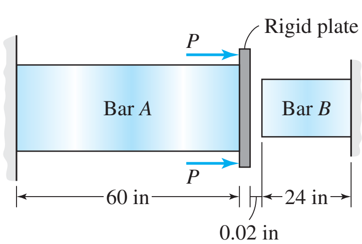


Figure P2.8

2.9 The average normal strain in bar *A* due to the application of the forces in Figure P2.8, was found to be 2500 μ in./in. Determine the normal strain in bar *B*.

2.10 The average normal strain in bar *B* due to the application of the forces in Figure P2.8 was found to be -4000 μ in./in. Determine the normal strain in bar *A*.

2.11 The rigid bar *BD* in Figure P2.11 pivots about support *C*. Due to the application of force *P*, point *B* moves upward by 0.06 in. If the length of bar *A* is 24 in., determine the average normal strain in bar *A*.

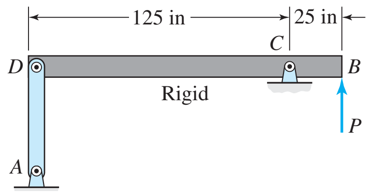


Figure P2.11

2.12 The average normal strain in bar *A* due to the application of force *P* in Figure P2.11 was found to be –6000 μ in./in. If the length of bar *A* is 36 in., determine the movement of point *B*.

2.13 The rigid bar *BD* in Figure P2.13 pivots about support *C*. Due to the application of force *P*, point *B* imoves upward by 0.06 in. If the length of bar *A* is 24 in., determine the average normal strain in bar *A*.

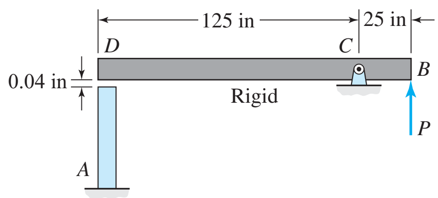


Figure P2.13

2.14 The average normal strain in bar *A* due to the application of force *P* in Figure P2.13 was found to be –6000 μ in./in. If the length of bar *A* is 36 in., determine the movement of point *B*.

2.15 The rigid bar *BED* in Figure P2.15 pivots about support *C*. Due to the application of force *P*, point *B* moves upward by 0.06 in. If the lengths of bars *A* and *F* are 24 in., determine the average normal strain in bars *A* and *F*.

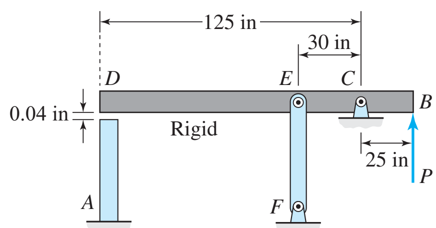


Figure P2.15

2.16 The average normal strain in bar *A* due to the application of force *P* in Figure P2.15 was found to be –5000 μ in./in. If the lengths of bars *A* and *F* are 36 in., determine the movement of point *B* and the average normal strain in bar *F*.

2.17 The average normal strain in bar *F* due to the application of force *P*, in Figure P2.15 was found to be -2000 μ in./in. If the lengths of bars *A* and *F* are 36 in., determine the movement of point *B* and the average normal strain in bar *A*.

2.18 The rigid bar *BD* in Figure P2.18 pivots about support *C*. Due to the application of force *P*, point *B* moves left by 0.75 mm. If the length of bar *A* is 1.2 m, determine the average normal strain in bar *A*.

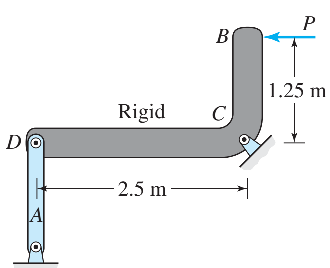


Figure P2.18

2.19 The average normal strain in bar *A* due to the application of force *P* in Figure P2.18 was found to be –2000 μ m /m. If the length of bar *A* is 2 m, determine the movement of point *B*.

2.20 The rigid bar *BD* in Figure P2.20 pivots about support *C*. Due to the application of force *P*, point *B* moves left by 0.75 mm. If the length of bar *A* is 1.2 m, determine the average normal strain in bar *A*.

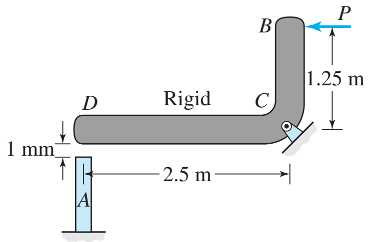


Figure P2.20

2.21 The average normal strain in bar *A* due to the application of force *P* in Figure P2.20 was found to be –2000 μ m/m. If the length of bar *A* is 2 m, determine the movement of point *B*.

2.22 The rigid bar *BD* in Figure P2.22 pivots about support *C*. Due to the application of force *P*, point *B* moves left by 0.75 mm. If the lengths of bars *A* and *F* are 1.2 m, determine the average normal strains in bars *A* and *F*.

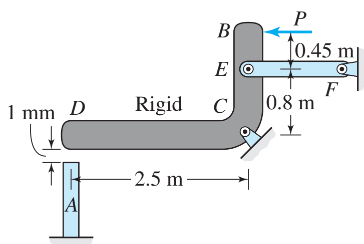


Figure P2.22

2.23 The average normal strain in bar *A* due to the application of force *P* in Figure P2.22 was found to be –2500 μ m/m. Bars *A* and *F* are 2 m long. Determine the movement of point *B* and the average normal strain in bar *F*.

2.24 The average normal strain in bar *F* due to the application of force *P* in Figure P2.22 was found to be 1000 μ m/m. Bars *A* and *F* are 2 m long. Determine the movement of point *B* and the average normal strain in bar *A*.

2.25 Two bars of equal lengths of 400 mm are welded to rigid plates at right angles. The right angles between the bars and the plates are preserved as the rigid plates are rotated by an angle of ψ as shown in Figure P2.25. The distance between the bars is $h = 50$ mm. The average normal strains in bars AB and CD were determined as -2500 µ mm/mm and 3500 µ mm/mm, respectively. Determine the radius of curvature R and the angle ψ.

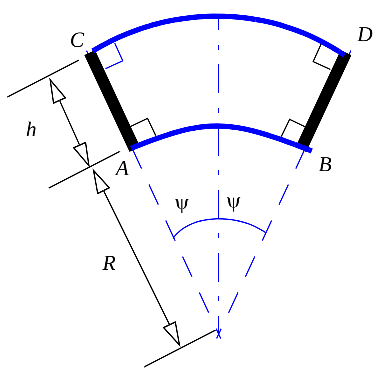


Figure P2.25

2.26 Two bars of equal lengths of 30 in. are welded to rigid plates at right angles. The right angles between the bars and the plates are preserved as the rigid plates are rotated by an angle of $\psi = 1.25^o$ as shown in Figure P2.25. The distance between the bars is h = 2 in. If the average normal strain in bar *AB* is -1500 µ in./in., determine the strain in bar *CD*.

2.27 Two bars of equal lengths of 48 in. are welded to rigid plates at right angles. The right angles between the bars and the plates are preserved as the rigid plates are rotated by an angle of ψ as shown in Figure P2.27. The average normal strains in bars *AB* and *CD* were determined as -2000 μ in./in. and 1500 μ in./in., respectively. Determine the location h of a third bar *EF* that should be placed such that it has zero normal strain.

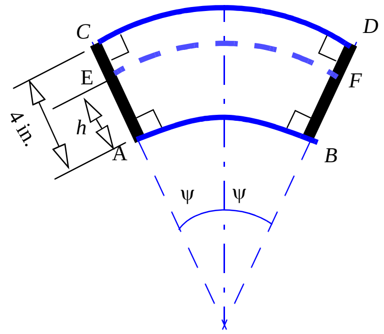


Figure P2.27

Average shear strains

2.28 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.28. Determine the average shear strain at point *A*.

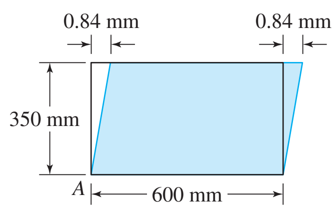


Figure P2.28

2.29 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.29. Determine the average shear strain at point *A*.

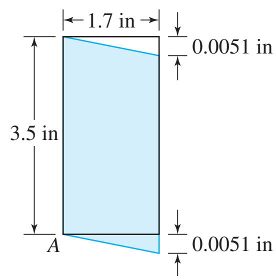


Figure P2.29

2.30 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.30. Determine the average shear strain at point *A*.

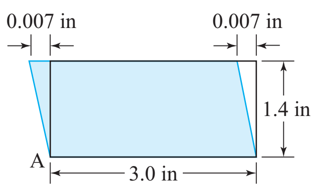


Figure P2.30

2.31 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.31. Determine the average shear strain at point *A*.

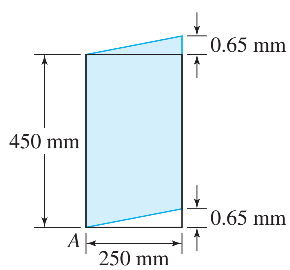


Figure P2.31

2.32 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.32. Determine the average shear strain at point *A*.

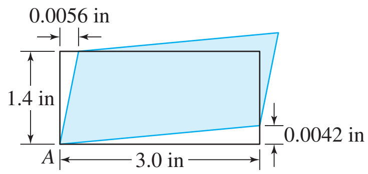


Figure P2.32

2.33 A rectangular plastic plate deforms into a shaded shape, as shown in Figure P2.33. Determine the average shear strain at point *A*.

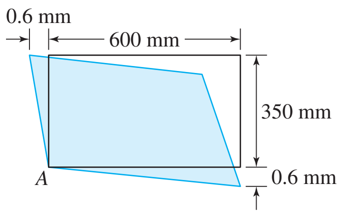


Figure P2.33

2.34 A thin triangular plate ABC forms a right angle at point A, as shown in Figure P2.34. During deformation, point A moves vertically down by δ_A = 0.005 in. Determine the average shear strains at point A.

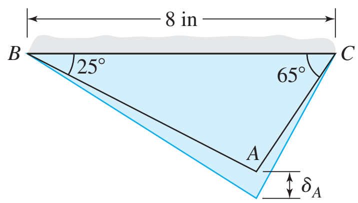


Figure P2.34

2.35 A thin triangular plate ABC forms a right angle at point A, as shown in Figure P2.35. During deformation, point A moves vertically down by δ_A = 0.006 in. Determine the average shear strains at point A.

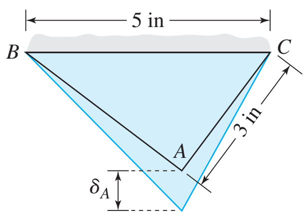


Figure P2.35

2.36 A thin triangular plate ABC forms a right angle at point A, as shown in Figure P2.36. During deformation, point A moves vertically down by δ_A = 0.75 mm. Determine the average shear strains at point A.

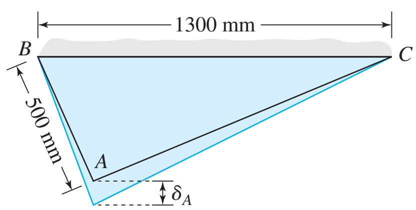


Figure P2.36

2.37 A thin triangular plate *ABC* forms a right angle at point *A*. During deformation, point A moves horizontally by δ_A =0.005 in., as shown in Figure P2.37. Determine the average shear strains at point *A*.

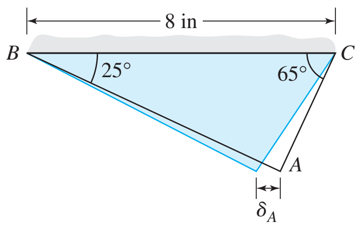


Figure P2.37

2.38 A thin triangular plate *ABC* forms a right angle at point *A*. During deformation, point *A* moves horizontally by δ_A =0.008 in., as shown in Figure P2.38. Determine the average shear strains at point *A*.

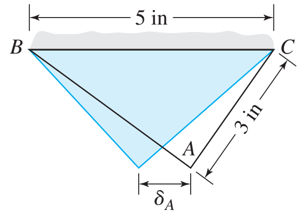


Figure P2.38

2.39 A thin triangular plate *ABC* forms a right angle at point *A*. During deformation, point *A* moves horizontally by δ_A =0.90 mm, as shown in Figure P2.39. Determine the average shear strains at point *A*.

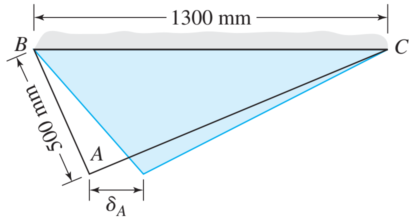


Figure P2.39

2.40 Bar *AB* is bolted to a plate along the diagonal as shown in Figure P2.40. The plate experiences an average strain in the *x* direction $\varepsilon = 500\ \mu\text{in.}/\text{in.}$. Determine the average normal strain in the bar *AB*.

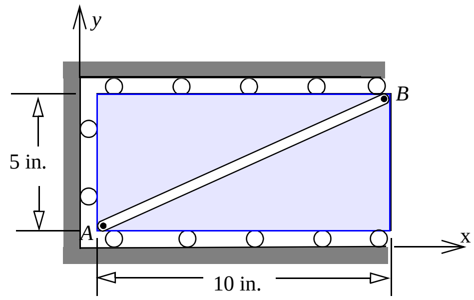


Figure P2.40

2.41 Bar *AB* is bolted to a plate along the diagonal as shown in Figure P2.40. The plate experiences an average strain in the *y* direction $\varepsilon = -1200\ \mu\text{mm}/\text{mm}$. Determine the average normal strain in the bar *AB*.

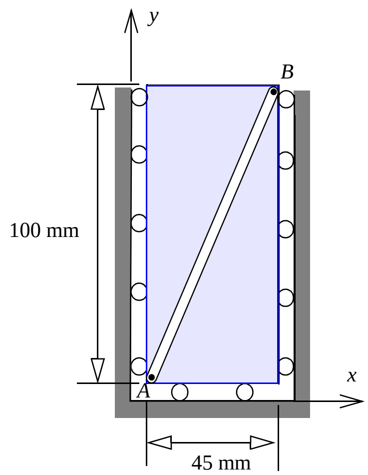


Figure P2.41

2.42 A right angle bar *ABC* is welded to a plate as shown in Figure P2.42. Points *B* are fixed. The plate experiences an average strain in the *x* direction $\varepsilon = -1000 \mu\text{mm/mm}$. Determine the average normal strain in *AB*.

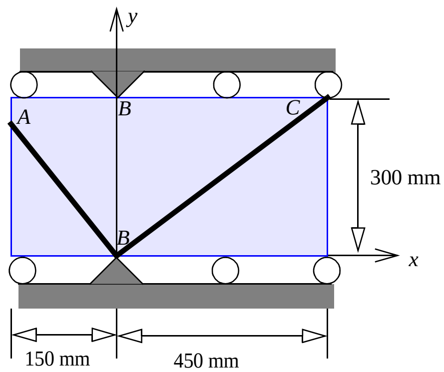


Figure P2.42

2.43 A right angle bar *ABC* is welded to a plate as shown in Figure P2.42. Points *B* are fixed. The plate experiences an average strain in the *x* direction $\varepsilon = 700 \ \mu\text{mm/mm}$. Determine the average normal strain in *BC*.

2.44 A right angle bar *ABC* is welded to a plate as shown in Figure P2.42. Points *B* are fixed. The plate experiences an average strain in the *x* direction $\varepsilon = -800 \ \mu\text{mm/mm}$. Determine the average shear strain at point *B* in the bar.

2.45 A right angle bar *ABC* is welded to a plate as shown in Figure P2.45. Points *B* are fixed. The plate experiences an average strain in the *y* direction $\varepsilon = 800 \ \mu\text{in./ in.}$ Determine the average normal strain in *AB*.

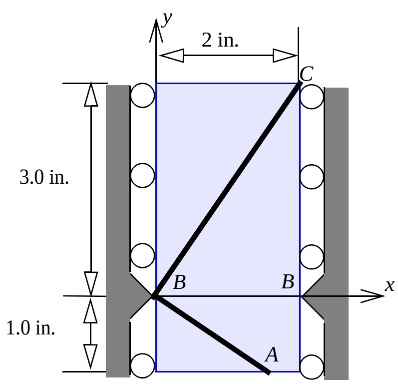


Figure P2.45

2.46 A right angle bar *ABC* is welded to a plate as shown in Figure P2.45. Points *B* are fixed. The plate experiences an average strain in the *y* direction $\varepsilon = -500 \ \mu\text{in./ in.}$ Determine the average normal strain in *BC*.

2.47 A right angle bar *ABC* is welded to a plate as shown in Figure P2.45. Points *B* are fixed. The plate experiences an average strain in the *y* direction $\varepsilon = 600 \ \mu\text{in./ in.}$ Determine the average shear strain at *B* in the bar.

2.48 The diagonals of two squares form a right angle at point *A* in Figure P2.48. The two rectangles are pulled horizontally to a deformed shape, shown by colored lines. The displacements of points *A* and *B* are $\delta_A = 0.4$ mm and $\delta_B = 0.8$ mm. Determine the average shear strain at point *A*.

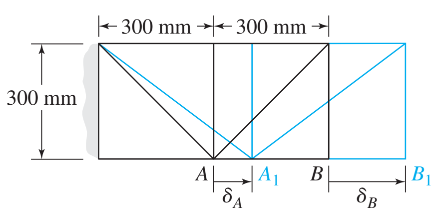


Figure P2.48

2.49 The diagonals of two squares form a right angle at point A in Figure P2.48. The two rectangles are pulled horizontally to a deformed shape, shown by colored lines. The displacements of points A and B are $\delta_A = 0.3$ mm and $\delta_B = 0.9$ mm. Determine the average shear strain at point A $\delta_A = 0.3$ mm and $\delta_B = 0.9$ mm.

Small-strain approximations

2.50 The roller at P slides in the slot by the given amount shown in Figure P2.50. Determine the strains in bar AP by (a) finding the deformed length of AP without the small-strain approximation, (b) using Equation (2.6), and (c) using Equation (2.7).

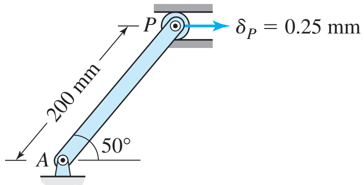


Figure P2.50

2.51 The roller at P slides in the slot by the given amount shown in Figure P2.51. Determine the strains in bar AP by (a) finding the deformed length of AP without small-strain approximation, (b) using Equation (2.6), and (c) using Equation (2.7).

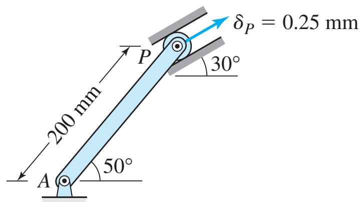


Figure P2.51

2.52 The roller at P slides in a slot by the amount shown in Figure P2.52. Determine the deformation in bars AP and BP using the small-strain approximation.

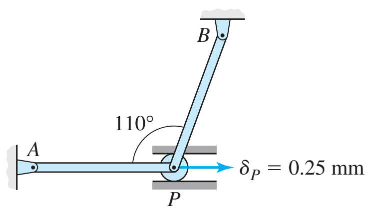


Figure P2.52

2.53 The roller at P slides in a slot by the amount shown in Figure P2.53. Determine the deformation in bars AP and BP using the small-strain approximation.

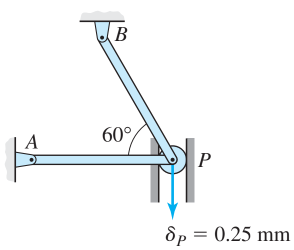


Figure P2.53

2.54 The roller at P slides in a slot by the amount shown in Figure P2.54. Determine the deformation in bars AP and BP using the small-strain approximation.

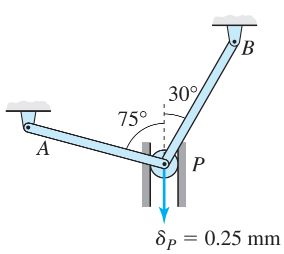


Figure P2.54

2.55 The roller at P slides in a slot by the amount shown in Figure P2.55. Determine the deformation in bars AP and BP using the small-strain approximation.

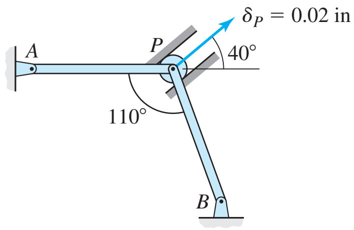


Figure P2.55

2.56 The roller at P slides in a slot by the amount shown in Figure P2.56. Determine the deformation in bars AP and BP using the small-strain approximation.

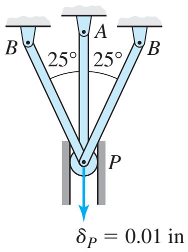


Figure P2.56

2.57 The roller at P slides in a slot by the amount shown in Figure P2.57. Determine the deformation in bars AP and BP using the small-strain approximation.

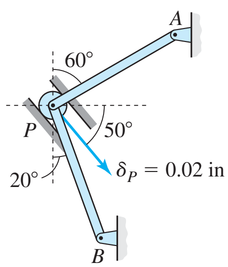


Figure P2.57

2.58 A gap of 0.004 in. exists between the rigid bar and bar *A* before the load *P* is applied in Figure P2.58. The rigid bar is hinged at point *C*. The strain in bar *A* due to force *P* was found to be –600 μ in./in. Determine the strain in bar *B*. The lengths of bars *A* and *B* are 30 in. and 50 in., respectively.

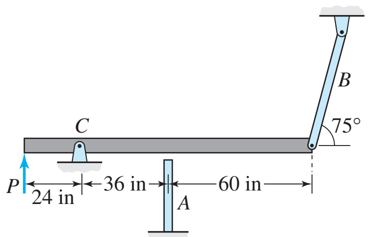


Figure P2.58

2.59 A gap of 0.004 in. exists between the rigid bar and bar *A* before the load *P* is applied in Figure P2.58. The rigid bar is hinged at point *C*. The strain in bar *B* due to force *P* was found to be 1500 μ in./in. Determine the strain in bar *A*. The lengths of bars *A* and *B* are 30 in. and 50 in., respectively.

Vector approach to small-strain approximation

2.60 The pin displacements of the truss in Figure P2.60 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.60. Determine the axial strains in members AB, BF, FG, and GB.

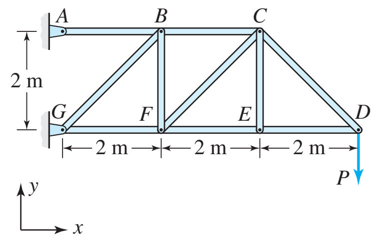


Figure P2.60

TABLE P2.60

u_B = 12.6 mm	v_B = –24.48 mm
u_C = 21.0 mm	v_C = –69.97 mm
u_D = –16.8 mm	v_D = –119.65 mm
u_E = –12.6 mm	v_E = –69.97 mm
u_F = –8.4 mm	v_F = –28.68 mm

2.61 The pin displacements of the truss in Figure P2.60 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.60. Determine the axial strains in members BC, CF, and FE.

2.62 The pin displacements of the truss in Figure P2.60 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.60. Determine the axial strains in members ED, DC, and CE.

2.63 The pin displacements of the truss in Figure P2.63 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.63. Determine the axial strains in members AB, BG, GA, and AH.

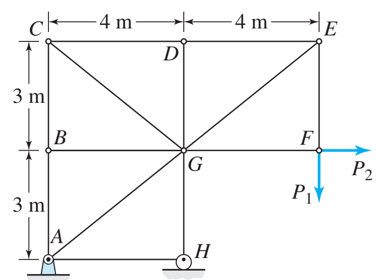


Figure P2.63

TABLE P2.63

u_B = 7.00 mm	v_B = 1.500 mm
u_C = 17.55 mm	v_C = 3.000 mm
u_D = 20.22 mm	v_D = –4.125 mm
u_E = 22.88 mm	v_E = –32.250 mm
u_F = 9.00 mm	v_F = –33.750 mm
u_G = 7.00 mm	v_G = –4.125 mm
u_H = 0	v_H = 0

2.64 The pin displacements of the truss in Figure P2.63 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.63.Determine the axial strains in members BC, CG, GB, and CD.

2.65 The pin displacements of the truss in Figure P2.63 were computed by the finite-element method. The displacements in x and y directions given by u and v are given in Table P2.63.Determine the axial strains in members GF, FE, EG, and DE.

2.66 Three poles are pin connected to a ring at P and to the supports on the ground. The ring slides on a vertical rigid pole by 2 in, as shown in Figure P2.66. The coordinates of the four points are as given. Determine the normal strain in each bar due to the movement of the ring.

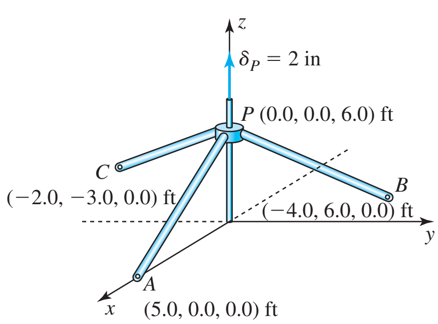


Figure P2.66

MoM in Action: Challenger Disaster

On January 28th, 1986, the space shuttle Challenger (Figure 2.21a) exploded just 73 seconds into the flight, killing seven astronauts. The flight was to have been the first trip for a civilian, the school-teacher Christa McAuliffe. Classrooms across the USA were preparing for the first science class ever taught from space. The explosion shocked millions watching the takeoff and a presidential commission was convened to investigate the cause. Shuttle flights were suspended for nearly two years.

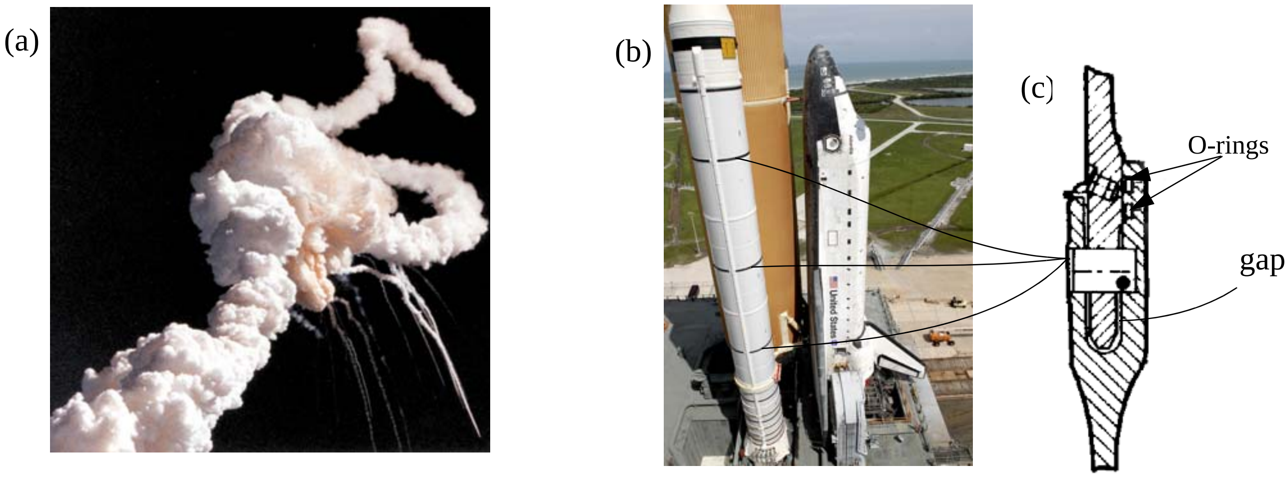


Figure 2.21 (a) Challenger explosion during flight (b) Shuttle Atlantis (c) O-ring joint.

The Presidential commission established that combustible gases from the solid rocket boosters had ignited, causing the explosion. These gases had leaked through the joint between the two lower segments of the boosters on the space shuttle's right side. The boosters of the *Challenger*, like those of the shuttle Atlantis (Figure 2.21b), were assembled using the O-ring joints illustrated in Figure 2.21c. When the gap between the two segments is 0.004 in. or less, the rubber O-rings are in contact with the joining surfaces and there is no chance of leak. At the time of launch, however, the gap was estimated to have exceeded 0.017 in.

But why? Apparently, prior launches had permanently enlarged diameter of the segments at some places, so that they were no longer round. Launch forces caused the segments to move further apart. Furthermore, the O-rings could not return to their uncompressed shape, because the material behavior alters dramatically with temperature. A compressed rubber O-ring at 78^o F is five times more responsive in returning to its uncompressed shape than an O-ring at 30^o F. The temperature around the joint varied from approximately 28^o F on the cold shady side to 50^o F in the sun.

Two engineers at Morton Thiokol, a contractor of NASA, had seen gas escape at a previous launch and had recommended against launching the shuttle when the outside air temperature is below 50^o F. Thiokol management initially backed their engineer's recommendation but capitulated to desire to please their main customer, NASA. The NASA managers felt under political pressure to establish the space shuttle as a regular, reliable means of conducting scientific and commercial missions in space. Roger Boisjoly, one of the Thiokol engineers was awarded the Prize for Scientific Freedom and Responsibility by American Association for the Advancement of Science for his professional integrity and his belief in engineer's rights and responsibilities.

The accident came about because the deformation at launch was in excess of the design's allowable deformation. An administrative misjudgment of risk assessment and the potential benefits had overruled the engineers.

2.5 STRAIN COMPONENTS

Let *u*, v, and *w* be the displacements in the *x*, *y*, and *z* directions, respectively. Figure 2.22 and Equations (2.9a) through (2.9i) define average engineering strain components:

$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x} \tag{2.9a}$$

$$\varepsilon_{yy} = \frac{\Delta v}{\Delta y} \tag{2.9b}$$

$$\varepsilon_{zz} = \frac{\Delta w}{\Delta z} \tag{2.9c}$$

$$\gamma_{xy} = \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x} \tag{2.9d}$$

$$\gamma_{yx} = \frac{\Delta v}{\Delta x} + \frac{\Delta u}{\Delta y} = \gamma_{xy} \tag{2.9e}$$

$$\gamma_{yz} = \frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y} \tag{2.9f}$$

$$\gamma_{zy} = \frac{\Delta w}{\Delta y} + \frac{\Delta v}{\Delta z} = \gamma_{yz} \tag{2.9g}$$

$$\gamma_{zx} = \frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z} \tag{2.9h}$$

$$\gamma_{xz} = \frac{\Delta u}{\Delta z} + \frac{\Delta w}{\Delta x} = \gamma_{zx} \tag{2.9i}$$

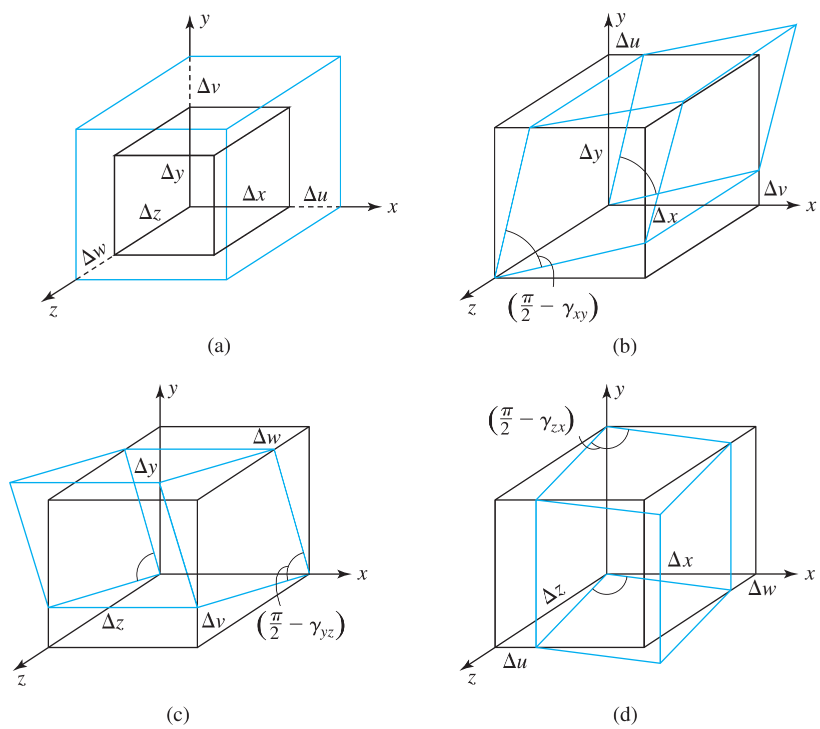


Figure 2.22 (*a*) Normal strains. (*b*) Shear strain γ_{xy}. (*c*) Shear strain γ_{yz}. (*d*) Shear strain γ_{zx}.

Equations (2.9a) through (2.9i) show that strain at a point has nine components in three dimensions, but only six are independent because of the symmetry of shear strain. The symmetry of shear strain makes intuitive sense. The change of angle between the *x* and *y* directions is obviously the same as between the *y* and *x* directions. In Equations (2.9a) through (2.9i) the first subscript is the direction of displacement and the second the direction of the line element. But because of the symmetry of shear strain, the order of the subscripts is immaterial. Equation (2.10) shows the components as an engineering strain matrix. The matrix is symmetric because of the symmetry of shear strain.

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_{yy} & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_{zz} \end{bmatrix} \tag{2.10}$$

2.5.1 Plane Strain

Plane strain is one of two types of two-dimensional idealizations in mechanics of materials. In Chapter 1 we saw the other type, plane stress. We will see the difference between the two types of idealizations in Chapter 3. By two-dimensional we imply that one of the coordinates does not play a role in the solution of the problem. Choosing z to be that coordinate, we set all strains with subscript z to be zero, as shown in the strain matrix in Equation (2.11). Notice that in plane strain, four components of strain are needed though only three are independent because of the symmetry of shear strain.

$$\begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.11}$$

The assumption of plane strain is often made in analyzing very thick bodies, such as points around tunnels, mine shafts in earth, or a point in the middle of a thick cylinder, such as a submarine hull. In thick bodies we can expect a point has to push a lot of material in the thickness direction to move. Hence the strains in the this direction should be small. It is not zero, but it is small enough to be neglected. Plane strain is a mathematical approximation made to simplify analysis.

EXAMPLE 2.9

Displacements u and v in x and y directions, respectively, were measured at many points on a body by the geometric Moiré method (See Section 2.7). The displacements of four points on the body of Figure 2.23 are as given. Determine strains ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = -0.0100 \text{ mm} \qquad v_A = 0.0100 \text{ mm}$$
$$u_B = -0.0050 \text{ mm} \qquad v_B = -0.0112 \text{ mm}$$
$$u_C = 0.0050 \text{ mm} \qquad v_C = 0.0068 \text{ mm}$$
$$u_D = 0.0100 \text{ mm} \qquad v_D = 0.0080 \text{ mm}$$

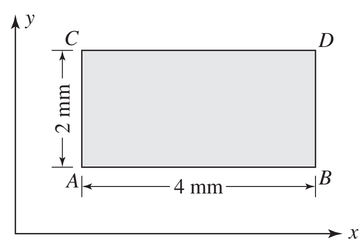


Figure 2.23 Undeformed geometry in Example 2.9.

PLAN

We can use point A as our reference point and calculate the relative movement of points B and C and find the strains from Equations (2.9a), (2.9b), and (2.9d).

SOLUTION

The relative movements of points B and C with respect to A are

$$u_B - u_A = 0.0050 \text{ mm} \qquad v_B - v_A = -0.0212 \text{ mm} \tag{E1}$$

$$u_C - u_A = 0.0150 \text{ mm} \qquad v_C - v_A = -0.0032 \text{ mm} \tag{E2}$$

The normal strains ε_{xx} and ε_{yy} can be calculated as

$$\varepsilon_{xx} = \frac{u_B - u_A}{x_B - x_A} = \frac{0.0050 \text{ mm}}{4 \text{ mm}} = 0.00125 \text{ mm/mm} \tag{E3}$$

$$\varepsilon_{yy} = \frac{v_C - v_A}{y_C - y_A} = \frac{-0.0032 \text{ mm}}{2 \text{ mm}} = -0.0016 \text{ mm/mm} \qquad \text{(E4)}$$

ANS. $\varepsilon_{xx} = 1250\ \mu\text{mm/mm} \qquad \varepsilon_{yy} = -1600\ \mu\text{mm/mm}$

From Equation (2.9d) the shear strain can be found as

$$\gamma_{xy} = \frac{v_B - v_A}{x_B - x_A} + \frac{u_C - u_A}{y_C - y_A} = \frac{-0.0212 \text{ mm}}{4 \text{ mm}} + \frac{0.0150 \text{ mm}}{2 \text{ mm}} = 0.0022 \text{ rad} \qquad \text{(E5)}$$

ANS. $\gamma_{xy} = 2200\ \mu\text{rads}$

COMMENT

1. Figure 2.24 shows an exaggerated deformed shape of the rectangle. Point A moves to point A_1; similarly, the other points move to B_1, C_1, and D_1. By drawing the undeformed rectangle from point A, we can show the relative movements of the three points. We could have calculated the length of A_1B from the Pythagorean theorem as $A_1B_1 = \sqrt{(4 - 0.005)^2 + (-0.0212)^2} = 3.995056$ mm, which would yield the following strain value:

$$\varepsilon_{xx} = \frac{A_1B_1 - AB}{AB} = 1236\ \mu\text{mm/mm}.$$

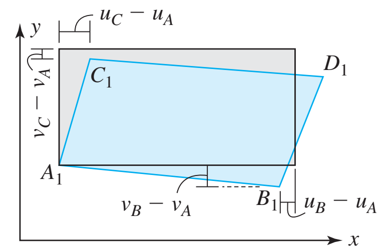


Figure 2.24 Elaboration of comment.

The difference between the two calculations is 1.1%. We will have to perform similar tedious calculations to find the other two strains if we want to gain an additional accuracy of 1% or less. But notice the simplicity of the calculations that come from a small-strain approximation.

2.6 STRAIN AT A POINT

In Section 2.5 the lengths Δx, Δy, and Δz were finite. If we shrink these lengths to zero in Equations (2.9a) through (2.9i), we obtain the definition of strain at a point. Because the limiting operation is in a given direction, we obtain partial derivatives and not the ordinary derivatives:

$$\varepsilon_{xx} = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) = \frac{\partial u}{\partial x} \qquad \text{(2.12a)}$$

$$\varepsilon_{yy} = \lim_{\Delta y \to 0}\left(\frac{\Delta v}{\Delta y}\right) = \frac{\partial v}{\partial y} \qquad \text{(2.12b)}$$

$$\varepsilon_{zz} = \lim_{\Delta z \to 0}\left(\frac{\Delta w}{\Delta z}\right) = \frac{\partial w}{\partial z} \qquad \text{(2.12c)}$$

$$\gamma_{xy} = \gamma_{yx} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left(\frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}\right) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \text{(2.12d)}$$

$$\gamma_{yz} = \gamma_{zy} = \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}}\left(\frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}\right) = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \text{(2.12e)}$$

$$\gamma_{zx} = \gamma_{xz} = \lim_{\substack{\Delta x \to 0 \\ \Delta z \to 0}}\left(\frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}\right) = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \qquad \text{(2.12f)}$$

Equations (2.12a) through (2.12f) show that engineering strain has two subscripts, indicating both the direction of deformation and the direction of the line element that is being deformed. Thus it would seem that engineering strain is also a second-order tensor. However, unlike stress, engineering strain does not satisfy certain coordinate transformation laws, which we will study in Chapter 9. Hence it is not a second-order tensor but is related to it as follows:

$$\text{tensor normal strains} = \text{engineering normal strains}; \qquad \text{tensor shear strains} = \frac{\text{engineering shear strains}}{2}$$

In Chapter 9 we shall see that the factor $1/2$, which changes engineering shear strain to tensor shear strain, plays an important role in strain transformation.

2.6.1 Strain at a Point on a Line

In axial members we shall see that the displacement u is only a function of x. Hence the partial derivative in Equation (2.12a) becomes an ordinary derivative, and we obtain

$$\boxed{\varepsilon_{xx} = \frac{du}{dx}(x)} \tag{2.13}$$

If the displacement is given as a function of x, then we can obtain the strain as a function of x by differentiating. If strain is given as a function of x, then by integrating we can obtain the deformation between two points —that is, the relative displacement of two points. If we know the displacement of one of the points, then we can find the displacement of the other point. Alternatively stated, the integration of Equation (2.13) generates a constant of integration. To determine it, we need to know the displacement at a point on the line.

EXAMPLE 2.10

Calculations using the finite-element method (see Section 4.8) show that the displacement in a quadratic axial element is given by

$$u(x) = 125.0(x^2 - 3x + 8)10^{-6} \text{ cm}, \qquad 0 \le x \le 2 \text{ cm}$$

Determine the normal strain ε_{xx} at $x = 1$ cm.

PLAN

We can find the strain by using Equation at any x and obtain the final result by substituting the value of $x = 1$.

SOLUTION

Differentiating the given displacement, we obtain the strain as shown in Equation (E1).

$$\varepsilon_{xx}(x=1) = \left.\frac{du}{dx}\right|_{x=1} = 125.0(2x-3)10^{-6}\Big|_{x=1} = -125(10^{-6}) \tag{E1}$$

ANS. $\varepsilon_{xx}(x = 1) = -125\ \mu$

EXAMPLE 2.11

Figure 2.25 shows a bar that has axial strain $\varepsilon_{xx} = K(L - x)$ due to its own weight. K is a constant for a given material. Find the total extension of the bar in terms of K and L.

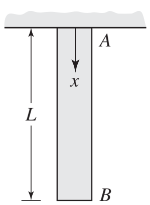


Figure 2.25 Bar in Example 2.11.

PLAN

The elongation of the bar corresponds to the displacement of point *B*. We start with Equation (2.13) and integrate to obtain the relative displacement of point *B* with respect to *A*. Knowing that the displacement at point *A* is zero, we obtain the displacement of point *B*.

SOLUTION

We substitute the given strain in Equation (2.13):

$$\varepsilon_{xx} = \frac{du}{dx} = K(L-x) \tag{E1}$$

Integrating Equation (E1) from point *A* to point *B* we obtain

$$\int_{u_A}^{u_B} du = \int_{x_A=0}^{x_B=L} K(L-x)\,dx \qquad \text{or} \qquad u_B - u_A = K\left(Lx - \frac{x^2}{2}\right)\Bigg|_0^L = K\left(L^2 - \frac{L^2}{2}\right) \tag{E2}$$

Since point *A* is fixed, the displacement $u_A = 0$ and we obtain the displacement of point *B*.

ANS. $u_B = (KL^2)/2$

COMMENTS

1. From strains we obtain deformation, that is relative displacement $u_B - u_A$. To get the absolute displacement we choose a point on the body that did not move.
2. We could integrate Equation (E1) to obtain $u(x) = K(Lx - x^2/2) + C_1$. Using the condition that the displacement u at $x = 0$ is zero, we obtain the integration constant $C_1 = 0$. We could then substitute $x = L$ to obtain the displacement of point *B*. The integration constant C_1 represents rigid-body translation, which we eliminate by fixing the bar to the wall.

Consolidate your knowledge

1. Explain in your own words deformation, strain, and their relationship without using equations.

QUICK TEST 2.1

Time: 15 minutes/Total: 20 points

Grade yourself using the answers in Appendix E. Each problem is worth 2 points.

1. What is the difference between displacement and deformation?
2. What is the difference between Lagrangian and Eulerian strains?
3. In decimal form, what is the value of normal strain that is equal to 0.3%?
4. In decimal form, what is the value of normal strain that is equal to 2000 μ?
5. Does the right angle increase or decrease with positive shear strains?
6. If the left end of a rod moves more than the right end in the negative *x* direction, will the normal strain be negative or positive? Justify your answer.
7. Can a 5% change in length be considered to be small normal strain? Justify your answer.
8. How many nonzero strain components are there in three dimensions?
9. How many nonzero strain components are there in plane strain?
10. How many independent strain components are there in plane strain?

PROBLEM SET 2.2

Strain components

2.67 A rectangle deforms into the colored shape shown in Figure P2.67. Determine ε_{xx}, ε_{yy}, and γ_{xy} at point *A*.

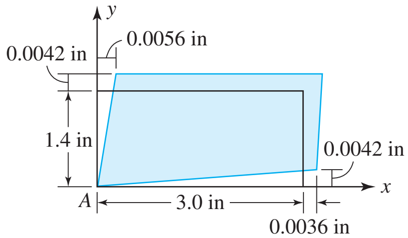


Figure P2.67

2.68 A rectangle deforms into the colored shape shown in Figure P2.68. Determine ε_{xx}, ε_{yy}, and γ_{xy} at point *A*.

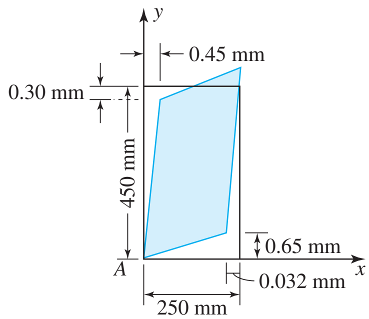


Figure P2.68

2.69 A rectangle deforms into the colored shape shown in Figure P2.69. Determine ε_{xx}, ε_{yy}, and γ_{xy} at point *A*.

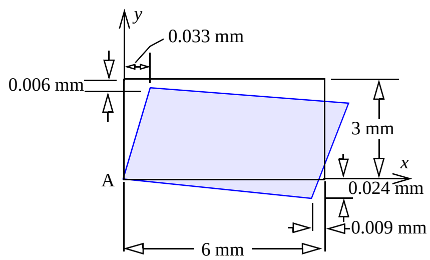


Figure P2.69

2.70 Displacements *u* and v in *x* and *y* directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as give below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

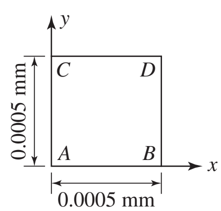


Figure P2.70

$$u_A = 0.500\mu\text{mm} \qquad v_A = -1.000\mu\text{mm}$$
$$u_B = 1.125\mu\text{mm} \qquad v_B = -1.3125\mu\text{mm}$$
$$u_C = 0 \qquad v_C = -1.5625\mu\text{mm}$$
$$u_D = 0.750\mu\text{mm} \qquad v_D = -2.125\mu\text{mm}$$

2.71 Displacements *u* and v in *x* and *y* directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as given below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = 0.625\mu\text{mm} \qquad v_A = -0.3125\mu\text{mm}$$
$$u_B = 1.500\mu\text{mm} \qquad v_B = -0.5000\mu\text{mm}$$
$$u_C = 0.250\mu\text{mm} \qquad v_C = -1.125\mu\text{mm}$$
$$u_D = 1.250\mu\text{mm} \qquad v_D = -1.5625\mu\text{mm}$$

2.72 Displacements u and v in x and y directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as given below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = -0.500\,\mu\text{mm} \qquad v_A = -0.5625\,\mu\text{mm}$$
$$u_B = 0.250\,\mu\text{mm} \qquad v_B = -1.125\,\mu\text{mm}$$
$$u_C = -1.250\,\mu\text{mm} \qquad v_C = -1.250\,\mu\text{mm}$$
$$u_D = -0.375\,\mu\text{mm} \qquad v_D = -2.0625\,\mu\text{mm}$$

2.73 Displacements u and v in x and y directions, respectively, were measured by the Moiré interferometry method at many points on a body. The displacements of four points shown in Figure P2.70 are as given below. Determine the average values of the strain components ε_{xx}, ε_{yy}, and γ_{xy} at point A.

$$u_A = 0.250\,\mu\text{mm} \qquad v_A = -1.125\,\mu\text{mm}$$
$$u_B = 1.250\,\mu\text{mm} \qquad v_B = -1.5625\,\mu\text{mm}$$
$$u_C = -0.375\,\mu\text{mm} \qquad v_C = -2.0625\,\mu\text{mm}$$
$$u_D = 0.750\,\mu\text{mm} \qquad v_D = -2.7500\,\mu\text{mm}$$

Strain at a point

2.74 In a tapered circular bar that is hanging vertically, the axial displacement due to its weight was found to be

$$u(x) = \left(-19.44 + 1.44x - 0.01x^2 - \frac{933.12}{72 - x}\right)10^{-3} \text{ in.}$$

Determine the axial strain ε_{xx} at $x = 24$ in.

2.75 In a tapered rectangular bar that is hanging vertically, the axial displacement due to its weight was found to be

$$u(x) = [7.5(10^{-6})x^2 - 25(10^{-6})x - 0.15\ln(1 - 0.004x)] \text{ mm}$$

Determine the axial strain ε_{xx} at $x = 100$ mm.

2.76 The axial displacement in the quadratic one-dimensional finite element shown in Figure P2.76 is given below. The displacements of nodes 1, 2, and 3 are u_1, u_2, and u_3, respectively. Determine the strain at node 2.

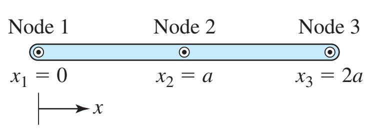


Figure P2.76

$$u(x) = \frac{u_1}{2a^2}(x-a)(x-2a) - \frac{u_2}{a^2}(x)(x-2a) + \frac{u_3}{2a^2}(x)(x-a)$$

2.77 The strain in the tapered bar due to the applied load in Figure P2.77 was found to be $\varepsilon_{xx} = 0.2/(40 - x)^2$. Determine the extension of the bar.

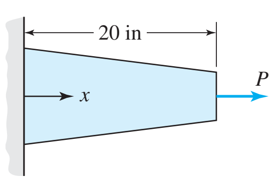


Figure P2.77

2.78 The axial strain in a bar of length L was found to be

$$\varepsilon_{xx} = \frac{KL}{(4L - 3x)} \qquad 0 \le x \le L$$

where K is a constant for a given material, loading, and cross-sectional dimension. Determine the total extension in terms of K and L.

2.79 The axial strain in a bar of length L due to its own weight was found to be

$$\varepsilon_{xx} = K\left[4L - 2x - \frac{8L^3}{(4L-2x)^2}\right] \qquad 0 \le x \le L,$$

where K is a constant for a given material and cross-sectional dimension. Determine the total extension in terms of K and L.

2.80 A bar has a tapered and a uniform section securely fastened, as shown in Figure P2.80. Determine the total extension of the bar if the axial strain in each section is

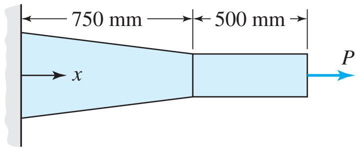


Figure P2.80

$$\varepsilon = \frac{1500 \times 10^3}{1875 - x}\,\mu, \qquad 0 \le x \le 750 \text{ mm}$$

$$\varepsilon = 1500\,\mu, \qquad 750 \text{ mm} \le x \le 1250 \text{ mm}$$

Stretch yourself

2.81 N axial bars are securely fastened together. Determine the total extension of the composite bar shown in Figure P2.81 if the strain in the i^{th} section is as given.

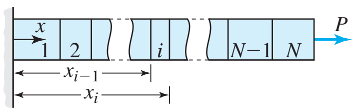


Figure P2.81

$$\varepsilon_i = a_i, \qquad x_{i-1} \le x \le x_i$$

2.82 True strain ε_T is calculated from $d\varepsilon_T = du/(L_0 + u)$, where u is the deformation at any given instant and L_0 is the original undeformed length. Thus the increment in true strain is the ratio of change in length at any instant to the length at that given instant. If ε represents engineering strain, show that at any instant the relationship between true strain and engineering strain is given by the following equation:

$$\varepsilon_T = \ln(1 + \varepsilon) \tag{2.14}$$

2.83 The displacements in a body are given by

$$u = [0.5(x^2 - y^2) + 0.5xy](10^{-3}) \text{ mm} \qquad v = [0.25(x^2 - y^2) - xy](10^{-3}) \text{ mm}$$

Determine strains ε_{xx}, ε_{yy}, and γ_{xy} at $x = 5$ mm and $y = 7$ mm.

2.84 A metal strip is to be pulled and bent to conform to a rigid surface such that the length of the strip OA fits the arc OB of the surface shown in Figure P2.84. The equation of the surface is $f(x) = 0.04x^{3/2}$ in. and the length OA = 9 in. Determine the average normal strain in the metal strip.

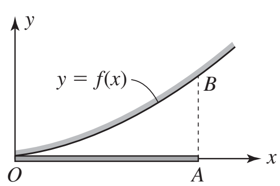


Figure P2.84

2.85 A metal strip is to be pulled and bent to conform to a rigid surface such that the length of the strip OA fits the arc OB of the surface shown in Figure P2.84. The equation of the surface is $f(x) = 625x^{3/2}$ μmm and the length OA = 200 mm. Determine the average normal strain in the metal strip.

Computer problems

2.86 A metal strip is to be pulled and bent to conform to a rigid surface such that the length of the strip OA fits the arc OB of the surface shown in Figure P2.84. The equation of the surface is $f(x) = (0.04x^{3/2} - 0.005x)$ in. and the length $OA = 9$ in. Determine the average normal strain in the metal strip. Use numerical integration.

2.87 Measurements made along the path of the stretch cord that is stretched over the canoe in Problem 2.4 (Figure P2.87) are shown in Table P2.87. The *y* coordinate was measured to the closest $\frac{1}{32}$ in. Between points *A* and *B* the cord path can be approximated by a straight line. Determine the average strain in the stretch cord if its original length it is 40 in. Use a spread sheet and approximate each 2-in. *x* interval by a straight line.

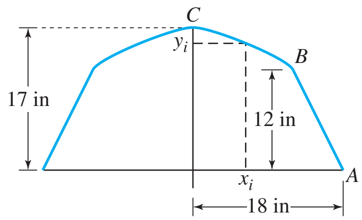


Figure P2.87

TABLE P2.87

x_i	y_i
0	17
2	$16\frac{30}{32}$
4	$16\frac{29}{32}$
6	$16\frac{19}{32}$
8	$16\frac{3}{32}$
10	$15\frac{16}{32}$
12	$14\frac{24}{32}$
14	$13\frac{28}{32}$
$x_B = 16$	$y_B = 12$
$x_A = 18$	$y_A = 0$

2.7* CONCEPT CONNECTOR

Like stress there are several definitions of strains. But unlike stress which evolved from intuitive understanding of strength to a mathematical definition, the development of concept of strain was mostly mathematical as described briefly in Section 2.7.1.

Displacements at different points on a solid body can be measured or analyzed by a variety of methods. One modern experimental technique is Moiré Fringe Method discussed briefly in Section 2.7.2.

2.7.1 History: The Concept of Strain

Normal strain, as a ratio of deformation over length, appears in experiments conducted as far back as the thirteenth century. Thomas Young (1773–1829) was the first to consider shear as an elastic strain, which he called *detrusion*. Augustin Cauchy (1789–1857), who introduced the concept of stress we use in this book (see Section 1.6.1), also introduced the mathematical definition of engineering strain given by Equations (2.12a) through (2.12f). The nonlinear Lagrangian strain written in tensor form was introduced by the English mathematician and physicist George Green (1793–1841) and is today called Green's strain tensor. The nonlinear Eulerian strain tensor, introduced in 1911 by E. Almansi, is also called Almansi's strain tensor. Green's and Almansi's strain tensors are often referred to as strain tensors in Lagrangian and Eulerian coordinates, respectively.

2.7.2 Moiré Fringe Method

Moiré fringe method is an experimental technique of measuring displacements that uses light interference produced by two equally spaced gratings. Figure 2.26 shows equally spaced parallel bars in two gratings. The spacing between the bars is called the *pitch*. Suppose initially the bars in the grating on the right overlap the spacings of the left. An observer on the right will be in a dark region, since no light ray can pass through both gratings. Now suppose that left grating moves, with a displacement less than the spacing between the bars. We will then have space between each pair of bars, resulting in regions of dark and light. These lines of light and dark lines are called *fringes*. When the left grating has moved through one pitch, the observer will once more be in the dark. By counting the number of times the regions of light and dark (i.e., the number of fringes passing this point) and multiplying by the pitch, we can obtain the displacement.

Note that any motion of the left grating parallel to the direction of the bars will not change light intensity. Hence displacements calculated from Moiré fringes are always perpendicular to the lines in the grating. We will need a grid of perpendicular lines to find the two components of displacements in a two-dimensional problem.

The left grating may be cemented, etched, printed, photographed, stamped, or scribed onto a specimen. Clearly, the order of displacement that can be measured depends on the number of lines in each grating. The right grating is referred to as the *reference grating*.

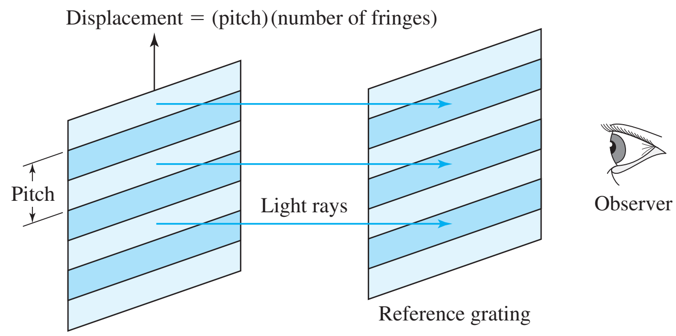


Figure 2.26 Destructive light interference by two equally spaced gratings.

Figure 2.26 illustrates light interference produced mechanical and is called *geometric Moiré method.* This method is used for displacement measurements in the range of 1 mm to as small as 10 μm, which corresponds to a grid of 1 to 100 lines per millimeter. In U.S. customary units the range is from 0.1 in. to as small as 0.001 in., corresponding to grids having from 10 to 1000 lines per inch.

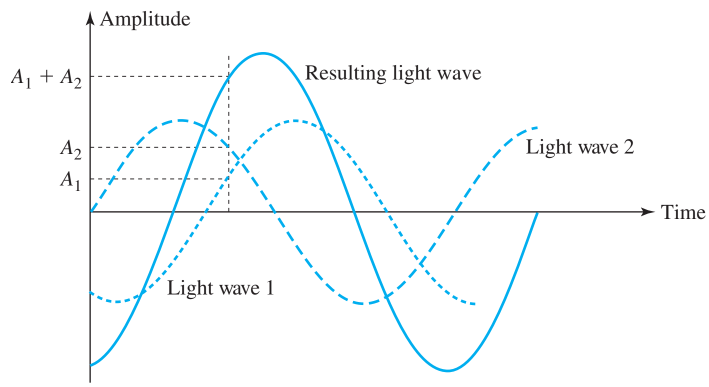


Figure 2.27 Superposition of two light waves.

Light interference can also be produced optically and techniques based on optical light interference are termed *optical interferometry.* Consider two light rays of the same frequency arriving at a point, as shown in Figure 2.27. The amplitude of the resulting light wave is the sum of the two amplitudes. If the crest of one light wave falls on the trough of another light wave, then the resulting amplitude will be zero, and we will have darkness at that point. If the crests of two waves arrive at the same time, then we will have light brighter than either of the two waves alone. This addition and subtraction, called *constructive* and *destructive* interference, is used in interferometry for measurements in a variety of ways.

In Moiré interferometry, for example, a reference grid may be created by the reflection of light from a grid fixed to the specimen, using two identical light sources. As the grid on the specimen moves, the reflective light and the incident light interfere constructively and destructively to produce Moiré fringes. Displacements as small as 10^{-5} in., corresponding to 100,000 lines per inch, can be measured. In the metric system, the order of displacements is 25×10^{-5} mm, which corresponds to 4000 lines per millimeter.

In an experiment to study mechanically fastened composites, load was applied on one end of the joint and equilibrated by applying a load on the lower hole, as shown in Figure 2.28*a*. Moiré fringes parallel to the applied load on the top plate are shown in Figure 2.28*b*.

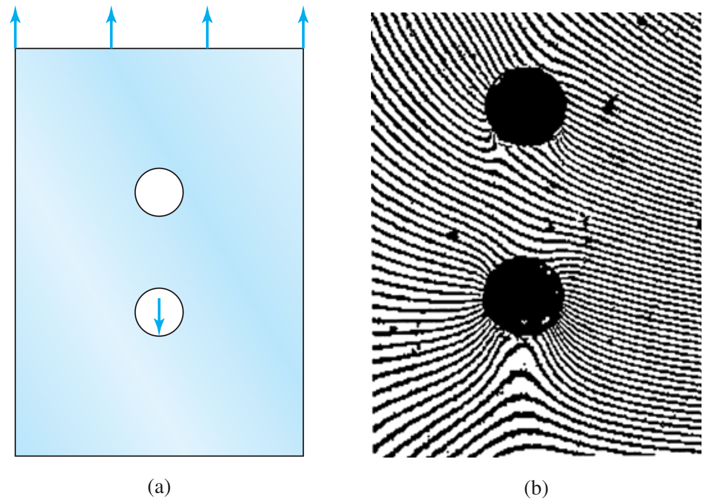


Figure 2.28 Deformation of a grid obtained from optical Moiré interferometry.

2.8 CHAPTER CONNECTOR

In this chapter we saw that the relation between displacement and strains is derived by studying the geometry of the deformed body. However, whenever we approximate a deformed body, we make assumptions regarding the displacements of points on the body. The simplest approach is to assume that each component of displacement is either a constant in the direction of coordinate axis, or else a linear function of the coordinate. From the displacements we can then obtain the strains.

The strain–displacement relation is independent of material properties. In the next chapter we shall introduce material properties and the relationship between stresses and strains. Thus, from displacements we first deduce the strains. From these we will deduce stress variations, from which we can find the internal forces. Finally, we relate the internal forces to external forces, as we did in Chapter 1. We shall see the complete logic in Chapter 3.

We will study strains again in Chapter 9, on strain transformation which relates strains in different coordinate systems. This is important as both experimental measurements and strains analyses are usually performed in a coordinate system chosen to simplify calculations. Developing a discipline of drawing deformed shapes has the same importance as drawing a free-body diagrams for calculating forces. These drawing provide an intuitive understanding of deformation and strain, as well as reduce mistakes in calculations.

POINTS AND FORMULAS TO REMEMBER

- The total movement of a point with respect to a fixed reference coordinate is called displacement.
- The relative movement of a point with respect to another point on the body is called deformation.
- The displacement of a point is the sum of rigid body motion and motion due to deformation.
- Lagrangian strain is computed from deformation by using the original undeformed geometry as the reference geometry.
- Eulerian strain is computed from deformation by using the final deformed geometry as the reference geometry.
- $\varepsilon = \dfrac{L_f - L_0}{L_0}$ (2.1) $\varepsilon = \dfrac{\delta}{L_0}$ (2.2) $\varepsilon = \dfrac{u_B - u_A}{x_B - x_A}$ (2.3)
- where ε is the average normal strain, L_0 is the original length of a line, L_f is the final length of a line, δ is the deformation of the line, and u_A and u_B are displacements of points x_A and x_B, respectively.
- Elongations result in positive normal strains. Contractions result in negative normal strains.
- $\gamma = \pi/2 - \alpha$ (2.4)

  where α is the final angle measured in radians and $\pi/2$ is the original right angle.

- Decreases in angle result in positive shear strain. Increases in angle result in negative shear strain.
- Small-strain approximation may be used for strains less than 0.01.
- Small-strain calculations result in linear deformation analysis.
- Small normal strains are calculated by using the deformation component in the original direction of the line element, regardless of the orientation of the deformed line element.
- In small shear strain (γ) calculations the following approximation may be used for the trigonometric functions:
- $\tan\gamma \approx \gamma$ $\sin\gamma \approx \gamma$ $\cos\gamma \approx 1$
- In small strain,
- $\delta = \bar{\mathbf{D}}_{AP} \cdot \bar{\mathbf{i}}_{AP}$ (2.7)
- where $\bar{\mathbf{D}}_{AP}$ is the deformation vector of the bar *AP* and $\bar{\mathbf{i}}_{AP}$ is the unit vector in the original direction of the bar *AP.*
- The same reference point must be used in the calculations of the deformation vector and the unit vector.

Average strain

$$\varepsilon_{xx} = \frac{\Delta u}{\Delta x} \quad \gamma_{xy} = \gamma_{yx} = \frac{\Delta u}{\Delta y} + \frac{\Delta v}{\Delta x}$$
$$\varepsilon_{yy} = \frac{\Delta v}{\Delta y} \quad \gamma_{yz} = \gamma_{zy} = \frac{\Delta v}{\Delta z} + \frac{\Delta w}{\Delta y}$$
$$\varepsilon_{zz} = \frac{\Delta w}{\Delta z} \quad \gamma_{zx} = \gamma_{xz} = \frac{\Delta w}{\Delta x} + \frac{\Delta u}{\Delta z}$$

(2.9a) through (2.9i)

Strain at a point

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \quad \gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} \quad \gamma_{yz} = \gamma_{zy} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$
$$\varepsilon_{zz} = \frac{\partial w}{\partial z} \quad \gamma_{zx} = \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(2.12a) through (2.12f)

- where *u*, v, and *w* are the displacements of a point in the *x*, *y*, and *z* directions, respectively.
- Shear strain is symmetric.
- In three dimensions there are nine strain components but only six are independent.
- In two dimensions there are four strain components but only three are independent.
- If *u* is only a function of *x*,

$$\varepsilon_{xx} = \frac{du(x)}{dx} \qquad (2.13)$$

CHAPTER THREE
MECHANICAL PROPERTIES OF MATERIALS

Learning objectives

1. Understand the qualitative and quantitative descriptions of mechanical properties of materials.
2. Learn the logic of relating deformation to external forces.

The ordinary wire and rubber stretch cord in Figure 3.1 have the same undeformed length and are subjected to the same loads. Yet the rubber deforms significantly more, which is why we use rubber cords to tie luggage on top of a car. As the example shows, before we can relate deformation to applied forces, we must first describe the mechanical properties of materials.

Figure 3.1 Material impact on deformation.

In engineering, adjectives such as *elastic*, *ductile*, or *tough* have very specific meanings. These terms will be our *qualitative* description of materials. Our *quantitative* descriptions will be the equations relating stresses and strains. Together, these description form the material model (Figure 3.2). The parameters in the material models are determined by the least-square method (see Appendix B.3) to fit the best curve through experimental observations. In this chapter, we develop a simple model and learn the logic relating deformation to forces. In later chapters, we shall apply this logic to axial members, shafts, and beams and obtain formulas for stresses and deformations.

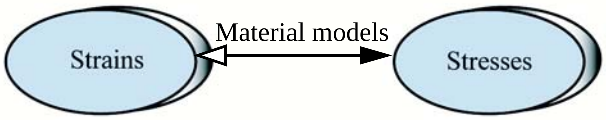


Figure 3.2 Relationship of stresses and strains.

3.1 MATERIALS CHARACTERIZATION

The American Society for Testing and Materials (ASTM) specifies test procedures for determining the various properties of a material. These descriptions are guidelines used by experimentalists to obtain reproducible results for material properties. In this section, we study the tension and compression tests, which allow us to determine many parameters relating stresses and strains.

3.1.1 Tension Test

In the **tension test**, standard specimen are placed in a tension-test machine, where they are gripped at each end and pulled in the axial direction. Figure 3.3 shows two types of standard geometry: a specimen with a rectangular cross-section and specimen with a circular cross-section.

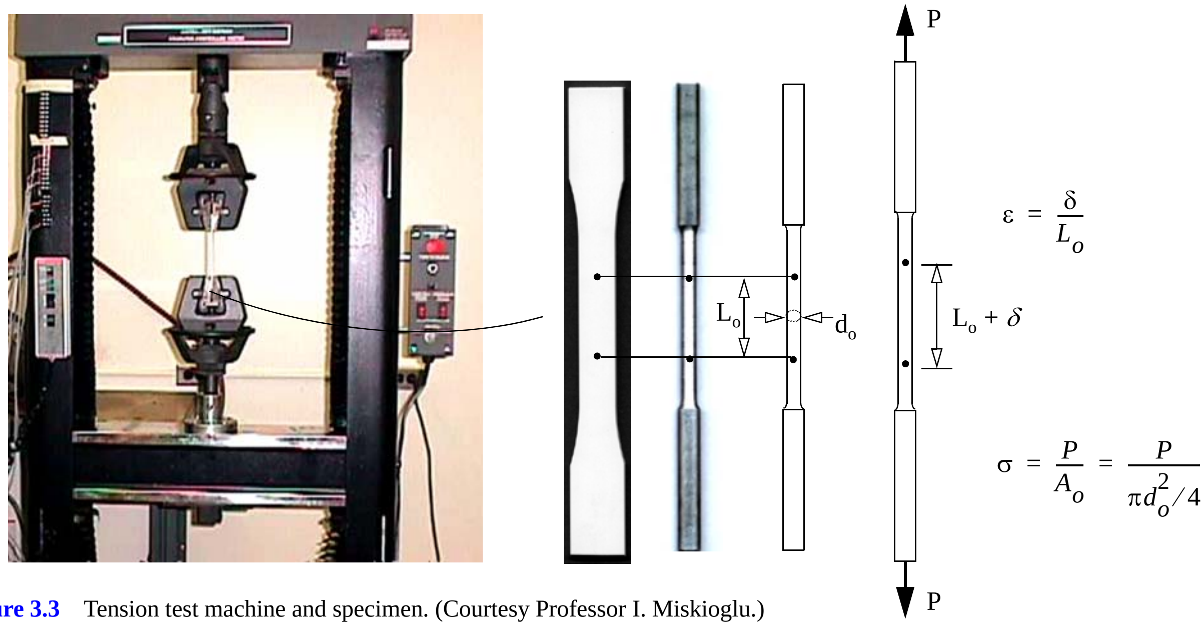


Figure 3.3 Tension test machine and specimen. (Courtesy Professor I. Miskioglu.)

Two marks are made in the central region, separated by the **gage length** L_0. The deformation δ is movement of the two marks. For metals, such as aluminum or steel, ASTM recommends a gage length L_0 = 2 in. and diameter d_0 = 0.5 in. The normal strain ε is the deformation δ divided by L_0.

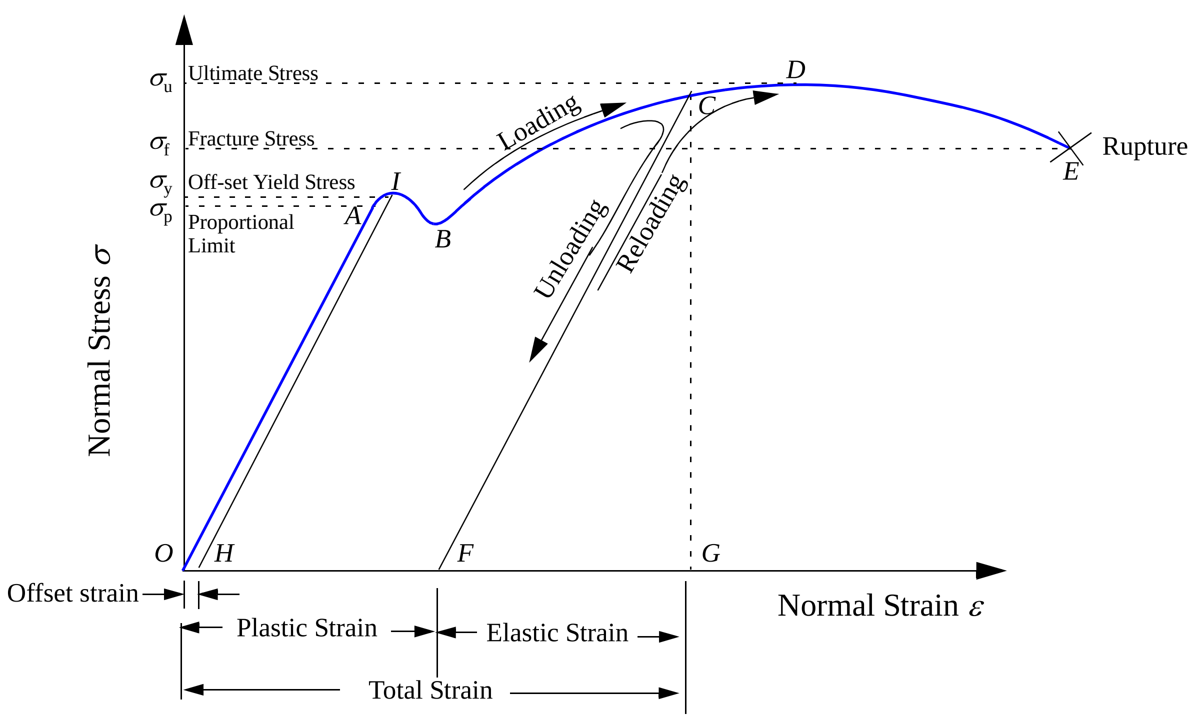


Figure 3.4 Stress–strain curve.

The tightness of the grip, the symmetry of the grip, friction, and other local effects are assumed and are observed to die out rapidly with the increase in distance from the ends. This dissipation of local effects is further facilitated by the gradual change in the cross-section. The specimen is designed so that its central region is in a uniform state of axial stress. The normal stress is calculated by dividing the applied force P by the area of cross section A_0, which can be found from the specimen's width or diameter.

The tension test may be conducted by controlling the force P and measuring the corresponding deformation δ. Alternatively, we may control the deformation δ by the movement of the grips and measuring the corresponding force P. The values of force P and deformation δ are recorded, from which normal stress σ and normal strain ε are calculated. Figure 3.4 shows a typical stress–strain (σ-ε) curve for metal.

As the force is applied, initially a straight line (OA) is obtained. The end of this linear region is called the **proportional limit**. For some metals, the stress may then decrease slightly (the region *AB*), before increasing once again. The largest stress (point *D* on the curve) is called the **ultimate stress**. In a force-controlled experiment, the specimen will suddenly break at the ultimate stress. In a displacement-controlled experiment we will see a decrease in stress (region *DE*). The stress at breaking point *E* is called **fracture** or **rupture stress**.

Elastic and plastic regions

If we load the specimen up to any point along line *OA*—or even a bit beyond—and then start unloading, we find that we retrace the stress–strain curve and return to point *O*. In this **elastic region**, the material regains its original shape when the applied force is removed.

If we start unloading only after reaching point *C*, however, then we will come down the straight line *FC*, which will be parallel to line *OA*. At point *F*, the *stress* is zero, but the *strain* is nonzero. *C* thus lies in the **plastic region** of the stress-strain curve, in which the material is deformed permanently, and the permanent strain at point *F* is the **plastic strain**. The region in which the material deforms permanently is called **plastic region**. The total strain at point C is sum the plastic strain (*OF*) and an additional **elastic strain** (*FG*)

The point demarcating the elastic from the plastic region is called the **yield point**. The stress at yield point is called the **yield stress**. In practice, the yield point may lie anywhere in the region *AB*, although for most metals it is close to the proportional limit. For many materials it may not even be clearly defined. For such materials, we mark a prescribed value of offset strain recommended by ASTM to get point *H* in Figure 3.4. Starting from *H* we draw a line (*HI*) parallel to the linear part (*OA*) of the stress–strain curve. **Offset yield stress** would correspond to a plastic strain at point *I*. Usually the offset strain is given as a percentage. A strain of 0.2% equals $\varepsilon = 0.002$ (as described in Chapter 2).

It should be emphasized that *elastic* and *linear* are two distinct material descriptions. Figure 3.5*a* shows the stress–strain curve for a soft rubber that can stretch several times its original length and still return to its original geometry. Soft rubber is thus *elastic* but *nonlinear* material.

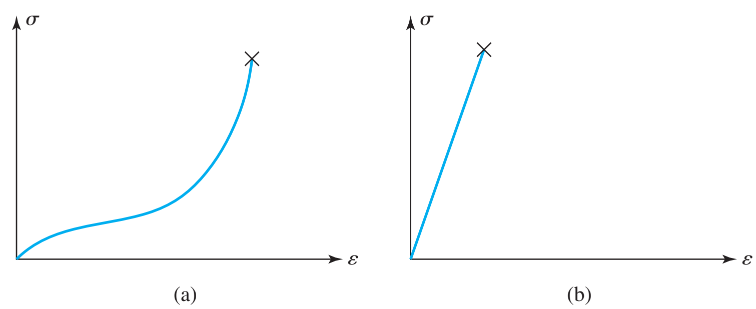


Figure 3.5 Examples of nonlinear and brittle materials. (*a*) Soft rubber. (*b*) Glass.

Ductile and brittle materials

Ductile materials, such as aluminum and copper, can undergo large *plastic* deformations before fracture. (Soft rubber can undergo large deformations but it is not a ductile material.) Glass, on the other hand, is **brittle**: it exhibits little or no plastic deformation as shown in Figure 3.5*b*. A material's ductility is usually described as percent elongation before rupture. The elongation values of 17% for aluminum and 35% for copper before rupture reflect the large plastic strains these materials undergo before rupture, although they show small elastic deformation as well.

Recognizing ductile and brittle material is important in design, in order to characterize failure as we shall see in Chapters 8 and 10. A ductile material usually yields when the maximum shear stress exceeds the yield *shear stress*. A brittle material usually ruptures when the maximum *tensile normal stress* exceeds the ultimate tensile stress.

Hard and soft materials

A material **hardness** is its resistance to scratches and indentation (*not* its strength). In Rockwell test, the most common hardness test, a hard indenter of standard shape is pressed into the material using a specified load. The depth of indentation is measured and assigned a numerical scale for comparing hardness of different materials.

A soft material can be made harder by gradually increasing its yield point by **strain hardening**. As we have seen, at point *C* in Figure 3.4 the material has a permanent deformation even after unloading. If the material now is reloaded, point *C* becomes the new yield point, as additional plastic strain will be observed only after stress exceeds this point. Strain hardening is used, for example, to make aluminum pots and pans more durable. In the manufacturing process, known as *deep drawing*, the aluminum undergoes large plastic deformation. Of course, as the yield point increases, the remaining plastic deformation before fracture decreases, so the material becomes more brittle.

True stress and true strain

We noted that stress decreases with increasing strain between the ultimate stress and rupture (region $DE\ in$ Figure 3.4). However, this decrease is seen only if we plot Cauchy's stress versus engineering strain. (Recall that Cauchy's stress is the load P divided by the original undeformed cross-sectional area.) An alternative is to plot true stress versus true strain, calculated using the actual, deformed cross-section and length (Section 1.6 and Problem 2.82). In such a plot, the stress in region DE continues to increase with increasing strain and just as in region *BD*.

Past ultimate stress a specimen also undergoes a sudden decease in cross-sectional area called **necking**. Figure 3.6 shows necking in a broken specimen from a tension test.

Figure 3.6 Specimen showing necking. (Courtesy Professor J. B. Ligon.)

3.1.2 Material Constants

Hooke's law give the relationship between normal stress and normal strain for the linear region:

$$\boxed{\sigma = E\varepsilon} \tag{3.1}$$

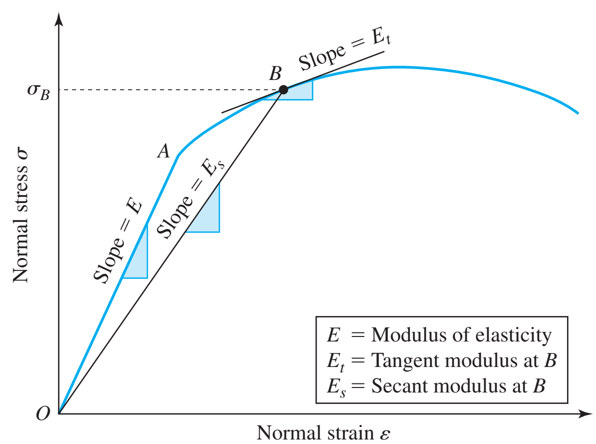


Figure 3.7 Different material moduli.

where *E* is called **modulus of elasticity** or **Young's modulus**. It represents the slope of the straight line in a stress–strain curve, as shown in Figure 3.7. Table 3.1 shows the moduli of elasticity of some typical engineering materials, with wood as a basis of comparison.

In the nonlinear regions, the stress–strain curve is approximated by a variety of equations as described in Section 3.11. The choice of approximation depends on the need of the analysis being performed. The two constants that are often used are shown in Figure 3.7. The slope of the tangent drawn to the stress–strain curve at a given stress value is called the **tangent modulus**. The slope of the line that joins the origin to the point on the stress–strain curve at a given stress value is called the **secant modulus.**

TABLE 3.1 Comparison of moduli of elasticity for typical materials

Material	Modulus of Elasticity (10^3 ksi)	Modulus Relative to Wood
Rubber	0.12	0.06
Nylon	0.60	0.30
Adhesives	1.10	0.55
Soil	1.00	0.50
Bones	1.86	0.93
Wood	2.00	1.00
Concrete	4.60	2.30
Granite	8.70	4.40
Glass	10.00	5.00
Aluminum	10.00	5.00
Steel	30.00	15.00

Figure 3.8 shows that the elongation of a cylindrical specimen in the longitudinal direction (direction of load) causes contraction in the lateral (perpendicular to load) direction and vice versa. The ratio of the two normal strains is a material constant called the **Poisson's ratio**, designated by the Greek letter ν (nu):

$$\boxed{\nu = -\left(\frac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{longitudinal}}}\right)} \tag{3.2}$$

Poisson's ratio is a dimensionless quantity that has a value between 0 and $\frac{1}{2}$ for most materials, although some composite materials can have negative values for ν. The theoretical range for Poisson's ration is $-1 \le \nu \le \frac{1}{2}$.

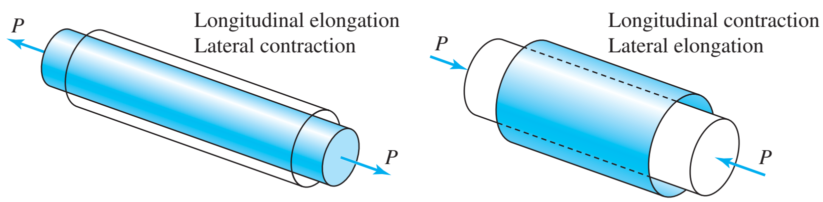


Figure 3.8 Poisson effect.

To establish the relationship between shear stress and shear strain, a torsion test is conducted using a machine of the type shown in Figure 3.9. On a plot of shear stress τ versus shear strain γ, we obtain a curve similar to that shown in Figure 3.4. In the linear region

$$\boxed{\tau = G\gamma} \tag{3.3}$$

where G is the **shear modulus of elasticity** or **modulus of rigidity**.

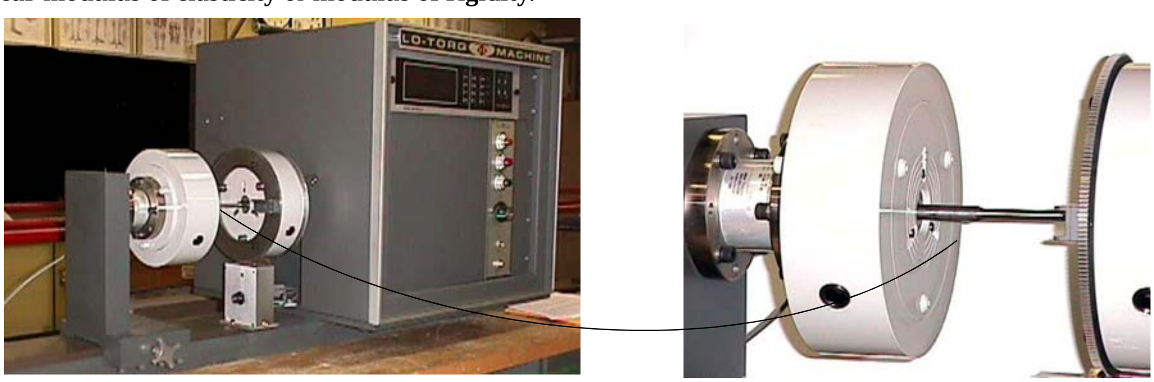


Figure 3.9 Torsion testing machine. (Courtesy Professor I. Miskioglu.)

3.1.3 Compression Test

We can greatly simplify analysis by assuming material behavior to be the same in tension and compression. This assumption of similar tension and compression properties works well for the values of material constants (such as E and ν). Hence the stress and deformation formulas developed in this book can be applied to members in tension and in compression. However, the compressive strength of many brittle materials can be very different from its tensile strength. In ductile materials as well the stress reversal from tension to compression in the plastic region can cause failure.

Figure 3.10*a* shows the stress–strain diagrams of two brittle materials. Notice the moduli of elasticity (the slopes of the lines) is the same in tension and compression. However, the compressive strength of cast iron is four times its tensile strength, while concrete can carry compressive stresses up to 5 ksi but has negligible tensile strength. Reinforcing concrete with steel bars can help, because the bars carry most of the tensile stresses.

Figure 3.10*b* shows the stress–strain diagrams for a ductile material such as mild steel. If compression test is conducted without unloading, then behavior under tension and compression is nearly identical: modulus of elasticity, yield stress, and ultimate stress are much the same. However, if material is loaded past the yield stress (point *A*), up to point *B* and then unloaded, the stress-strain diagram starts to curve after point *C* in the compressive region

Suppose we once more reverse loading direction, but starting at point *D*, which is at least $2\sigma_{yield}$ below point *B*, and ending at point *F*, where there is no applied load. The plastic strain is now less than that at point *C*. In fact, it is conceivable that the loading–unloading cycles can return the material to point *O* with no plastic strain. Does that mean we have the same material as the one we started with? No! The internal structure of the material has been altered significantly. Breaking of the material below the ultimate stress by load cycle reversal in the plastic region is called the **Bauschinger effect**. Design therefore usually precludes cyclic loading into the plastic region. Even in the elastic region, cyclic loading can cause failure due to fatigue (see Section 3.10).

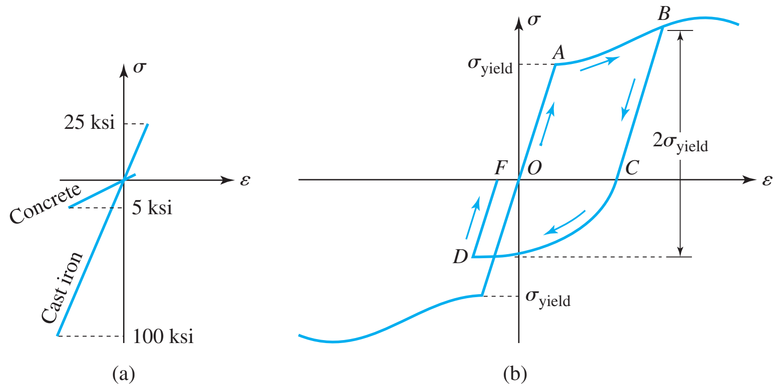


Figure 3.10 Differences in tension and compression. (*a*) Brittle material. (*b*) Ductile material.

EXAMPLE 3.1

A tension test was conducted on a circular specimen of titanium alloy. The gage length of the specimen was 2 in. and the diameter in the test region before loading was 0.5 in. Some of the data from the tension test are given in Table 3.2, where P is the applied load and δ is the corresponding deformation. Calculate the following quantities: (a) Stress at proportional limit. (b) Ultimate stress. (c) Yield stress at offset strain of 0.4%. (d) Modulus of elasticity. (e) Tangent and secant moduli of elasticity at a stress of 136 ksi. (f) Plastic strain at a stress of 136 ksi.

TABLE 3.2 Tension test data in Example 3.1

#	P (kips)	δ (10^{-3} in.)
1	0.0	0.0
2	5.0	3.2
3	15.0	9.5
4	20.0	12.7
5	24.0	15.3
6	24.5	15.6
7	25.0	15.9
8	25.2	16.9
9	25.4	19.7
10	26.0	28.5
11	26.5	36.9
12	27.0	46.5
13	27.5	58.3
14	28.0	75.2
15	28.2	87.1
16	28.3	100.0
17	28.2	112.9
18	28.0	124.8

TABLE 3.3 Stress and strain in Example 3.1

#	σ (ksi)	ε (10^{-3})
1	0.0	0.0
2	25.5	1.6
3	76.4	4.8
4	101.9	6.4
5	122.2	7.6
6	124.8	7.8
7	127.3	8.0
8	128.3	8.5
9	129.9	10.5
10	132.4	14.3
11	135.0	18.4
12	137.5	23.3
13	140.1	29.1
14	142.6	37.6
15	143.6	43.5
16	144.0	50.0
17	143.6	56.5
18	142.6	62.4

PLAN

We can divide the column of load P by the cross-sectional area to get the values of stress. We can divide the column of deformation δ by the gage length of 2 in. to get strain. We can plot the values to obtain the stress–strain curve and calculate the quantities, as described in Section 3.1.

SOLUTION

We divide the load column by the cross-sectional area $A = \pi(0.5 \text{ in.})^2/4 = 0.1964 \text{ in.}^2$ to obtain stress σ, and the deformation column by the gage length of 2 in. to obtain strain ε, as shown in Table 3.3, which is obtained using a spread sheet. Figure 3.11 shows the corresponding stress–strain curve.

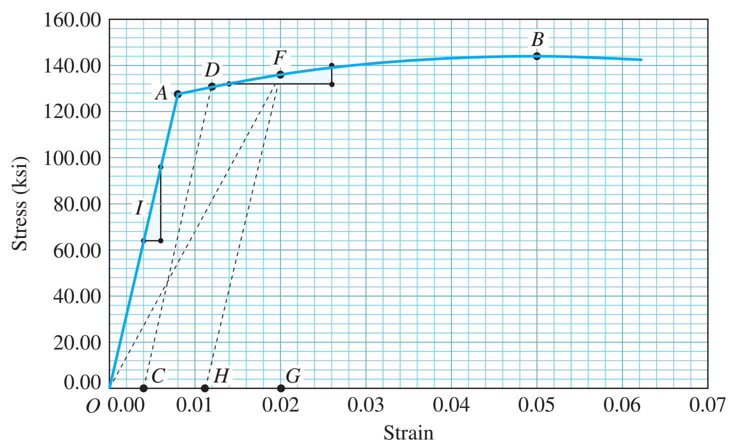


Figure 3.11 Stress–strain curve for Example 3.1.

(a) Point A is the proportional limit in Figure 3.11. The stress at point A is:

ANS. $\sigma_{prop} = 128 \text{ ksi.}$

(b) The stress at point B in Figure 3.11 is the ultimate as it is largest stress on the stress–strain curve.

ANS. $\sigma_{ult} = 144 \text{ ksi.}$

(c) The offset strain of 0.004 (or 0.4%) corresponds to point C. We can draw a line parallel to OA from point C, which intersects the stress–strain curve at point D. The stress at point D is the offset yield stress

ANS. $\sigma_{yield} = 132 \text{ ksi.}$

(d) The modulus of elasticity E is the slope of line OA. Using the triangle at point I we can find E,

$$E = \frac{96\text{ ksi} - 64\text{ ksi}}{0.006 - 0.004} = 16(10^3)\text{ ksi} \tag{E1}$$

ANS. $E = 16{,}000$ ksi

(e) At point F the stress is 136 ksi. We can find the tangent modulus by finding the slope of the tangent at F,

$$E_t = \frac{140\text{ ksi} - 132\text{ ksi}}{0.026 - 0.014} = 666.67\text{ ksi} \tag{E2}$$

ANS. $E_t = 666.7$ ksi

(f) We can use triangle OFG to calculate the slope of OF to obtain secant modulus of elasticity at 136 ksi.

$$E_s = \frac{136\text{ ksi} - 0}{0.02 - 0} = 6800\text{ ksi} \tag{E3}$$

ANS. $E_s = 6800$ ksi

(g) To find the plastic strain at 136 ksi, we draw a line parallel to OA through point F. Following the description in Figure 3.4, OH represents the plastic strain. We know that the value of plastic strain will be between 0.01 and 0.012. We can do a more accurate calculation by noting that the plastic strain OH is the total strain OG minus the elastic strain HG. We find the elastic strain by dividing the stress at F (136 ksi) by the modulus of elasticity E:

$$\varepsilon_{\text{plastic}} = \varepsilon_{\text{total}} - \varepsilon_{\text{elastic}} = 0.02 - \frac{136\text{ ksi}}{16{,}000\text{ ksi}} = 0.0115 \tag{E4}$$

ANS. $\varepsilon_{\text{plastic}} = 11{,}500\ \mu$

3.1.4* Strain Energy

In the design of springs and dampers, the energy stored or dissipated is as significant as the stress and deformation. In designing automobile structures for crash worthiness, for example, we must consider how much kinetic energy is dissipated through plastic deformation. Some failure theories too, are based on energy rather than on maximum stress or strain. Minimum-energy principles are thus an important alternative to equilibrium equations and can often simplify our calculation.

The energy stored in a body due to deformation is the **strain energy,** U, and the strain energy per unit volume is the **strain energy density**, U_0:

$$U = \int_V U_0\, dV \tag{3.4}$$

where V is the volume of the body. Geometrically, U_0 is the area underneath the stress–strain curve up to the point of deformation. From Figure 3.12,

$$U_0 = \int_0^{\varepsilon} \sigma\, d\varepsilon \tag{3.5}$$

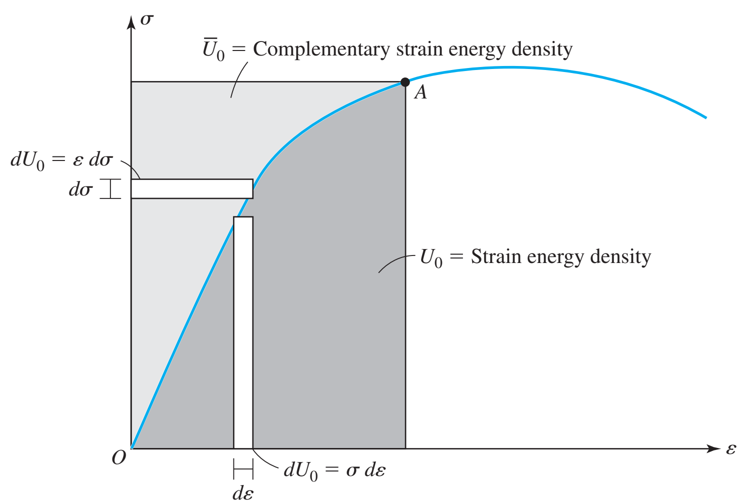


Figure 3.12 Energy densities.

The strain energy density has the same dimensions as stress since strain is dimensionless, but the units of strain energy density are different —$N \cdot m/m^3$, J/m^3, $in. \cdot lb/in.^3$, or $ft \cdot lb/ft^3$. Figure 3.12 also shows the **complementary strain energy density** $\bar{U}_0$, defined as

$$\bar{U}_0 = \int_0^{\sigma} \varepsilon \, d\sigma \tag{3.6}$$

The strain energy density at the yield point is called **modulus of resilience** (Figure 3.13a). This property is a measure of the recoverable (elastic) energy per unit volume that can be stored in a material. Since a spring is designed to operate in the elastic range, the higher the modulus of resilience, the more energy it can store.

The strain energy density at rupture is called **modulus of toughness**. This property is a measure of the energy per unit volume that can be absorbed by a material without breaking and is important in resistance to cracks and crack propagation. Whereas a *strong* material has high ultimate stress, a *tough* material has large area under the stress–strain curve, as seen in Figure 3.13*c*. It should be noted that strain energy density, complementary strain energy density, modulus of resilience, and modulus of toughness all have units of energy per unit volume.

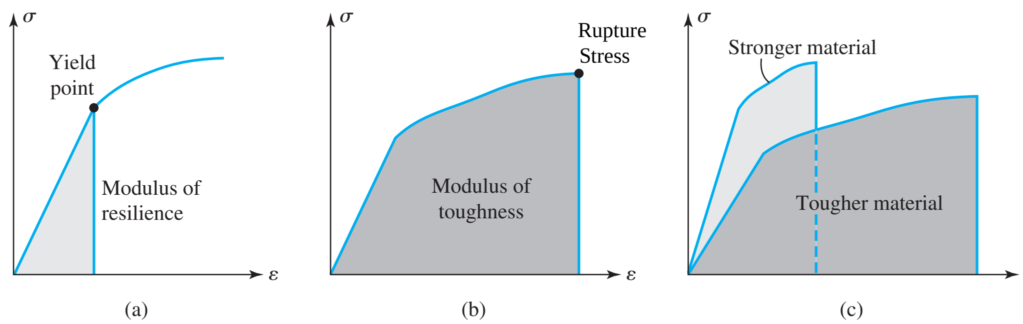


Figure 3.13 Energy-related moduli.

Linear Strain Energy Density

Most engineering structures are designed to function without permanent deformation. Thus most of the problems we will work with involve linear–elastic material. *Normal* stress and strain in the linear region are related by Hooke's law. Substituting $\sigma = E\varepsilon$ in Equation (3.5) and integrating, we obtain $U_0 = \int_0^{\varepsilon} E\varepsilon \, d\varepsilon = E\varepsilon^2/2$, which, again using Hooke's law, can be rewritten as

$$\boxed{U_0 = \frac{1}{2}\sigma\varepsilon} \tag{3.7}$$

Equation (3.7) reflects that the strain energy density is equal to the area of the triangle underneath the stress–strain curve in the linear region. Similarly, Equation (3.8) can be written using the *shear* stress–strain curve:

$$\boxed{U_0 = \frac{1}{2}\tau\gamma} \tag{3.8}$$

Strain energy, and hence strain energy density, is a scalar quantity. We can add the strain energy density due to the individual stress and strain components to obtain

$$\boxed{U_0 = \frac{1}{2}[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]} \tag{3.9}$$

EXAMPLE 3.2

For the titanium alloy in Example 3.1, determine: (a) The modulus of resilience. Use proportional limit as an approximation for yield point. (b) Strain energy density at a stress level of 136 ksi. (c) Complementary strain energy density at a stress level of 136 ksi. (d) Modulus of toughness.

PLAN

We can identify the proportional limit, the point on curve with stress of 136 ksi and the rupture point and calculate the areas under the curve to obtain the quantities of interest.

SOLUTION

Figure 3.11 is redrawn as Figure 3.14.
Point A is the proportional limit we can use to approximate the yield point in Figure 3.14. The area of the triangle OAA_1 can be calculated as shown in Equation (E1) and equated to modulus of resilience.

$$AOA_1 = \frac{128 \times 0.008}{2} = 0.512 \quad \text{(E1)}$$

ANS. The modulus of resilience is 0.512 in.·kips/in.^{3}.

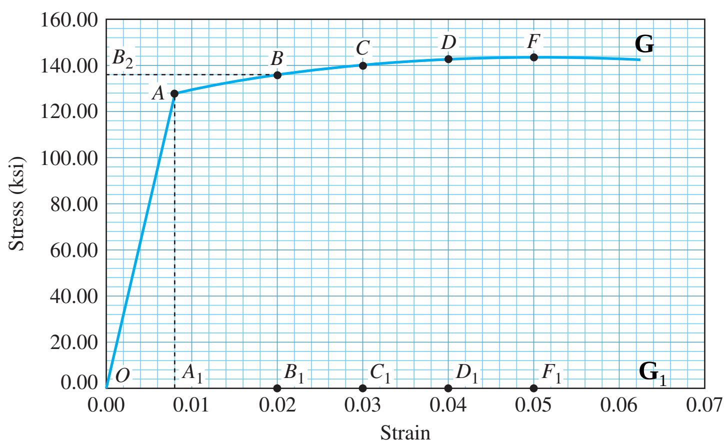


Figure 3.14 Area under curve in Example 3.2.

Point B in Figure 3.14 is at 136 ksi. The strain energy density at point B is the area AOA_1 plus the area AA_1BB_1. The area AA_1BB_1 can be approximated as the area of a trapezoid and found as

$$AA_1BB_1 = \frac{(128 + 136)\,0.012}{2} = 1.584 \quad \text{(E2)}$$

The strain energy density at B (136 ksi) is $U_B = 0.512 + 1.584$

ANS. $U_B = 2.1 \text{ in.} \cdot \text{kips/in.}^3$

The complementary strain energy density at B can be found by subtracting U_B from the area of the rectangle OB_2BB_1. Thus, $\bar{U}_B = 136 \times 0.02 - 2.1$.

ANS. $\bar{U}_B = 0.62 \text{ in.} \cdot \text{kips/in.}^3$

The rupture stress corresponds to point G on the graph. The area underneath the curve in Figure 3.14 can be calculated by approximating the curve as a series of straight lines AB, BC, CD, DF, and FG.

$$BB_1CC_1 = \frac{(136 + 140)\,0.010}{2} = 1.38 \quad \text{(E3)}$$

$$CC_1DD_1 = \frac{(140 + 142)\,0.010}{2} = 1.41 \quad \text{(E4)}$$

$$DD_1FF_1 = \frac{(142 + 144)\,0.010}{2} = 1.43 \quad \text{(E5)}$$

$$FF_1GG_1 = \frac{(144 + 142)\,0.012}{2} = 1.716 \quad \text{(E6)}$$

The total area is the sum of the areas given by Equations (E1) through (E6), or 8.032.

ANS. The modulus of toughness is 8.03 in.·kips/in.^{3}.

COMMENTS

1. Approximation of the curve by a straight line for the purpose of finding areas is the same as using the trapezoidal rule of integration.

2. In Table 3.3, there were many data points between the points shown by letters *A* through *G* in Figure 3.14. We can obtain more accurate results if we approximate the curve between two data points by a straight line. This would become tedious unless we use a spread sheet as discussed in Appendix B.1.

3.2 THE LOGIC OF THE MECHANICS OF MATERIALS

We now have all the pieces in place for constructing the logic that is used for constructing theories and obtain formulas for the simplest one-dimensional structural members, such as in this book, to linear or nonlinear structural members of plates and shells seen in graduate courses. In Chapter 1 we studied the two steps of relating stresses to internal forces and relating internal forces to external forces. In Chapter 2 we studied the relationship of strains and displacements. Finally, in Section 3.1 we studied the relationship of stresses and strains. In this section we integrate all these concepts, to show the logic of structural analysis.

Figure 3.15 shows how we relate displacements to external forces. It is possible to start at any point and move either clockwise (shown by the filled arrows ➞) or counterclockwise (shown by the hollow arrows ⇾). No one arrow directly relates displacement to external forces, because we cannot relate the two without imposing limitations and making assumptions regarding the geometry of the body, material behavior, and external loading.

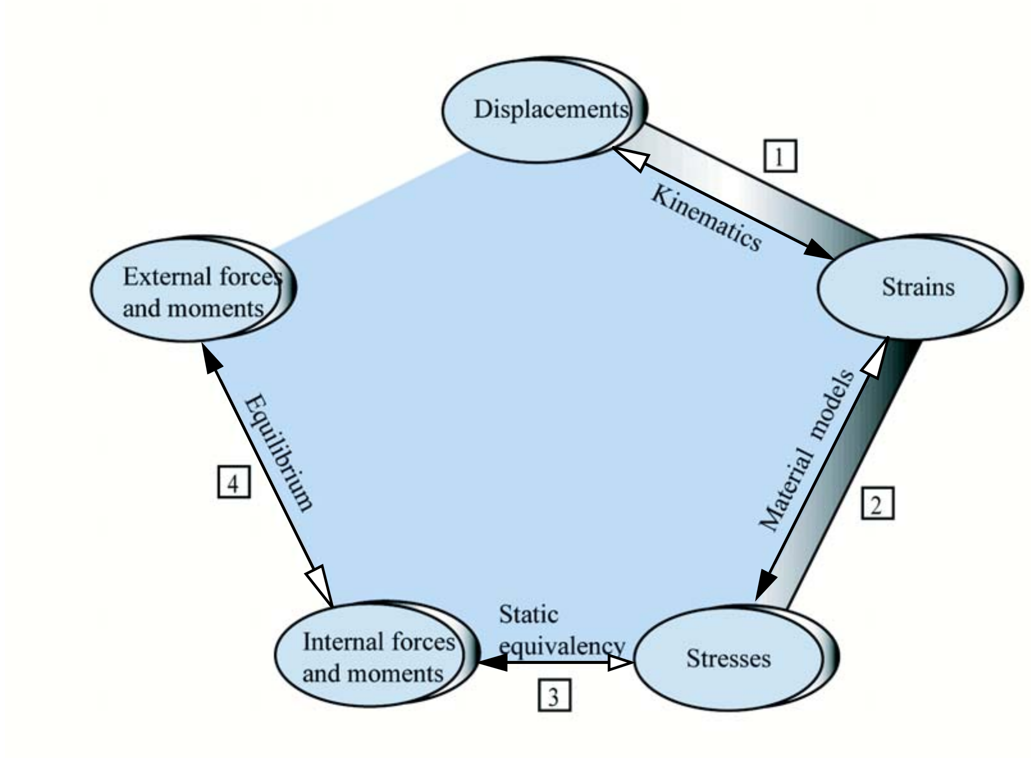


Figure 3.15 Logic in structural analysis.

The starting point in the logical progression depends on the information we have or can deduce about a particular variable. If the material model is simple, then it is possible to deduce the behavior of stresses, as we did in Chapter 1. But as the complexity in material models grows, so does the complexity of stress distributions, and deducing stress distribution becomes increasingly difficult. Unlike stresses, displacements can be measured directly or observed or deduced from geometric considerations. Later chapters will develop theories for axial rods, torsion of shafts, and bending of beams by approximating displacements and relating these displacements to external forces and moments using the logic shown in Figure 3.15.

Examples 3.3 and 3.4 demonstrate logic of problem solving shown in Figure 3.15. Its modular character permits the addition of complexities without changing the logical progression of derivation, as demonstrated by Example 3.5.

EXAMPLE 3.3

A rigid plate is attached to two 10 mm × 10 mm square bars (Figure 3.16). The bars are made of hard rubber with a shear modulus G = 1.0 MPa. The rigid plate is constrained to move horizontally due to action of the force F. If the horizontal movement of the plate is 0.5 mm, determine the force F assuming uniform shear strain in each bar.

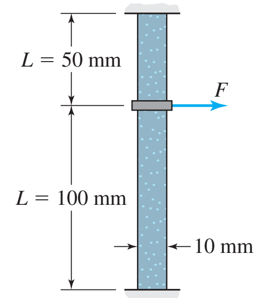


Figure 3.16 Geometry in Example 3.3.

PLAN

We can draw an approximate deformed shape and calculate the shear strain in each bar. Using Hooke's law we can find the shear stress in each bar. By multiplying the shear stress by the area we can find the equivalent internal shear force. By drawing the free-body diagram of the rigid plate we can relate the internal shear force to the external force F and determine F.

SOLUTION

1. *Strain calculation:* Figure 3.17*a* shows an approximate deformed shape. Assuming small strain we can find the shear strain in each bar:

$$\tan\gamma_{AB} \approx \gamma_{AB} = \frac{0.5\text{ mm}}{100\text{ mm}} = 5000\ \mu\text{rad} \tag{E1}$$

$$\tan\gamma_{CD} \approx \gamma_{CD} = \frac{0.5\text{ mm}}{50\text{ mm}} = 10{,}000\ \mu\text{rad} \tag{E2}$$

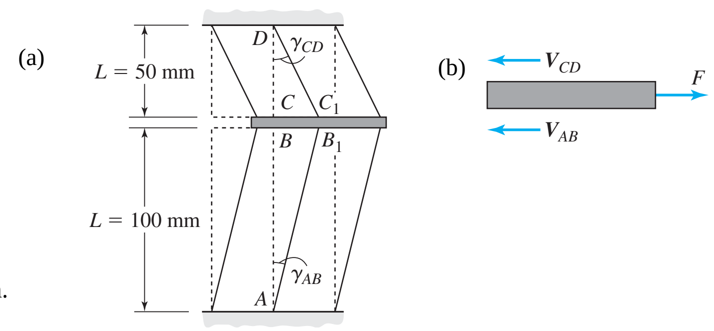


Figure 3.17 (a) Deformed geometry. (b) Free-body diagram.

2. *Stress calculation:* From Hooke's law $\tau = G\gamma$ we can find the shear stress in each bar:

$$\tau_{AB} = (10^6\text{ N/m}^2)(5000)(10^{-6}) = 5000\text{ N/m}^2 \tag{E3}$$

$$\tau_{CD} = (10^6\text{ N/m}^2)(10{,}000)(10^{-6}) = 10{,}000\text{ N/m}^2 \tag{E4}$$

3. *Internal force calculation:* The cross-sectional area of the bar is $A = 100\text{ mm}^2 = 100(10^{-6})\text{ m}^2$. Assuming uniform shear stress, we can find the shear force in each bar:

$$\boldsymbol{V}_{AB} = \tau_{AB}A = (5000\text{ N/m}^2)(100)(10^{-6})\text{ m}^2 = 0.5\text{ N} \tag{E5}$$

$$\boldsymbol{V}_{CD} = \tau_{CD}A = (10{,}000\text{ N/m}^2)(100)(10^{-6})\text{ m}^2 = 1.0\text{ N} \tag{E6}$$

4. *External force calculation:* We can make imaginary cuts on either side of the rigid plate and draw the free-body diagram as shown in Figure 3.17*b*. From equilibrium of the rigid plate we can obtain the external force F as

$$F = \boldsymbol{V}_{AB} + \boldsymbol{V}_{CD} = 1.5\text{ N} \tag{E7}$$

ANS. $F = 1.5\text{ N}$

EXAMPLE 3.4*

The steel bars (E = 200 GPa) in the truss shown in Figure 3.18 have cross-sectional area of 100 mm^2. Determine the forces F_1 and F_2 if the displacements u and v of the pins in the x and y directions, respectively, are as given below.

$$u_B = -0.500\text{ mm} \qquad v_B = -2.714\text{ mm}$$
$$u_C = -1.000\text{ mm} \qquad v_C = -6.428\text{ mm}$$
$$u_D = 1.300\text{ mm} \qquad v_D = -2.714\text{ mm}$$

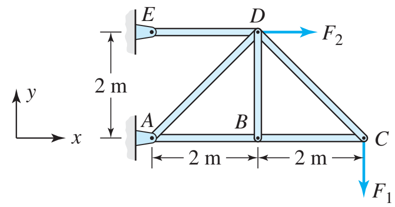


Figure 3.18 Pin displacements in Example 3.4.

PLAN

We can find strains using small-strain approximation as in Example 2.8. Following the logic in Figure 3.15 we can find stresses and then the internal force in each member. We can then draw free-body diagrams of joints C and D to find the forces F_1 and F_2.

SOLUTION

1. *Strain calculations:* The strains in the horizontal and vertical members can be found directly from the displacements,

$$\varepsilon_{AB} = \frac{u_B - u_A}{L_{AB}} = -0.250(10^{-3})\text{ m/m} \qquad \varepsilon_{BC} = \frac{u_C - u_B}{L_{BC}} = -0.250(10^{-3})\text{ m/m}$$
$$\varepsilon_{ED} = \frac{u_D - u_E}{L_{ED}} = 0.650(10^{-3})\text{ m/m} \qquad \varepsilon_{BD} = \frac{v_D - v_B}{L_{BD}} = 0 \tag{E1}$$

For the inclined member AD we first find the relative displacement vector $\bar{\mathbf{D}}_{AD}$ and then take a dot product with the unit vector $\bar{\mathbf{i}}_{AD}$, to obtain the deformation of AD as

$$\bar{\mathbf{D}}_{AD} = (u_D\bar{\mathbf{i}} + v_D\bar{\mathbf{j}}) - (u_A\bar{\mathbf{i}} + v_A\bar{\mathbf{j}}) = (1.3\bar{\mathbf{i}} - 2.714\bar{\mathbf{j}})\text{ mm}$$
$$\bar{\mathbf{i}}_{AD} = \cos 45\ \bar{\mathbf{i}} + \sin 45\ \bar{\mathbf{j}} = 0.707\bar{\mathbf{i}} + 0.707\bar{\mathbf{j}} \tag{E2}$$

$$\delta_{AD} = \bar{\mathbf{D}}_{AD}\cdot\bar{\mathbf{i}}_{AD} = (1.3\text{ mm})(0.707) + (-2.714\text{ mm})(0.707) = -1.000\text{ mm} \tag{E3}$$

The length of AD is $L_{AD} = 2.828\text{ m}$ we obtain the strain in AD as

$$\varepsilon_{AD} = \frac{\delta_{AD}}{L_{AD}} = \frac{-1.000(10^{-3})\text{ m}}{2.828\text{ m}} = -0.3535(10^{-3})\text{ m/m} \tag{E4}$$

Similarly for member CD we obtain

$$\bar{\mathbf{D}}_{CD} = (u_D\bar{\mathbf{i}} + v_D\bar{\mathbf{j}}) - (u_C\bar{\mathbf{i}} + v_C\bar{\mathbf{j}}) = (2.3\ \bar{\mathbf{i}} + 3.714\bar{\mathbf{j}})\text{ mm}$$
$$\bar{\mathbf{i}}_{CD} = -\cos 45\ \bar{\mathbf{i}} + \sin 45\ \bar{\mathbf{j}} = -0.707\bar{\mathbf{i}} + 0.707\bar{\mathbf{j}} \tag{E5}$$

$$\delta_{CD} = \bar{\mathbf{D}}_{CD}\cdot\bar{\mathbf{i}}_{CD} = (2.3\text{ mm})(-0.707) + (3.714\text{ mm})(0.707) = 1.000\text{ mm} \tag{E6}$$

The length of CD is $L_{CD} = 2.828\text{ m}$ and we obtain the strain in CD as

$$\varepsilon_{CD} = \frac{\delta_{CD}}{L_{CD}} = \frac{1.000(10^{-3})\text{ m}}{2.828\text{ m}} = 0.3535(10^{-3})\text{ m/m} \tag{E7}$$

2. *Stress calculations:* From Hooke's law $\sigma = E\varepsilon$, we can find stresses in each member:

$$\sigma_{AB} = (200\times10^9\text{N/m}^2)(-0.250\times10^{-3}) = 50\text{ MPa }(C)$$
$$\sigma_{BC} = (200\times10^9\text{N/m}^2)(-0.250\times10^{-3}) = 50\text{ MPa }(C)$$
$$\sigma_{ED} = (200\times10^9\text{N/m}^2)(0.650\times10^{-3}) = 130\text{ MPa}(T)$$
$$\sigma_{BD} = (200\times10^9\text{N/m}^2)(0.000\times10^{-3}) = 0 \tag{E8}$$
$$\sigma_{AD} = (200\times10^9\text{N/m}^2)(-0.3535\times10^{-3}) = 70.7\text{ MPa }(C)$$
$$\sigma_{CD} = (200\times10^9\text{N/m}^2)(0.3535\times10^{-3}) = 70.7\text{ MPa }(T)$$

3. *Internal force calculations:* The internal normal force can be found from $N = \sigma A$, where the cross-sectional area is $A = 100\times10^{-6}$ m^2. This yields the following internal forces:

$$N_{AB} = 5 \text{ kN (C)} \qquad N_{BC} = 5 \text{ kN (C)}$$
$$N_{ED} = 13.0 \text{ kN (T)} \qquad N_{BD} = 0 \tag{E9}$$
$$N_{AD} = 7.07 \text{ kN (C)} \qquad N_{CD} = 7.07 \text{ kN (T)}$$

4. *External forces:* We draw free-body diagrams of pins *C* and *D* as shown in Figure 3.19.

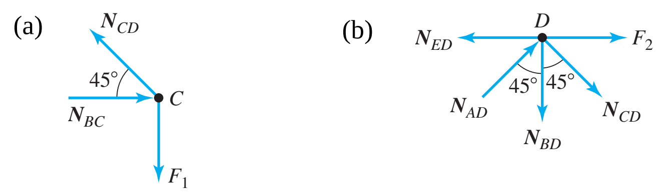


Figure 3.19 Free-body diagram of joint (a) *C* (b) *D*.

By equilibrium of forces in *y* direction in Figure 3.19*a*

$$N_{CD} \sin 45° - F_1 = 0 \tag{E10}$$

ANS. $F_1 = 5 \text{ kN}$

By equilibrium of forces in *x* direction in Figure 3.19*b*

$$F_2 + N_{CD} \sin 45° + N_{AD} \sin 45° - N_{ED} = 0 \tag{E11}$$

ANS. $F_2 = 3 \text{ kN}$

COMMENTS

1. Notice the direction of the internal forces. Forces that are pointed into the joint are compressive and the forces pointed away from the joint are tensile.
2. We used force equilibrium in only one direction to determine the external forces. We can use the equilibrium in the other direction to check our results. By equilibrium of forces in the x-direction in Figure 3.19a we obtain:

$$N_{BC} = = N_{CD} \cos 45° = 7.07 \text{ kN} \cos 45° = 5 \text{ kN}$$

which checks with the value we calculated. The forces in the *y* direction in Figure 3.19*b* must also be in equilibrium. With N_{BD} equal to zero we obtain N_{AD} should be equal to $N_{CD,}$ which checks with the values calculated.

EXAMPLE 3.5

A canoe on top of a car is tied down using rubber stretch cords, as shown in Figure 3.20*a*. The undeformed length of the stretch cord is 40 in. The initial diameter of the cord is $d = 0.5$ in. and the modulus of elasticity of the cord is $E = 510$ psi. Assume that the path of the stretch cord over the canoe can be approximated as shown in Figure 3.20*b*. Determine the approximate force exerted by the cord on the carrier of the car.

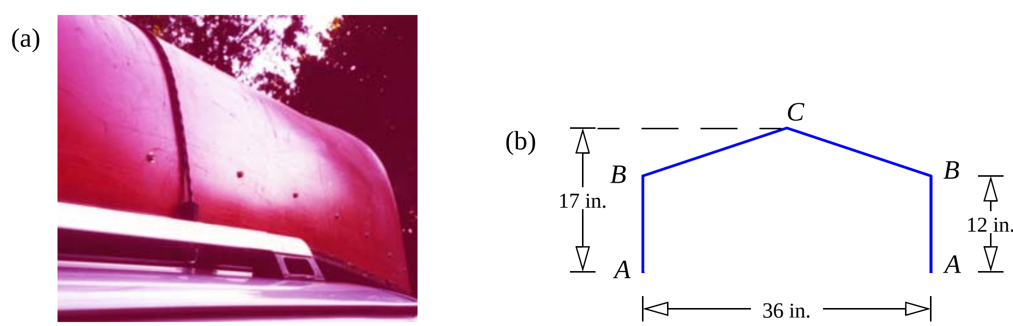


Figure 3.20 Approximation of stretch cord path on top of canoe in Example 3.5.

PLAN

We can find the stretched length L_f of the cord from geometry. Knowing L_f and $L_0 = 40$ in., we can find the average normal strain in the cord from Equation (2.1). Using the modulus of elasticity, we can find the average normal stress in the cord from Hooke's law, given by Equation (3.1). Knowing the diameter of the cord, we can find the cross-sectional area of the cord and multiply it by the normal stress to obtain the tension in the cord. If we make an imaginary cut in the cord just above *A*, we see that the tension in the cord is the force exerted on the carrier.

SOLUTION

1. *Strain calculations:* We can find the length *BC* using Figure 3.21a from the Pythagorean theorem:

$$BC = \sqrt{(5\text{ in.})^2 + (18\text{ in.})^2} = 18.68\text{ in.} \tag{E1}$$

Noting the symmetry, we can find the total length L_f of the stretched cord and the average normal strain:

$$L_f = 2(AB + BC) = 61.36\text{ in.} \tag{E2}$$

$$\varepsilon = \frac{L_f - L_0}{L_0} = \frac{61.36\text{ in.} - 40\text{ in.}}{40\text{ in.}} = 0.5341\text{ in./in.} \tag{E3}$$

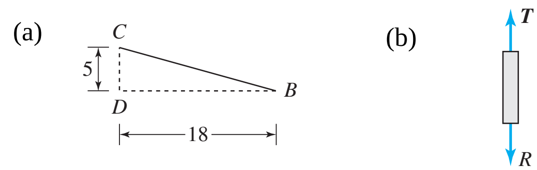


Figure 3.21 Calculations in Example 3.5 of (a) length (b) reaction force

2. *Stress calculation:* From Hooke's law we can find the stress as

$$\sigma = E\varepsilon = (510\ psi)(0.5341) = 272.38\ psi \tag{E4}$$

3. *Internal force calculations:* We can find the cross-sectional area from the given diameter $d = 0.5$ in. and multiply it with the stress to obtain the internal tension,

$$A = \frac{\pi d^2}{4} = \frac{\pi(0.5\text{ in.})^2}{4} = 0.1963\text{ in.}^2 \tag{E5}$$

$$\boldsymbol{T} = \sigma A = (0.1963\text{ in.}^2)(272.38\ psi) = 53.5\ lb \tag{E6}$$

4. *Reaction force calculation:* We can make a cut just above A and draw the free-body diagram as shown in Figure 3.21*b* to calculate the force R exerted on the carrier,

$$R = \boldsymbol{T} \tag{E7}$$

ANS. $\boldsymbol{T} = 53.5$ lb

COMMENTS

1. Unlike in the previous two examples, where relatively accurate solutions would be obtained, in this example we have large strains and several other approximations, as elaborated in the next comment. The only thing we can say with some confidence is that the answer has the right order of magnitude.

2. The following approximations were made in this example:

(a) The path of the cord should have been an inclined straight line between the carrier rail and the point of contact on the canoe, and then the path should have been the contour of the canoe.

(b) The strain along the cord is nonuniform, which we approximated by a uniform average strain.

(c) The stress–strain curve of the rubber cord is nonlinear. Thus as the strain changes along the length, so does the modulus of elasticity E, and we need to account for this variation of E in the calculation of stress.

(d) The cross-sectional area for rubber will change significantly with strain and must be accounted for in the calculation of the internal tension.

3. Depending on the need of our accuracy, we can include additional complexities to address the error from the preceding approximations.

(a) Suppose we did a better approximation of the path as described in part (2a) but made no other changes. In such a case the only change would be in the calculation of L_f in Equation (E2) (see Problem 2.87), but the rest of the equations would remain the same.

(b) Suppose we make marks on the cord every 2 in. before we stretch it over the canoe. We can then measure the distance between two consecutive marks when the cord is stretched. Now we have L_f for each segment and can repeat the calculation for each segment (see Problem 3.68).

(c) Suppose, in addition to the above two changes, we have the stress–strain curve of the stretch cord material. Now we can use the tangent modulus in Hooke's law for each segment, and hence we can get more accurate stresses in each segment. We can then calculate the internal force as before (see Problem 3.69).

(d) Rubber has a Poisson's ratio of 0.5. Knowing the longitudinal strain from Equation (E3) for each segment, we can compute the transverse strain in each segment and find the diameter of the cord in the stretched position in each segment. This will give us a more accurate area of cross section, and hence a more accurate value of internal tension in the cord (see Problem 3.70).

4. These comments demonstrate how complexities can be added one at a time to improve the accuracy of a solution. In a similar manner, we shall derive theories for axial members, shafts, and beams in Chapters 4 through 6, to which complexities can be added as asked of you in "Stretch yourself" problems. Which complexity to include depends on the individual case and our need for accuracy.

Consolidate your knowledge

1. In your own words, describe the tension test and the quantities that can be calculated from the experiment.

QUICK TEST 3.1

Time: 15 minutes/Total: 20 points

Grade yourself using the answers given in Appendix E. Each question is worth two points.

1. What are the typical units of modulus of elasticity and Poisson's ratio in the metric system?
2. Define offset yield stress.
3. What is strain hardening?
4. What is necking?
5. What is the difference between proportional limit and yield point?
6. What is the difference between a brittle material and a ductile material?
7. What is the difference between linear material behavior and elastic material behavior?
8. What is the difference between strain energy and strain energy density?
9. What is the difference between modulus of resilience and modulus of toughness?
10. What is the difference between a strong material and a tough material?

3.3 FAILURE AND FACTOR OF SAFETY

There are many types of failures. The breaking of the ship *S.S. Schenectady* (Chapter 1) was a failure of strength, whereas the failure of the O-ring joints in the shuttle *Challenger* (Chapter 2) was due to excessive deformation. **Failure** implies that a component or a structure does not perform the function for which it was designed.

A machine component may interfere with other moving parts because of excessive deformation; a chair may feel rickety because of poor joint design; a gasket seal leaks because of insufficient deformation of the gasket at some points; lock washers may not deform enough to provide the spring force needed to keep bolted joints from becoming loose; a building undergoing excessive deformation may become aesthetically displeasing. These are examples of failure caused by too little or too much deformation.

The stiffness of a structural element depends on the modulus of elasticity of the material as well as on the geometric properties of the member, such as cross-sectional area, area moments of inertia, polar moments of inertia, and the length of the components. The use of carpenter's glue in the joints of a chair to prevent a rickety feeling is a simple example of increasing joint and structure stiffness by using adhesives.

Prevention of a component fracture is an obvious design objective based on strength. At other times, our design objective may be avoid to making a component *too* strong. The adhesive bond between the lid and a sauce bottle must break so that the bottle may be opened by hand; shear pins must break before critical components get damaged; the steering column of an automobile must collapse rather than impale the passenger in a crash. Ultimate normal stress is used for assessing failure due to breaking or rupture particularly for brittle materials.

Permanent deformation rather than rupture is another stress-based failure. Dents or stress lines in the body of an automobile; locking up of bolts and screws because of permanent deformation of threads; slackening of tension wires holding a structure in place—in each of these examples, plastic deformation is the cause of failure. Yield stress is used for assessing failure due to plastic deformation, particularly for ductile materials.

A support in a bridge may fail, but the bridge can still carry traffic. In other words, the failure of a component does not imply failure of the entire structure. Thus the strength of a structure, or the deflection of the entire structure, may depend on a large number of variables. In such cases loads on the structure are used to characterize failure. Failure loads may be based on the stiffness, the strength, or both.

A margin of safety must be built into any design to account for uncertainties or a lack of knowledge, lack of control over the environment, and the simplifying assumptions made to obtain results. The measure of this margin of safety is the factor of safety K_{safety} defined as

$$\boxed{K_{\text{safety}} = \frac{\text{failure-producing value}}{\text{computed (allowable) value}}} \tag{3.10}$$

Equation (3.10) implies that the factor of safety must always be *greater than 1*. The numerator could be the failure deflection, failure stress, or failure load and is assumed known. In analysis, the denominator is determined, and from it the factor of safety is found. In design, the factor of safety is specified, and the variables affecting the denominator are determined such that the denominator value is not exceeded. Thus in design the denominator is often referred to as the allowable value.

Several issues must be considered in determining the appropriate factor of safety in design. No single issue dictates the choice. The value chosen is a compromise among various issues and is arrived at from experience.

Material or operating costs are the primary reason for using a low factor of safety, whereas liability cost considerations push for a greater factor of safety. A large fixed cost could be due to expensive material, or due to large quantity of material used to meet a given factor of safety. Greater weight may result in higher fuel costs. In the aerospace industries the operating costs supersede material costs. Material costs dominate the furniture industry. The automobile industry seeks a compromise between fixed and running costs. Though liability is a consideration in all design, the building industry is most conscious of it in determining the factor of safety.

Lack of control or lack of knowledge of the operating environment also push for higher factors of safety. Uncertainties in predicting earthquakes, cyclones, or tornadoes, for examples, require higher safety factors for the design of buildings located in regions prone to these natural calamities. A large scatter in material properties, as usually seen with newer materials, is another uncertainty pushing for higher factor of safety.

Human safety considerations not only push the factor of safety higher but often result in government regulations of the factors of safety, as in building codes.

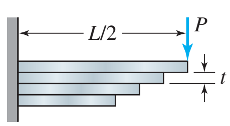
This list of issues affecting the factor of safety is by no means complete, but is an indication of the subjectivity that goes into the choice of the factor of safety. The factors of safety that may be recommended for most applications range from 1.1 to 6.

EXAMPLE 3.6

In the leaf spring design in Figure 3.22 the formulas for the maximum stress σ and deflection δ given in Equation (3.11) are derived from theory of bending of beams (see Example 7.4):

$$\sigma = \frac{3PL}{nbt^2} \qquad \delta = \frac{3PL^3}{4Enbt^3} \tag{3.11}$$

where P is the load supported by the spring, L is the length of the spring, n is the number of leaves b is the width of each leaf, t is the thickness of each leaf, and E is the modulus of elasticity. A spring has the following data: $L = 20$ in., $b = 2$ in., $t = 0.25$ in., and $E = 30{,}000$ ksi. The failure stress is $\sigma_{failure} = 120$ ksi, and the failure deflection is $\delta_{failure} = 0.5$ in. The spring is estimated to carry a maximum force P = 250 lb and is to have a factor of safety of $K_{safety} = 4$. (a) Determine the minimum number of leaves. (b) For the answer in part (a) what is the real factor of safety?



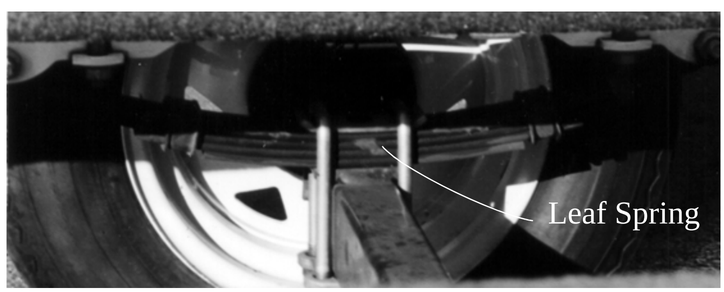


Figure 3.22 Leaf spring in Example 3.6.

PLAN

(a) The allowable stress and allowable deflection can be found from Equation (3.10) using the factor of safety of 4. Equation (3.11) can be used with two values of n to ensure that the allowable values of stress and deflection are not exceeded. The higher of the two values of n is the minimum number of leaves in the spring design. (b) Substituting n in Equation (3.11), we can compute the maximum stress and deflection and obtain the two factors of safety from Equation (3.10). The lower value is the real factor of safety.

SOLUTION

(a) The allowable values for stress and deflection can be found from Equation (3.10) as:

$$\sigma_{allow} = \frac{\sigma_{failure}}{K_{safety}} = \frac{120 \text{ ksi}}{4} = 30 \text{ ksi} \tag{E1}$$

$$\delta_{allow} = \frac{\delta_{failure}}{K_{safety}} = \frac{0.5 \text{ in.}}{4} = 0.125 \text{ in.} \tag{E2}$$

Substituting the given values of the variables in the stress formula in Equation (3.11), we obtain the maximum stress, which should be less than the allowable stress. From this we can obtain one limitation on n:

$$\sigma = \frac{3PL}{nbt^2} = \frac{3(250 \text{ lb})(20 \text{ in.})}{n(2 \text{ in.})(0.25 \text{ in.})^2} = \frac{120(10^3) \text{ psi}}{n} \le 30(10^3) \text{ psi} \qquad \text{or} \tag{E3}$$

$$n \ge 4 \tag{E4}$$

Substituting the given values in the deflection formula, in Equation (3.11), we obtain the maximum deflection, which should be less than the allowable stress we thus obtain one limitation on n:

$$\delta = \frac{3PL^3}{4Enbt^3} = \frac{3(250 \text{ lb})(20 \text{ in.})^3}{4(30 \times 10^6 \text{ psi})(n)(2 \text{ in.})(0.25 \text{ in.})^3} = \frac{1.6 \text{ in.}}{n} \le 0.125 \text{ in.} \qquad \text{or} \tag{E5}$$

$$n \ge 12.8 \tag{E6}$$

The minimum number of leaves that will satisfy Equations (E4) and (E6) is our answer.

ANS. $n = 13$

(b) Substituting $n = 13$ in Equations (E3) and (E5) we find the computed values of stress and deflection and the factors of safety from Equation (3.10).

$$\sigma_{comp} = \frac{120(10^3) \text{ psi}}{13 \text{ psi}} = 9.23(10^3) \text{ psi} \qquad K_\sigma = \frac{\sigma_{failure}}{\sigma_{comp}} = \frac{120(10^3) \text{ psi}}{9.23(10^3) \text{ psi}} = 13 \tag{E7}$$

$$\delta_{comp} = \frac{1.6 \text{ in.}}{13} = 0.1232 \text{ in} \qquad K_\delta = \frac{\delta_{failure}}{\delta_{comp}} = \frac{0.5 \text{ in.}}{0.1232 \text{ in.}} = 4.06 \tag{E8}$$

The factor of safety for the system is governed by the lowest factor of safety, which in our case is given by Equation (E8).

ANS. $K_\delta = 4.06$

COMMENTS

1. This problem demonstrates the difference between the allowable values, which are used in design decisions based on a specified factor of safety, and computed values, which are used in analysis for finding the factor of safety.
2. For purposes of design, formulas are initially obtained based on simplified models, as shown in Figure 3.22. Once the preliminary relationship between variables has been established, then complexities are often incorporated by using factors determined experimentally. Thus the deflection of the spring, accounting for curvature, end support, variation of thickness, and so on is given by $\delta = K(3PL^3/4Enbt^3)$, where K is determined experimentally as function of the complexities not accounted for in the simplified model. This comment highlights how the mechanics of materials provides a guide to developing formulas for complex realities.

PROBLEM SET 3.1

Stress–strain curves

***3.1 -3.5** A tensile test specimen having a diameter of 10 mm and a gage length of 50 mm was tested to fracture. The stress–strain curve from the tension test is shown in Figure P3.3. The lower plot is the expanded region OAB and associated with the strain values given on the lower scale. Solve Problems 3.1 through 3.5.*

3.1 Determine (a) the ultimate stress; (b) the fracture stress; (c) the modulus of elasticity; (d) the proportional limit; (e) the offset yield stress at 0.2%; (f) the tangent modulus at stress level of 420 MPa; (g) the secant modulus at stress level of 420 MPa.

3.2 Determine the axial force acting on the specimen when it is extended by (a) 0.2 mm; (b) 4.0 mm.

3.3 Determine the extension of the specimen when the axial force on the specimen is 33 kN.

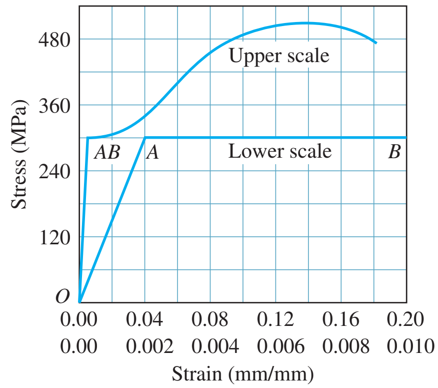


Figure P3.3

3.4 Determine the total strain, the elastic strain, and the plastic strain when the axial force on the specimen is 33 kN.

3.5 After the axial load was removed, the specimen was observed to have a length of 54 mm. What was the maximum axial load applied to the specimen?

3.6—3.10 A tensile test specimen having a diameter of $\frac{5}{8}$ in. and a gage length of 2 in. was tested to fracture. The stress–strain curve from the tension test is shown in Figure P3.6. The lower plot is the expanded region OAB and associated with the strain values given on the lower scale. Solve Problems 3.6 through 3.10 using this graph.

3.6 Determine (a) the ultimate stress; (b) the fracture stress; (c) the modulus of elasticity; (d) the proportional limit.(e)the offset yield stress at 0.1%; (f) the tangent modulus at the stress level of 72 kips; (g) the secant modulus at the stress level of 72 kips.

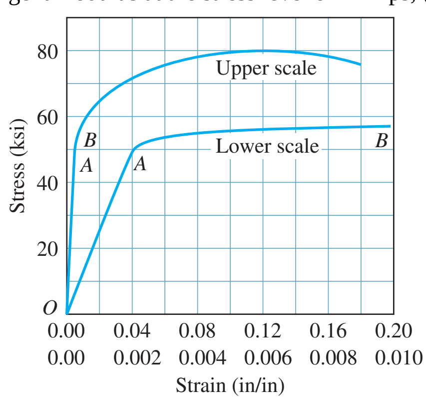


Figure P3.6

3.7 Determine the axial force acting on the specimen when it is extended by (a) 0.006 in.; (b) 0.120 in.

3.8 Determine the extension of the specimen when the axial force on the specimen is (a) 10 kips. (b) 20 kips.

3.9 Determine the total strain, the elastic strain, and the plastic strain when the axial force on the specimen is 20 kips.

3.10 After the axial load was removed, the specimen was observed to have a length of 2.12 in. What was the maximum axial load applied to the specimen?

3.11 A typical stress-strain graph for cortical bone is shown in Figure P3.11. Determine (a) the modulus of elasticity; (b) the proportional limit; (c) the yield stress at 0.15% offset; (d) the secant modulus at stress level of 130 MPa; (d) the tangent modulus at stress level of 130 MPa; (e) the permanent strain at stress level of 130MPa. (f) If the shear modulus of the bone is 6.6 GPa, determine Poisson's ratio assuming the bone is isotropic. (g) Assuming the bone specimen was 200 mm long and had a material cross-sectional area of 250 mm^2, what is the elongation of the bone when a 20-kN force is applied?

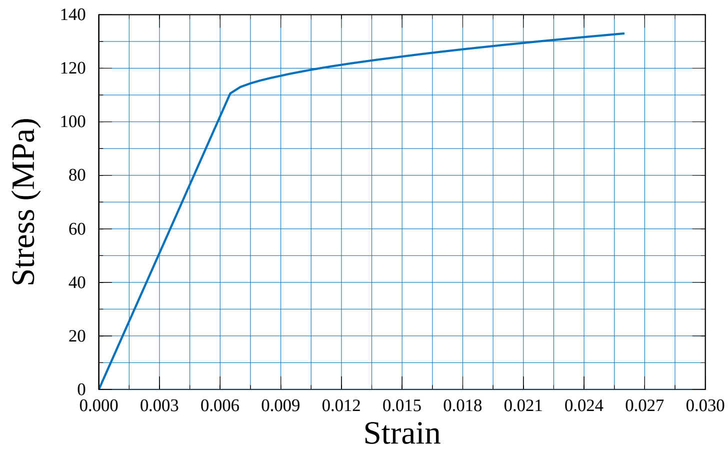


Figure P3.11

3.12 A 12 mm × 12 mm square metal alloy having a gage length of 50 mm was tested in tension. The results are given in Table P3.12. Draw the stress–strain curve and calculate the following quantities. (a) the modulus of elasticity. (b) the proportional limit. (c) the yield stress at 0.2% offset. (d) the tangent modulus at a stress level of 1400 MPa. (e) the secant modulus at a stress level of 1400 MPa. (f) the plastic strain at a stress level of 1400 MPa. (Use of a spreadsheet is recommended.)

TABLE P3.12

Load (kN)	Change in Length (mm)	Load (kN)	Change in Length (mm)
0.00	0.00	200.01	5.80
17.32	0.02	204.65	7.15
60.62	0.07	209.99	8.88
112.58	0.13	212.06	9.99
147.22	0.17	212.17	11.01
161.18	0.53	208.64	11.63
168.27	1.10	204.99	12.03
176.03	1.96	199.34	12.31
182.80	2.79	192.15	12.47
190.75	4.00	185.46	12.63
193.29	4.71	Break	

3.13 A mild steel specimen of 0.5 in. diameter and a gage length of 2 in. was tested in tension. The test results are reported Table P3.13. Draw the stress–strain curve and calculate the following quantities: (a) the modulus of elasticity; (b) the proportional limit; (c) the yield stress at 0.05% offset; (d) the tangent modulus at a stress level of 50 ksi; (e) the secant modulus at a stress level of 50 ksi; (f) the plastic strain at a stress level of 50 ksi. (Use of a spreadsheet is recommended.)

TABLE P3.13

Load (10^3 lb)	Change in Length (10^{-3} in.)	Load (10^3 lb)	Change in Length (10^{-3} in.)
0.00	0.00	11.18	112.10
3.11	1.28	11.72	140.40
7.24	2.96	11.99	161.21
7.50	3.06	12.27	192.65
7.70	8.76	12.41	214.22
7.90	19.05	12.55	245.93
8.16	28.70	12.70	283.47
8.46	37.73	12.77	316.36
8.82	47.18	12.84	363.10
9.32	59.06	12.04	385.34
9.86	70.85	11.44	396.03
10.40	84.23	10.71	406.42
10.82	97.85	9.96	414.72
		Break	

3.14 A rigid bar AB of negligible weight is supported by cable of diameter 1/4 in, as shown in Figure P3.14. The cable is made from a material that has a stress- strain curve shown in Figure P3.6. (a) Determine the extension of the cable when P = 2 kips. (b) What is the permanent deformation in BC when the load P is removed?

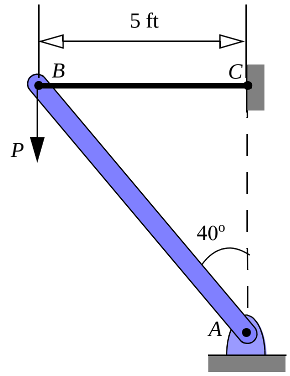


Figure P3.14

3.15 A rigid bar AB of negligible weight is supported by cable of diameter 1/4 in., as shown in Figure P3.14. The cable is made from a material that has a stress-strain curve shown in Figure P3.6. (a) Determine the extension of the cable when P =4.25 kips. (b) What is the permanent deformation in the cable when the load P is removed?

Material constants

3.16 A rectangular bar has a cross-sectional area of 2 in.^{2} and an undeformed length of 5 in., as shown in Figure 3.18. When a load P = 50,000 lb is applied, the bar deforms to a position shown by the colored shape. Determine the modulus of elasticity and the Poisson's ratio of the material.

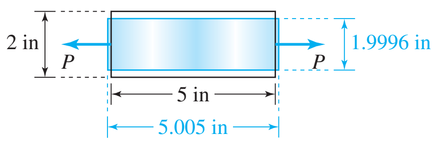


Figure 3.23

3.17 A force $P = 20$ kips is applied to a rigid plate that is attached to a square bar, as shown in Figure P3.24. If the plate moves a distance of 0.005 in., determine the modulus of elasticity.

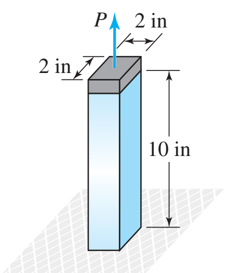


Figure 3.24

3.18 A force $P = 20$ kips is applied to a rigid plate that is attached to a square bar, as shown in Figure P3.25. If the plate moves a distance of 0.0125 in, determine the shear modulus of elasticity. Assume line AB remains straight.

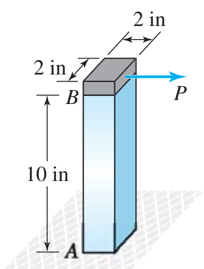


Figure 3.25

3.19 Two rubber blocks of length L and cross section dimension a x b are bonded to rigid plates as shown in Figure P3.19. Point A was observed to move downwards by 0.02 in. when the weight $W = 900$ lb was hung from the middle plate. Determine the shear modulus of elasticity using small strain approximation. Use L= 12 in., a = 3 in., and b = 2 in.

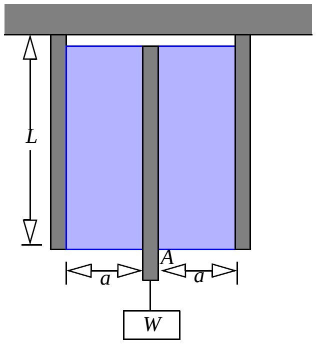


Figure P3.19

3.20 Two rubber blocks with a shear modulus of 1.0 MPa and length L and of cross section dimensions a x b are bonded to rigid plates as shown in Figure P3.19. Using the small-strain approximation, determine the displacement of point A, if a weight of 500 N is hung from the middle plate. Use L = 200 mm, a = 45 mm, and b = 60 mm.

3.21 Two rubber blocks with a shear modulus of 750 psi and length L and cross section dimension a x b are bonded to rigid plates as shown in Figure P3.19. If the allowable shear stress in the rubber is 15 psi, and allowable deflection is 0.03 in., determine the maximum weight W that can be hung from the middle plate using small strain approximation. Use L = 12 in., a = 2 in., and b = 3 in.

3.22 Two rubber blocks with a shear modulus of G, length L and cross section of dimensions a x b are bonded to rigid plates as shown in Figure P3.19. Obtain the shear stress in the rubber block and the displacement of point A in terms of G, L, W, a, and b.

3.23 A circular bar of 200-mm length and 20-mm diameter is subjected to a tension test. Due to an axial force of 77 kN, the bar is seen to elongate by 4.5 mm and the diameter is seen to reduce by 0.162 mm. Determine the modulus of elasticity and the shear modulus of elasticity.

3.24 A circular bar of 6-in. length and 1-in. diameter is made from a material with a modulus of elasticity E = 30,000 ksi and a Poisson's ratio $\nu = \frac{1}{3}$. Determine the change in length and diameter of the bar when a force of 20 kips is applied to the bar.

3.25 A circular bar of 400 mm length and 20 mm diameter is made from a material with a modulus of elasticity E = 180 GPa and a Poisson's ratio $\nu = 0.32$. Due to a force the bar is seen to elongate by 0.5 mm. Determine the change in diameter and the applied force.

3.26 A 25 mm × 25 mm square bar is 500 mm long and is made from a material that has a Poisson's ratio of $\frac{1}{3}$. In a tension test, the bar is seen to elongate by 0.75 mm. Determine the percentage change in volume of the bar.

3.27 A circular bar of 50 in. length and 1 in. diameter is made from a material with a modulus of elasticity E = 28,000 ksi and a Poisson's ratio $\nu = 0.32$. Determine the percentage change in volume of the bar when an axial force of 20 kips is applied.

3.28 An aluminum rectangular bar has a cross section of 25 mm × 50 mm and a length of 500 mm. The modulus of elasticity E = 70 GPa and the Poisson's ratio $\nu = 0.25$. Determine the percentage change in the volume of the bar when an axial force of 300 kN is applied.

3.29 A circular bar of length L and diameter d is made from a material with a modulus of elasticity E and a Poisson's ratio ν. Assuming small strain, show that the percentage change in the volume of the bar when an axial force P is applied and given as $400P(1-2\nu)/(E\pi d^2)$. Note the percentage change is zero when $\nu = 0.5$.

3.30 A rectangular bar has a cross-sectional dimensions $a \times b$ and a length L. The bar material has a modulus of elasticity E and a Poisson's ratio ν. Assuming small strain, show that the percentage change in the volume of the bar when an axial force P is applied given by $100P(1 - 2\nu)/Eab$. Note the percentage change is zero when $\nu = 0.5$.

Strain energy

3.31 What is the strain energy in the bar of Problem 3.16.?

3.32 What is the strain energy in the bar of Problem 3.17?

3.33 What is the strain energy in the bar of Problem 3.18?

3.34 A circular bar of length L and diameter of d is made from a material with a modulus of elasticity E and a Poisson's ratio ν. In terms of the given variables, what is the linear strain energy in the bar when axial load P is applied to the bar?

3.35 A rectangular bar has a cross-sectional dimensions $a \times b$ and a length L. The bar material has a modulus of elasticity E and a Poisson's ratio ν. In terms of the given variables, what is the linear strain energy in the bar when axial load P is applied to the bar?

3.36 For the material having the stress–strain curve shown in Figure P3.3, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 420 MPa; (c) the complementary strain energy density at a stress level of 420 MPa; (d) the modulus of toughness.

3.37 For the material having the stress–strain curve shown in Figure P3.6, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 72 ksi; (c) the complementary strain energy density at a stress level of 72 ksi; (d) the modulus of toughness.

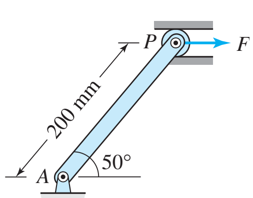
3.38 For the metal alloy given in Problem 3.12, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 1400 MPa; (c) the complementary strain energy density at a stress level of 1400 MPa; (d) the modulus of toughness.

3.39 For the mild steel given in Problem 3.13, determine (a) the modulus of resilience (using the proportional limit to approximate the yield point); (b) the strain energy density at a stress level of 50 ksi; (c) the complementary strain energy density at a stress level of 50 ksi; (d) the modulus of toughness.

Logic in mechanics

3.40 The roller at P slides in the slot by an amount $\delta_P = 0.25$ mm due to the force F, as shown in Figure P3.40. Member AP has a cross-sectional area $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Determine the force applied F.

Figure P3.40



3.41 The roller at P slides in the slot by an amount $\delta_P = 0.25$ mm due to the force F, as shown in Figure P3.41. Member AP has a cross-sectional area $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Determine the applied force F.

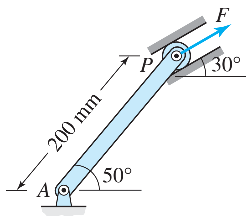


Figure P3.41

3.42 A roller slides in a slot by the amount $\delta_P = 0.01$ in. in the direction of the force F, as shown in Figure P3.42. Each bar has a cross-sectional area $A = 0.2$ in.^{2} and a modulus of elasticity $E = 30{,}000$ ksi. Bars AP and BP have lengths $L_{AP} = 8$ in. and $L_{BP} = 10$ in., respectively. Determine the applied force F.

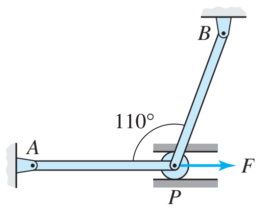


Figure P3.42

3.43 A roller slides in a slot by the amount $\delta_P = 0.25$ mm in the direction of the force F, as shown in Figure P3.43. Each bar has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars AP and BP have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the applied force F.

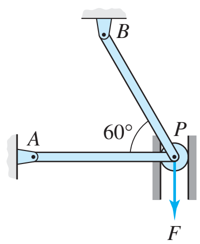


Figure P3.43

3.44 A roller slides in a slot by the amount $\delta_P = 0.25$ mm in the direction of the force F as shown in Figure P3.44. Each bar has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Bars AP and BP have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm, respectively. Determine the applied force F.

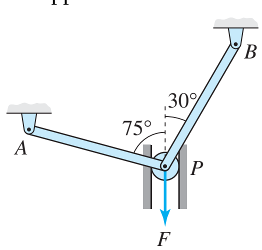
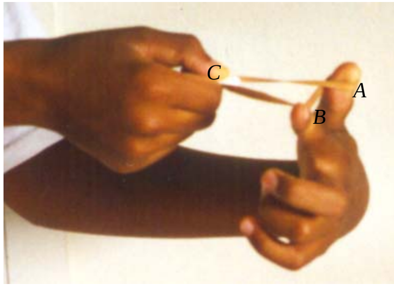


Figure P3.44

3.45 A little boy shoots paper darts at his friends using a rubber band that has an unstretched length of 7 in. The piece of rubber band between points A and B is pulled to form the two sides AC and CB of a triangle, as shown in Figure P3.45. Assume the same normal strain in AC and CB, and the rubber band around the thumb and forefinger is a total of 1 in. The cross-sectional area of the band is $\frac{1}{128}$ in.^{2}, and the rubber has a modulus of elasticity $E = 150$ psi. Determine the approximate force F and the angle θ at which the paper dart leaves the boy's hand.



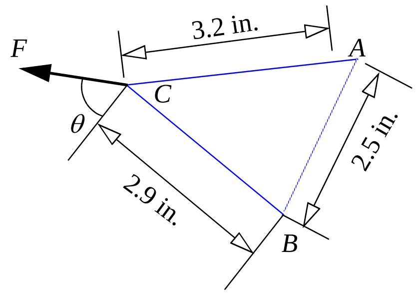


Figure P3.45

3.46 Three poles are pin connected to a ring at P and to the supports on the ground. The coordinates of the four points are given in Figure P3.46. All poles have cross-sectional areas $A = 1$ in.^{2} and a modulus of elasticity $E = 10{,}000$ ksi. If under the action of force F the ring at P moves vertically by the distance $\delta_P = 2$ in., determine the force F.

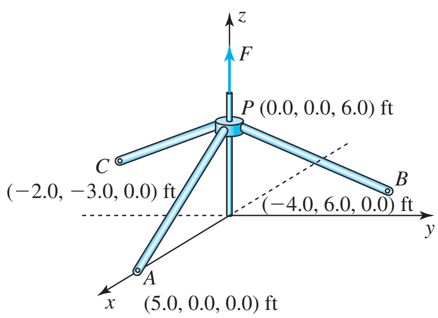


Figure P3.46

3.47 A gap of 0.004 in. exists between a rigid bar and bar A before a force F is applied (Figure P3.47). The rigid bar is hinged at point C. Due to force F the strain in bar A was found to be –500 μin/in. The lengths of bars A and B are 30 in. and 50 in., respectively. Both bars have cross-sectional areas $A = 1$ in.^{2} and a modulus of elasticity $E = 30{,}000$ ksi. Determine the applied force F.

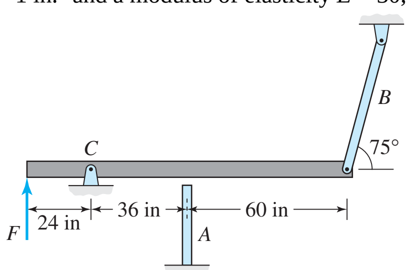


Figure P3.47

3.48 The cable between two poles shown in Figure P3.48 is taut before the two traffic lights are hung on it. The lights are placed symmetrically at 1/3 the distance between the poles. The cable has a diameter of 1/16 in. and a modulus of elasticity of 28,000 ksi. Determine the weight of the traffic lights if the cable sags as shown. Use the entire cable to calculate the average values.

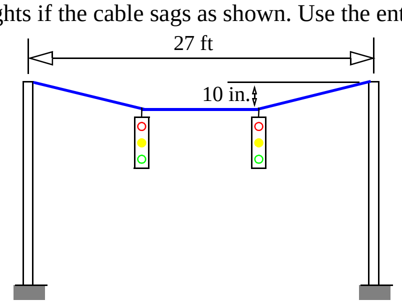


Figure P3.48

3.49 A steel bolt (E_s= 200 GPa) of 25 mm diameter passes through an aluminum (E_{al} = 70 GPa) sleeve of thickness 4 mm and outside diameter of 48 mm as shown in Figure P3.49. Due to the tightening of the nut the rigid washers move towards each other by 0.75 mm. (a) Determine the average normal stress in the sleeve and the bolt. (b) What is the extension of the bolt?

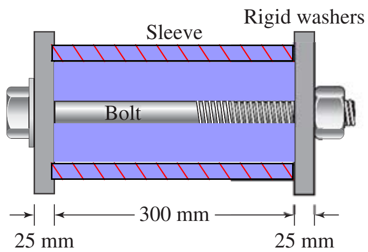


Figure P3.49

3.50 The pins in the truss shown in Figure P3.50 are displaced by u and v in the x and y directions, respectively, as shown in Table P3.50. All rods in the truss have cross-sectional areas $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Determine the external forces P_1 and P_2 in the truss.

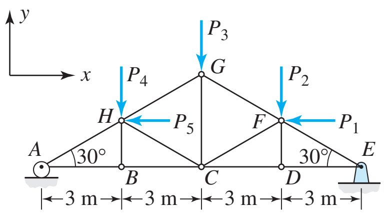


Figure P3.50

TABLE P3.50

u_A = –4.6765 mm	v_A = 0
u_B = –3.3775 mm	v_B = –8.8793 mm
u_C = –2.0785 mm	v_C = –9.7657 mm
u_D = –1.0392 mm	v_D = –8.4118 mm
u_E = 0.0000 mm	v_E = 0.0000 mm
u_F = –3.2600 mm	v_F = –8.4118 mm
u_G = –2.5382 mm	v_G = –9.2461 mm
u_H = –1.5500 mm	v_H = –8.8793 mm

3.51 The pins in the truss shown in Figure P3.50 are displaced by u and v in the x and y directions, respectively, as shown in Table P3.50. All rods in the truss have cross-sectional areas $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Determine the external force P_3 in the truss.

3.52 The pins in the truss shown in Figure P3.50 are displaced by u and v in the x and y directions, respectively, as shown in Table P3.50. All rods in the truss have cross-sectional areas $A = 100\ \text{mm}^2$ and a modulus of elasticity $E = 200$ GPa. Determine the external forces P_4 and P_5 in the truss.

Factor of safety

3.53 A joint in a wooden structure shown in Figure P3.53 is to be designed for a factor of safety of 3. If the average failure stress in shear on the surface *BCD* is 1.5 ksi and the average failure bearing stress on the surface *BEF* is 6 ksi, determine the smallest dimensions h and d to the nearest $\frac{1}{16}$ in.

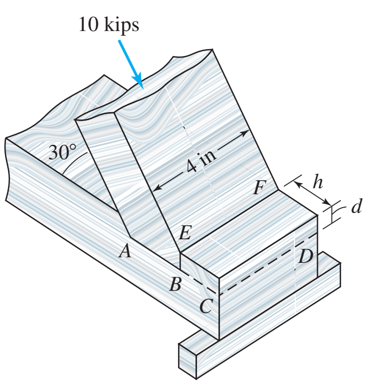


Figure P3.53

3.54 A 125 kg light is hanging from a ceiling by a chain as shown in Figure P3.54. The links of the chain are loops made from a thick wire. Determine the minimum diameter of the wire to the nearest millimeter for a factor of safety of 3. The normal failure stress for the wire is 180 MPa.

Figure P3.54

3.55 A light is hanging from a ceiling by a chain as shown in Figure P3.54. The links of the chain are loops made from a thick wire with a diameter of $\frac{1}{8}$ in. The normal failure stress for the wire is 25 ksi. For a factor of safety of 4, determine the maximum weight of the light to the nearest pound.

3.56 Determine the maximum weight *W* that can be suspended using cables, as shown in Figure P3.56, for a factor of safety of 1.2. The cable's fracture stress is 200 MPa, and its diameter is 10 mm.

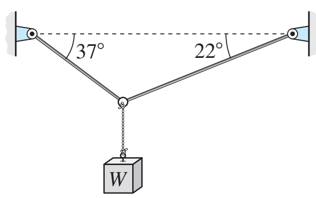


Figure P3.56

3.57 The cable in Figure P3.56 has a fracture stress of 30 ksi and is used for suspending the weight W = 2500 lb. For a factor of safety of 1.25, determine the minimum diameter of the cables to the nearest $\frac{1}{16}$ in. that can be used.

3.58 An adhesively bonded joint in wood is fabricated as shown in Figure P3.58. For a factor of safety of 1.25, determine the minimum overlap length *L* and dimension *h* to the nearest $\frac{1}{8}$ in. The shear strength of the adhesive is 400 psi and the wood strength is 6 ksi in tension.

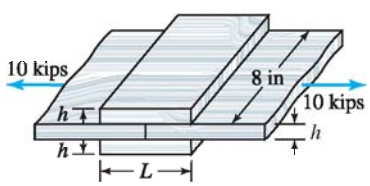


Figure P3.58

3.59 A joint in a truss has the configuration shown in Figure P3.59. Determine the minimum diameter of the pin to the nearest millimeter for a factor of safety of 2.0. The pin's failure stress in shear is 300 MPa.

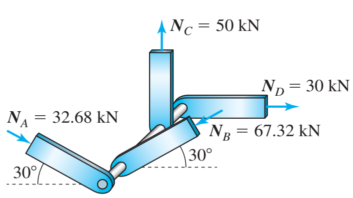
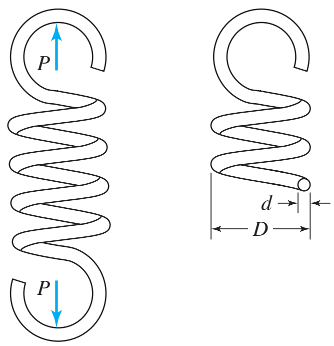


Figure P3.59

3.60 The shear stress on the cross section of the wire of a helical spring (Figure P3.60) is given by $\tau = K(8PC/\pi d^2)$, where P is the force on the spring, d is the diameter of the wire from which the spring is constructed, C is called the spring index, given by the ratio $C = D/d$, D is the diameter of the coiled spring, and K is called the *Wahl factor*, as given below.



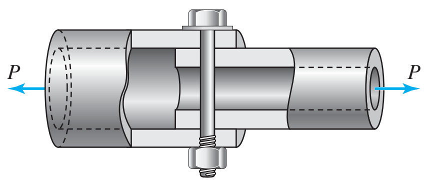
$$K = \frac{4C-1}{4C-4} + \frac{0.615}{C}$$

Figure P3.60

The spring is to be designed to resist a maximum force of 1200 N and must have a factor of safety of 1.1 in yield. The shear stress in yield is 350 MPa. Make a table listing admissible values of C and d for 4 mm $\le d \le$ 16 mm in steps of 2 mm.

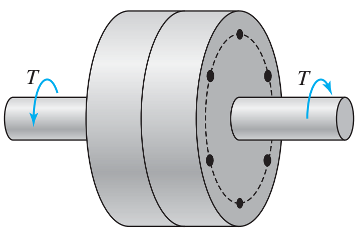
3.61 Two cast-iron pipes are held together by a steel bolt, as shown in Figure P3.61. The outer diameters of the two pipes are 2 in. and 2 3/4 in., and the wall thickness of each pipe is 1/4 in. The diameter of the bolt is 1/2 in. The yield strength of cast iron is 25 ksi in tension and steel is 15 ksi in shear. What is the maximum force P to the nearest pound this assembly can transmit for a factor of safety of 1.2?

Figure P3.61



3.62 A coupling of diameter 250-mm is assembled using 6 bolts of diameter 12.5 mm as shown in Figure P3.62. The holes for the bolts are drilled with center on a circle of diameter 200 mm. A factor of safety of 1.5 for the assembly is desired. If the shear strength of the bolts is 300 MPa, determine the maximum torque that can be transferred by the coupling.

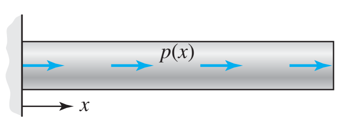
Figure P3.62



Stretch yourself

3.63 A circular rod of 15-mm diameter is acted upon by a distributed force *p(x)* that has the units of kN/m, as shown in Figure P3.63. The modulus of elasticity of the rod is 70 GPa. Determine the distributed force *p(x)* if the displacement *u(x)* in *x* direction is $u(x) = 30(x - x^2)10^{-6}$ m with x is measured in meters.

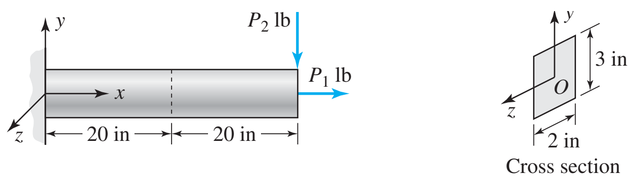
Figure P3.63



3.64 A circular rod of 15-mm diameter is acted upon by a distributed force *p(x)* that has the units of kN/m, as shown in Figure P3.63. The modulus of elasticity of the rod is 70 GPa. Determine the distributed force *p(x)* if the displacement *u(x)* in *x* direction is $u(x) = 50(x^2 - 2x^3)10^{-6}$ m with x is measured in meters.

3.65 Consider the beam shown in Figure P3.65. The displacement in the x direction due to the action of the forces, was found to be $u = [(60x + 80xy - x^2y)/180]10^{-3}$ in. The modulus of elasticity of the beam is 30,000 ksi. Determine the statically equivalent internal normal force $\boldsymbol{N}$ and the internal bending moment $\boldsymbol{M}_z$ acting at point O at a section at $x = 20$ in. Assume an unknown shear stress is acting on the cross-section.

Figure P3.65



Computer problems

3.66 Assume that the stress–strain curve *after* yield stress in Problem 3.12 is described by the quadratic equation $\sigma = a + b\varepsilon + c\varepsilon^2$. (a) Determine the coefficients a, b, and c by the least-squares method. (b) Find the tangent modulus of elasticity at a stress level of 1400 MPa.

3.67 Assume that the stress–strain curve *after* yield stress in Problem 3.13 is described by the quadratic equation $\sigma = a + b\varepsilon + c\varepsilon^2$. (a) Determine the coefficients a, b, and c by the least-squares method. (b) Find the tangent modulus of elasticity at a stress level of 50 ksi.

3.68 Marks were made on the cord used for tying the canoe on top of the car in Example 3.5. These marks were made every 2 in. to produce a total of 20 segments. The stretch cord is symmetric with respect to the top of the canoe. The starting point of the first segment is on the carrier rail of the car and the end point of the tenth segment is on the top of the canoe. The measured length of each segment is as shown in Table 3.68. Determine (a) the tension in the cord of each segment; (b) the force exerted by the cord on the carrier of the car. Use the modulus of elasticity $E = 510$ psi and the diameter of the stretch cord as 1/2 in.

TABLE P3.68

Segment Number	Deformed Length (inches)
1	3.4
2	3.4
3	3.4
4	3.4
5	3.4
6	3.4
7	3.1
8	2.7
9	2.3
10	2.2

3.69 Marks were made on the cord used for tying the canoe on top of the car in Example 3.5. These marks were made every 2 in. to produce a total of 20 segments. The stretch cord is symmetric with respect to the top of the canoe. The starting point of the first segment is on the carrier rail of the car and the end point of the tenth segment is on the top of the canoe. The measured length of each segment is as shown in Table 3.68. Determine (a) the tension in the cord of each segment; (b) the force exerted by the cord on the carrier of the car. Use the diameter of the stretch cord as 1/2 in. and the following equation for the stress–strain curve:

$$\sigma = \begin{cases} 1020\varepsilon - 1020\varepsilon^2 \text{ psi} & \varepsilon < 0.5 \\ 255 \text{ psi} & \varepsilon \geq 0.5 \end{cases}$$

3.70 Marks were made on the cord used for tying the canoe on top of the car in Example 3.5. These marks were made every 2 in. to produce a total of 20 segments. The stretch cord is symmetric with respect to the top of the canoe. The starting point of the first segment is on the carrier rail of the car and the end point of the tenth segment is on the top of the canoe. The measured length of each segment is as shown in Table 3.68. Determine: (a) the tension in the cord of each segment; (b) the force exerted by the cord on the carrier of the car. Use the Poisson's ratio $\nu = \frac{1}{2}$ and the initial diameter of 1/2 in. and calculate the diameter in the deformed position for each segment. Use the stress–strain relationship given in Problem 3.69.

3.4 ISOTROPY AND HOMOGENEITY

The description of a material as isotropic or homogeneous are acquiring greater significance with the development of new materials. In composites (See Section 3.12.3) two or more materials are combined together to produce a stronger or stiffer material. Both material descriptions are approximations influenced by several factors. As will be seen in this section four possible descriptions are: Isotropic–homogeneous; anisotropic–homogeneous; isotropic–nonhomogeneous; and anisotropic–nonhomogeneous.

The number of material constants that need to be measured depends on the material model we want to incorporate into our analysis. Any material model is the relationship between stresses and strains—the simplest model, a linear relationship. With no additional assumptions, the linear relationship of the six strain components to six stress components can be written

$$\begin{aligned}
\varepsilon_{xx} &= C_{11}\sigma_{xx} + C_{12}\sigma_{yy} + C_{13}\sigma_{zz} + C_{14}\tau_{yz} + C_{15}\tau_{zx} + C_{16}\tau_{xy} \\
\varepsilon_{yy} &= C_{21}\sigma_{xx} + C_{22}\sigma_{yy} + C_{23}\sigma_{zz} + C_{24}\tau_{yz} + C_{25}\tau_{zx} + C_{26}\tau_{xy} \\
\varepsilon_{zz} &= C_{31}\sigma_{xx} + C_{32}\sigma_{yy} + C_{33}\sigma_{zz} + C_{34}\tau_{yz} + C_{35}\tau_{zx} + C_{36}\tau_{xy} \\
\gamma_{yz} &= C_{41}\sigma_{xx} + C_{42}\sigma_{yy} + C_{43}\sigma_{zz} + C_{44}\tau_{yz} + C_{45}\tau_{zx} + C_{46}\tau_{xy} \\
\gamma_{zx} &= C_{51}\sigma_{xx} + C_{52}\sigma_{yy} + C_{53}\sigma_{zz} + C_{54}\tau_{yz} + C_{55}\tau_{zx} + C_{56}\tau_{xy} \\
\gamma_{xy} &= C_{61}\sigma_{xx} + C_{62}\sigma_{yy} + C_{63}\sigma_{zz} + C_{64}\tau_{yz} + C_{65}\tau_{zx} + C_{66}\tau_{xy}
\end{aligned} \tag{3.12}$$

Equation (3.12) implies that we need 36 material constants to describe the most general linear relationship between stress and strain. However, it can be shown that the matrix formed by the constants C_{ij} is symmetric (i.e., $C_{ij} = C_{ji}$, where i and j can be any number from 1 to 6). This symmetry is due to the requirement that the strain energy always be positive, but the proof is beyond the scope of this book. The symmetry reduces the maximum number of independent constants to 21 for the most general linear relationship between stress and strain. (Section 3.12.1 describes the controversy over the number of independent constants required in a linear stress–strain relationship.)

Equation (3.12) presupposes that the relation between stress and strain in the x direction is different from the relation in the y or z direction. Alternatively, Equation (3.12) implies that if we apply a force (stress) in the x direction and observe the deformation (strain), then this deformation will differ from the deformation produced if we apply the same force in the y direction. This phenomenon is not observable by the naked eye for most metals, but if we were to look at the metals at the crystal-size level, then the number of constants needed to describe the stress–strain relationship depends on the crystal structure. Thus we need to ask at what level we are conducting the analysis—eye level or crystal size? If we average the impact of the crystal structure at the eye level, then we have defined the simplest material. An **isotropic material** has stress–strain relationships that are independent of the orientation of the coordinate system at a point.

An **anisotropic material** is a material that is not isotropic. The most general anisotropic material requires 21 independent material constants to describe a linear stress–strain relationships. An isotropic body requires only two independent material constants to describe a linear stress–strain relationships (See Example 9.8 and Problem 9.81). Between the isotropic material and the most general anisotropic material lie several types of materials, which are discussed briefly in Section 3.11.2. The degree of difference in material properties with orientation, the scale at which the analysis is being conducted, and the kind of information that is desired from the analysis are some of the factors that influence whether we treat a material as isotropic or anisotropic.

There are many constants used to describe relate stresses and strains (see Problems 3.97 and 3.109), but for isotropic materials only two are independent. That is, all other constants can be found if any two constants are known. The three constants that we shall encounter most in this book are the modulus of elasticity E, the shear modulus of elasticity G, and the Poisson's ratio ν. In Example 9.8 we shall show that for isotropic materials

$$\boxed{G = \frac{E}{2(1+\nu)}} \tag{3.13}$$

Homogeneity is another approximation that is often used to describe a material behavior. A **homogeneous material** has same the material properties at all points in the body. Alternatively, if the material constants C_{ij} are functions of the coordinates x, y, or z, then the material is called nonhomogeneous.

Most materials at the atomic level, the crystalline level, or the grain-size level are nonhomogeneous. The treatment of a material as homogeneous or nonhomogeneous depends once more on the type of information that is to be obtained from the analysis. Homogenization of material properties is a process of averaging different material properties by an overall material property. Any body can be treated as a homogeneous body if the scale at which the analysis is conducted is made sufficiently large.

3.5 GENERALIZED HOOKE'S LAW FOR ISOTROPIC MATERIALS

The equations relating stresses and strains at a *point* in three dimensions are called the **generalized Hooke's law**. The generalized Hooke's law can be developed from the definitions of the three material constants E, ν, and G and the assumption of isotropy. No assumption of homogeneity needs to be made, as the generalized Hooke's law is a stress–strain relationship at a point. In Figure 3.26 normal stresses are applied one at a time. From the definition of the modulus of elasticity we can obtain the strain in the direction of the applied stress, which then is used to get the strains in the perpendicular direction by using the definition of Poisson's ratio. From Figure (3.26*a*), (3.26*b*), and (3.26*c*) we obtain

$$\varepsilon_{xx}^{(1)} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy}^{(1)} = -\nu\varepsilon_{xx}^{(1)} = -\nu\left(\frac{\sigma_{xx}}{E}\right) \qquad \varepsilon_{zz}^{(1)} = -\nu\varepsilon_{xx}^{(1)} = -\nu\left(\frac{\sigma_{xx}}{E}\right)$$

$$\varepsilon_{xx}^{(2)} = -\nu\varepsilon_{yy}^{(2)} = -\nu\left(\frac{\sigma_{yy}}{E}\right) \qquad \varepsilon_{yy}^{(2)} = \frac{\sigma_{yy}}{E} \qquad \varepsilon_{zz}^{(2)} = -\nu\varepsilon_{yy}^{(2)} = -\nu\left(\frac{\sigma_{yy}}{E}\right)$$

$$\varepsilon_{xx}^{(3)} = -\nu\varepsilon_{zz}^{(3)} = -\nu\left(\frac{\sigma_{zz}}{E}\right) \qquad \varepsilon_{yy}^{(3)} = -\nu\varepsilon_{zz}^{(3)} = -\nu\left(\frac{\sigma_{zz}}{E}\right) \qquad \varepsilon_{zz}^{(3)} = \frac{\sigma_{zz}}{E}$$

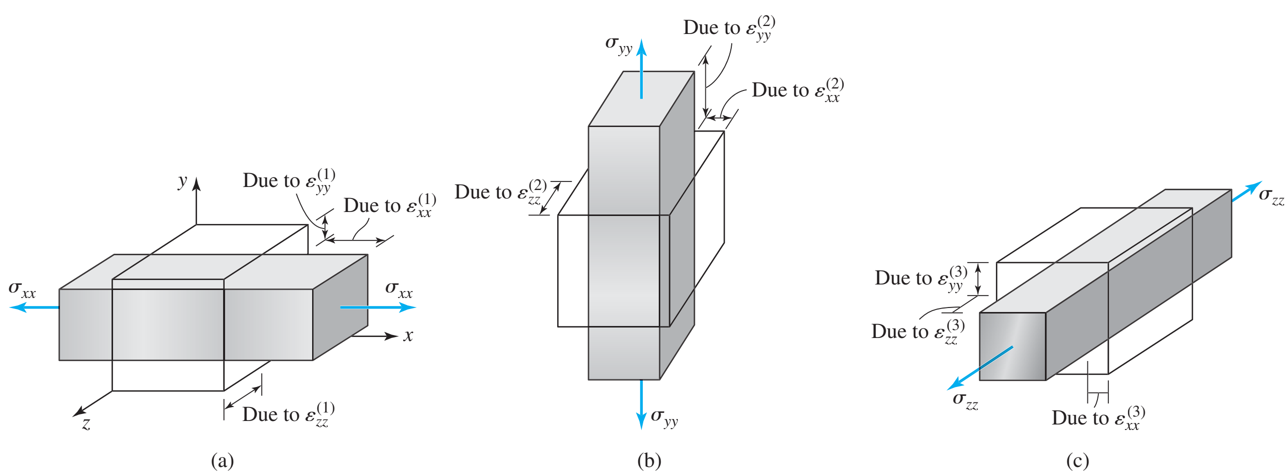


Figure 3.26 Derivation of the generalized Hooke's law.

The use of the same E and ν to relate stresses and strains in different directions implicitly assumes isotropy. Notice that no change occurs in the right angles from the application of normal stresses. Thus no shear strain is produced due to normal stresses in a fixed coordinate system for an isotropic material.

Assuming the material is linearly elastic, we can use the principle of superposition to obtain the total strain $\varepsilon_{ii} = \varepsilon_{ii}^{(1)} + \varepsilon_{ii}^{(2)} + \varepsilon_{ii}^{(3)}$, as shown in Equations (3.14a) through (3.14c). From the definition of shear modulus given in Equation (3.3), we obtain Equations (3.14d) through (3.14f).

Generalized Hooke's law:

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} \tag{3.14a}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} \tag{3.14b}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} \tag{3.14c}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \tag{3.14d}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G} \tag{3.14e}$$

$$\gamma_{zx} = \frac{\tau_{zx}}{G} \tag{3.14f}$$

The equations are valid for nonhomogeneous material. The nonhomogeneity will make the material constants E, ν, and G functions of the spatial coordinates. The use of Poisson's ratio to relate strains in perpendicular directions is valid not only for Cartesian coordinates but for any orthogonal coordinate system. Thus the generalized Hooke's law may be written for *any orthogonal* coordinate system, such as spherical and polar coordinate systems.

An alternative form[1] for Equations (3.14a) through (3.14c), which may be easier to remember, is the matrix form

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \tag{3.15}$$

3.6 PLANE STRESS AND PLANE STRAIN

In Chapters 1 and 2 two-dimensional problems of plane stress and plane strain, respectively. Taking the two definitions and using Equations (3.14a), (3.14b), (3.14c), and (3.14f), we obtain the matrices shown in Figure 3.27. The difference between the two idealizations of material behavior is in the zero and nonzero values of the normal strain and normal stress in the z direction. In plane stress $\sigma_{zz} = 0$, which from Equation (3.14c) implies that the normal strain in the z direction is $\varepsilon_{zz} = -\nu(\sigma_{xx} + \sigma_{yy})/E$. In plane strain $\varepsilon_{zz} = 0$, which from Equation (3.14c) implies that the normal stress in the z direction is $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$..

$$\text{Plane stress} \longrightarrow \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's law}} \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

$$\text{Plane strain} \longrightarrow \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy} & 0 \\ \gamma_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Generalized Hooke's law}} \begin{bmatrix} \sigma_{xx} & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_{yy} & 0 \\ 0 & 0 & \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \end{bmatrix}$$

Figure 3.27 Stress and strain matrices in plane stress and plane strain.

Figure 3.28 shows two plates on which only compressive normal stresses in the x and y directions are applied. The top and bottom surfaces on the plate in Figure 3.28*a* are free surfaces (plane stress), but because the plate is free to expand, the deformation (strain) in the z direction is not zero. The plate in Figure 3.28*b* is constrained from expanding in the z direction by

[1]Another alternative is $\varepsilon_{ii} = [(1+\nu)\sigma_{ii} - \nu I_1]/E$, where $I_1 = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$.

the rigid surfaces. As the material pushes on the plate, a reaction force develops, and this reaction force results in a nonzero value of normal stress in the z direction. Plane stress or plane strain are often approximations to simplify analysis. Plane stress approximation is often made for thin bodies, such as the metal skin of an aircraft. Plane strain approximation is often made for thick bodies, such as the hull of a submarine.

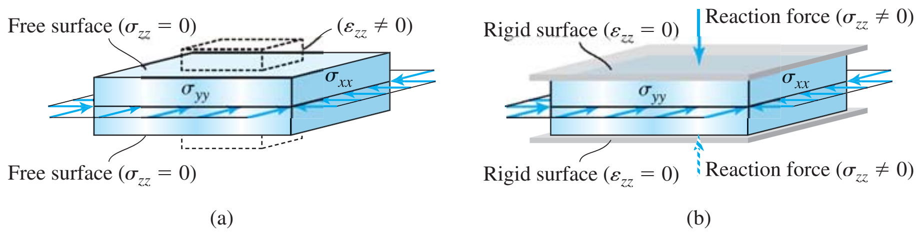


Figure 3.28 (*a*) Plane stress. (*b*) Plane strain.

It should be recognized that in plane strain and plane stress conditions there are only three independent quantities, even though the nonzero quantities number more than 3. For example, if we know σ_{xx}, σ_{yy}, and τ_{xy}, then we can calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, and σ_{zz} for plane stress and plane strain. Similarly, if we know ε_{xx}, ε_{yy}, and γ_{xy} then we can calculate σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}, and ε_{zz} for plane stress and plane strain. Thus in both plane stress and plane strain the number of independent stress or strain components is 3, although the number of nonzero components is greater than 3. Examples 3.7 and 3.8 elaborate on the difference between plane stress and plane strain conditions and the difference between nonzero and independent quantities.

EXAMPLE 3.7

The stresses at a point on steel were found to be σ_{xx} = 15 ksi (T), σ_{yy} = 30 ksi (C), and τ_{xy} = 25 ksi. Using E = 30,000 ksi and G = 12,000 ksi, determine the strains ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz} and the stress σ_{zz} assuming (a) the point is in a state of plane stress. (b) the point is in a state of plane strain.

PLAN

In both cases the shear strain is the same and can be calculated using Equation (3.14d). (a) For plane stress $\sigma_{zz} = 0$ and the strains ε_{xx}, ε_{yy}, and ε_{zz} can be found from Equations (3.14a), (3.14b), and (3.14c), respectively. (b) For plane strain $\varepsilon_{zz} = 0$ and Equation (3.14c) can be used to find σ_{zz}. The stresses σ_{xx}, σ_{yy}, and σ_{zz} can be substituted into Equations (3.14a) and (3.14b) to calculate the normal strains ε_{xx} and ε_{yy}.

SOLUTION

From Equation (3.14d)

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{25 \text{ ksi}}{12,000 \text{ ksi}} = 0.002083 \tag{E1}$$

ANS. $\gamma_{xy} = 2083\ \mu$

The Poisson's ratio can be found from Equation (3.13),

$$G = \frac{E}{2(1+\nu)} \quad \text{or} \quad 12,000 \text{ ksi} = \frac{30,000 \text{ ksi}}{2(1+\nu)} \quad \text{or} \quad \nu = 0.25 \tag{E2}$$

(a) *Plane stress*: The normal strains in the x, y, and z directions are found from Equations (3.14a), (3.14b), and (3.14c), respectively.

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} = \frac{15 \text{ ksi} - 0.25(-30 \text{ ksi})}{30,000 \text{ ksi}} = 750(10^{-6}) \tag{E3}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} = \frac{-30 \text{ ksi} - 0.25(15 \text{ ksi})}{30,000 \text{ ksi}} = -1125(10^{-6}) \tag{E4}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} = \frac{0 - 0.25(15 \text{ ksi} - 30 \text{ ksi})}{30,000 \text{ ksi}} = 125(10^{-6}) \tag{E5}$$

ANS. $\varepsilon_{xx} = 750\ \mu$ $\varepsilon_{yy} = -1125\ \mu$ $\varepsilon_{zz} = 125\ \mu$

(b) *Plane strain:* From Equation (3.14c), we have

$$\varepsilon_{zz} = \frac{[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]}{E} = 0 \qquad \text{or} \qquad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = 0.25(15\text{ ksi} - 30\text{ ksi}) = -3.75\text{ksi} \tag{E6}$$

ANS. $\sigma_{zz} = 3.75\text{ ksi (C)}$

The normal strains in the *x* and *y* directions are found from Equations (3.14a) and (3.14b),

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E} = \frac{15\text{ ksi} - 0.25(-30\text{ ksi} - 3.75\text{ ksi})}{30{,}000\text{ ksi}} = 781.2(10^{-6}) \tag{E7}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} = \frac{-30\text{ ksi} - 0.25(15\text{ ksi} - 3.75\text{ ksi})}{30{,}000\text{ ksi}} = -1094(10^{-6}) \tag{E8}$$

ANS. $\varepsilon_{xx} = 781.2\ \mu \qquad \varepsilon_{yy} = -1094\ \mu$

COMMENTS

1. The three independent quantities in this problem were σ_{xx}, σ_{yy}, and τ_{xy}. Knowing these we were able to find all the strains in plane stress and plane strain.
2. The difference in the values of the strains came from the zero value of σ_{zz} in plane stress and a value of -3.75 ksi in plane strain.

EXAMPLE 3.8

The strains at a point on aluminum ($E = 70$ GPa, $G = 28$ GPa, and $\nu = 0.25$) were found to be $\varepsilon_{xx} = 650\ \mu$, $\varepsilon_{yy} = 300\ \mu$, and $\gamma_{xy} = 750\ \mu$. . Determine the stresses σ_{xx}, σ_{yy}, and τ_{xy} and the strain ε_{zz} assuming the point is in plane stress.

PLAN

The shear strain can be calculated using Equation (3.14d). If we note that $\sigma_{zz} = 0$ and the strains ε_{xx} and ε_{yy} are given, the stresses σ_{xx} and σ_{yy} can be found by solving Equations (3.14a) and (3.14b) simultaneously. The strain ε_{zz} can then be found from Equation (3.14c).

SOLUTION

From Equation (3.14d),

$$\tau_{xy} = G\gamma_{xy} = (28 \times 10^{9}\text{N/m}^2)(750 \times 10^{-6}) = 21(10^6)\ \text{N/m}^2 \tag{E1}$$

ANS. $\tau_{xy} = 21\text{ MPa}$

Equations (3.14a) and (3.14b) can be rewritten with $\sigma_{zz} = 0$,

$$\sigma_{xx} - \nu\sigma_{yy} = E\varepsilon_{xx} = (70 \times 10^9\text{ N/m}^2)(650 \times 10^{-6}) \qquad \text{or} \tag{E2}$$

$$\sigma_{xx} - \nu\sigma_{yy} = 45.5\text{ MPa} \tag{E3}$$

$$\sigma_{yy} - \nu\sigma_{xx} = E\varepsilon_{yy} = (70 \times 10^9\text{ N/m}^2)(300 \times 10^{-6}) \qquad \text{or} \tag{E4}$$

$$\sigma_{yy} - 0.25\sigma_{xx} = 21\text{ MPa} \tag{E5}$$

Solving Equations (E3) and (E5) we obtain σ_{xx} and σ_{yy}.

ANS. $\sigma_{xx} = 54.1\text{ MPa (T)} \qquad \sigma_{yy} = 34.5\text{ MPa (T)}$

From Equation (3.14c) we obtain

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} = \frac{0 - 0.25(54.13 + 34.53)10^6}{70 \times 10^9} = -317(10^{-6}) \tag{E6}$$

ANS. $\varepsilon_{zz} = -317\ \mu$

COMMENTS

1. Equations (E3) and (E5) have a very distinct structure. If we multiply either equation by ν and add the product to the other equation, the result will be to eliminate one of the unknowns. Equation (3.17) in Problem 3.104 is developed in this manner and can be used for solving this problem. But this would imply remembering one more formula. We can avoid this by remembering the defined structure of Hooke's law, which is applicable to all types of problems and not just plane stress.
2. Equation (3.18) in Problem 3.105 gives $\varepsilon_{zz} = -[\nu/(1-\nu)](\varepsilon_{xx} + \varepsilon_{yy})$. Substituting $\nu = 0.25$ and $\varepsilon_{xx} = 650\ \mu$, $\varepsilon_{yy} = 300\ \mu$, we obtain $\varepsilon_{zz} = -(0.25/0.75)(650 + 300) = 316.7\ \mu$, as before. This formula is useful if we do not need to calculate stresses, and we will use it in Chapter 9.

QUICK TEST 3.2

Time: 15 minutes/Total: 20 points

Grade yourself using the answers given in Appendix E. Each question is worth two points.

1. What is the difference between an isotropic and a homogeneous material?

2. What is the number of independent material constants needed in a linear stress–strain relationship for an isotropic material?

3. What is the number of independent material constants needed in a linear stress–strain relationship for the most general anisotropic materials?

4. What is the number of independent *stress* components in plane stress problems?

5. What is the number of independent *strain* components in plane stress problems?

6. How many nonzero *strain* components are there in plane stress problems?

7. What is the number of independent *strain* components in plane strain problems?

8. What is the number of independent *stress* components in plane strain problems?

9. How many nonzero *stress* components are there in plane strain problems?

10. Is the value of E always greater than G, less than G, or does it depend on the material? Justify your answer.

PROBLEM SET 3.2

3.71 Write the generalized Hooke's law for isotropic material in cylindrical coordinates (r, θ, z).

3.72 Write the generalized Hooke's law for isotropic material in spherical coordinates (r, θ, ϕ).

In problems 3.73 through 3.78 two material constants and the stress components in the x, y plane are given. Calculate ε_{xx}, ε_{yy}, γ_{xy}, ε_{zz}, *and* σ_{zz} *(a) assuming plane stress; (b) assuming plane strain.*

3.73	E = 200 GPa	$\nu = 0.32$	σ_{xx} = 100 MPa (T)	σ_{yy} = 150 MPa (T)	τ_{xy} = –125 MPa
3.74	E = 70 GPa	G = 28 GPa	σ_{xx} = 225 MPa (C)	σ_{yy} = 125 MPa (T)	τ_{xy} = 150 MPa
3.75	E = 30,000 ksi	$\nu = 0.3$	σ_{xx} = 22 ksi (C)	σ_{yy} = 25 ksi (C)	τ_{xy} = –15 ksi
3.76	E = 10,000 ksi	G = 3900 ksi	σ_{xx} = 15 ksi (T)	σ_{yy} = 12 ksi (C)	τ_{xy} = –10 ksi
3.77	G = 15 GPa	$\nu = 0.2$	σ_{xx} = 300 MPa (C)	σ_{yy} = 300 MPa (T)	τ_{xy} = 150 MPa
3.78	E = 2000 psi	G = 800 psi	σ_{xx} = 100 psi (C)	σ_{yy} = 150psi (T)	τ_{xy} = 100 psi

In problems 3.79 through 3.84 two material constants and the strain components in the x, y plane are given. Calculate σ_{xx}, σ_{yy}, τ_{xy}, σ_{zz}, *and* ε_{zz} *assuming the point is in plane stress.*

3.79	E = 200 GPa	$\nu = 0.32$	$\varepsilon_{xx} = 500\ \mu$	$\varepsilon_{yy} = 400\ \mu$	$\gamma_{xy} = -300\ \mu$
3.80	E = 70 GPa	G = 28 GPa	$\varepsilon_{xx} = 2000\ \mu$	$\varepsilon_{yy} = -1000\ \mu$	$\gamma_{xy} = 1500\mu$
3.81	E = 30,000 ksi	$\nu = 0.3$	$\varepsilon_{xx} = -800\ \mu$	$\varepsilon_{yy} = -1000\ \mu$	$\gamma_{xy} = -500\ \mu$
3.82	E = 10,000 ksi	G = 3900 ksi	$\varepsilon_{xx} = 1500\ \mu$	$\varepsilon_{yy} = -1200\ \mu)$	$\gamma_{xy} = -1000\ \mu$
3.83	G = 15 GPa	$\nu = 0.2$	$\varepsilon_{xx} = -2000\ \mu$	$\varepsilon_{yy} = 2000\ \mu$	$\gamma_{xy} = 1200\ \mu$
3.84	E = 2000 psi	G = 800 psi	$\varepsilon_{xx} = 50\ \mu$	$\varepsilon_{yy} = 75\ \mu$	$\gamma_{xy} = -25\ \mu$

3.85 The cross section of the wooden piece that is visible in Figure P3.85 is 40 mm × 25 mm. The clamped length of the wooden piece in the vice is 125 mm. The modulus of elasticity of wood is E = 14 GPa and the Poisson's ratio ν = 0.3. The jaws of the vice exert a uniform pressure of 3.2 MPa on the wood. Determine the average change of length of the wood.

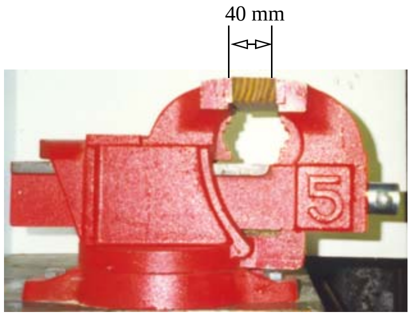


Figure P3.85

3.86 A thin plate ($E = 30{,}000$ ksi, $\nu = 0.25$) under the action of uniform forces deforms to the shaded position, as shown in Figure P3.86. Assuming plane stress, determine the average normal stresses in the *x* and *y* directions.

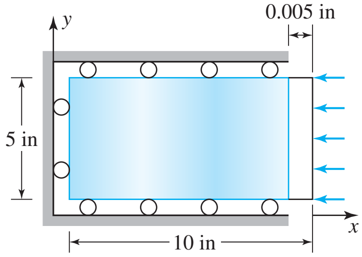


Figure P3.86

3.87 A thin plate (E=30,000 ksi, ν = 0.25) is subjected to a uniform stress σ = 10 ksi as shown in Figure P3.87. Assuming plane stress, determine (a) the average normal stress in *y* direction; (b) the contraction of the plate in *x* direction.

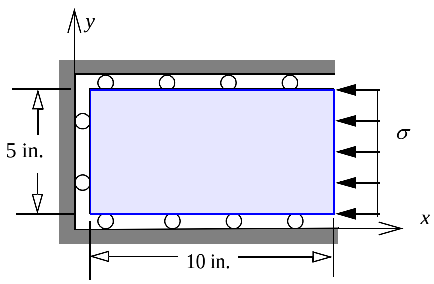


Figure P3.87

3.88 A rubber (E_R=300 psi and ν_R = 0.5) rod of diameter d_R =4 in. is placed in a steel (rigid) tube d_S =4.1 in. as shown in Figure P3.88. What is the smallest value of P that can be applied so that the space between the rubber rod and the steel tube would close.

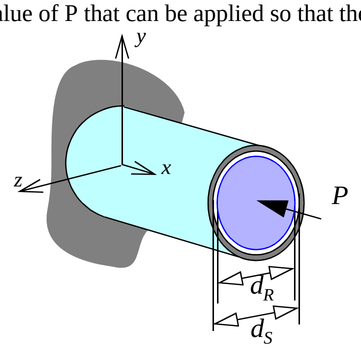


Figure P3.88

3.89 A rubber (E_R= 2.1GPa and ν_R = 0.5) rod of diameter d_R = 200 mm is placed in a steel (rigid) tube d_S = 204 mm as shown in Figure P3.88. If the applied force is P = 10 kN, determine the average normal stress in the *y* and *z* direction.

3.90 A 2 in. × 2 in. square with a circle inscribed is stressed as shown Figure P3.90. The plate material has a modulus of elasticity E = 10,000 ksi and a Poisson's ratio ν = 0.25. Assuming plane stress, determine the major and minor axes of the ellipse formed due to deformation.

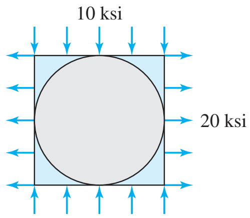


Figure P3.90

3.91 A 2 in. × 2 in. square with a circle inscribed is stressed as shown Figure P3.91. The plate material has a modulus of elasticity E = 10,000 ksi and a Poisson's ratio ν = 0.25. Assuming plane stress, determine the major and minor axes of the ellipse formed due to deformation.

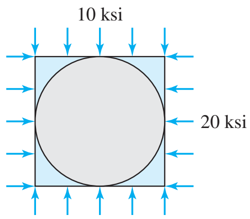


Figure P3.91

3.92 A 50 mm × 50 mm square with a circle inscribed is stressed as shown Figure P3.92. The plate material has a modulus of elasticity E = 70 GPa and a Poisson's ratio ν = 0.25. Assuming plane stress, determine the major and minor axes of the ellipse formed due to deformation.

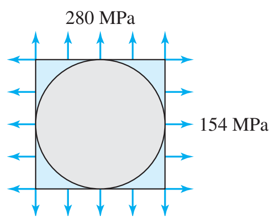


Figure P3.92

3.93 A rectangle inscribed on an aluminum (E = 10,000 ksi, ν = 0.25) plate is observed to deform into the colored shape shown in Figure P3.93. Determine the average stress components σ_{xx}, σ_{yy}, and τ_{xy} , assuming plane stress..

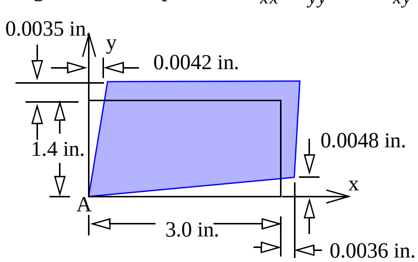


Figure P3.93

3.94 A rectangle inscribed on an steel (E = 210 GPa, ν = 0.28) plate is observed to deform into the colored shape shown in Figure P3.94. Determine the average stress components σ_{xx}, σ_{yy}, and τ_{xy} , assuming plane stress.

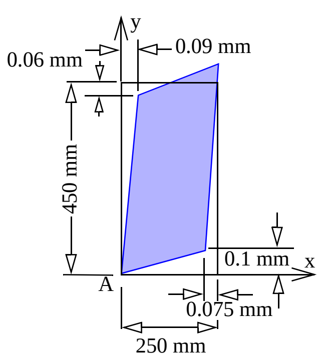


Figure P3.94

3.95 A 5 ft mean diameter spherical steel (E = 30,000 ksi, ν = 0.28) tank has a wall thickness of 3/4 in. Determine the increase in the mean diameter when the gas pressure inside the tank is 600 psi.

3.96 A steel (E = 210 GPa ν = 0.28) cylinder of mean diameter of 1 m and wall thickness of 10 mm has gas at 250 kPa. Determine the increase in the mean diameter due to gas pressure.

3.97 Derive the following relations of normal stresses in terms of normal strain from the generalized Hooke's law:

$$\begin{aligned}\sigma_{xx} &= [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}]\frac{E}{(1-2\nu)(1+\nu)}\\ \sigma_{yy} &= [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz} + \nu\varepsilon_{xx}]\frac{E}{(1-2\nu)(1+\nu)}\\ \sigma_{zz} &= [(1-\nu)\varepsilon_{zz} + \nu\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-2\nu)(1+\nu)}\end{aligned} \tag{3.16}$$

An alternative form that is easier to remember is $\sigma_{ii} = 2G\varepsilon_{ii} + \lambda(I_1)$, where i can be x, y, or z; $I_1 = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$; G is the shear modulus; and $\lambda = 2G\nu/(1-2\nu)$ is called *Lame's constant*, after G. Lame (1795–1870).

3.98 For a point in plane stress show that

$$\sigma_{xx} = (\varepsilon_{xx} + \nu\varepsilon_{yy})\frac{E}{1-\nu^2} \qquad \sigma_{yy} = (\varepsilon_{yy} + \nu\varepsilon_{xx})\frac{E}{1-\nu^2} \tag{3.17}$$

3.99 For a point in plane stress show that

$$\varepsilon_{zz} = -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{xx} + \varepsilon_{yy}) \tag{3.18}$$

3.100 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.79.

3.101 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.80.

3.102 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.81.

3.103 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.82.

3.104 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.83.

3.105 Using Equations (3.17) and (3.18), solve for σ_{xx}, σ_{yy}, and ε_{zz} in Problem 3.84.

3.106 For a point in plane strain show that

$$\varepsilon_{xx} = [(1-\nu)\sigma_{xx} - \nu\sigma_{yy}]\frac{1+\nu}{E} \qquad \varepsilon_{yy} = [(1-\nu)\sigma_{yy} - \nu\sigma_{xx}]\frac{1+\nu}{E} \tag{3.19}$$

3.107 For a point in plane strain show that

$$\sigma_{xx} = [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-2\nu)(1+\nu)} \qquad \sigma_{yy} = [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{xx}]\frac{E}{(1-2\nu)(1+\nu)} \tag{3.20}$$

3.108 A differential element subjected to only normal strains is shown in Figure P3.108. The ratio of change in a volume ΔV to the original volume V is called the volumetric strain ε_V, or *dilation*.

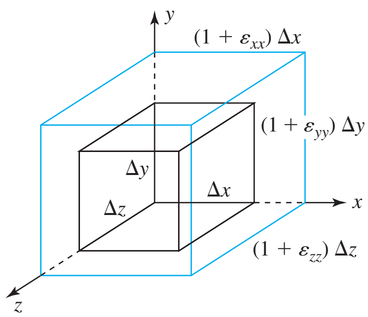


Figure P3.108

For small strain prove

$$\varepsilon_V = \frac{\Delta V}{V} = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \qquad (3.21)$$

3.109 Prove

$$p = -K\varepsilon_V \qquad p = -\left(\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3}\right) \qquad K = \frac{E}{3(1-2\nu)} \qquad (3.22)$$

where K is the *bulk modulus* and p is the *hydrostatic pressure* because at a point in a fluid the normal stresses in all directions are equal to $-p$. Note that at $\nu = \frac{1}{2}$ there is no change in volume, regardless of the value of the stresses. Such materials are called *incompressible materials*.

Stretch yourself

An orthotropic material (Section 3.12.3) has the following stress–strain relationship at a point in plane stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} \qquad (3.23)$$

Use Equations (3.23) to solve Problems 3.110 through 3.117 .

The stresses at a point on a free surface of an orthotropic material are given in Problems 3.110 through 3.113 . Also given are the material constants. Using Equations (3.23) solve for the strains ε_{xx}, ε_{yy}, and γ_{xy}.

Problem	σ_{xx}	σ_{yy}	τ_{xy}	E_x	E_y	ν_{xy}	G_{xy}
3.110	5 ksi (C)	8 ksi (T)	6 ksi	7500 ksi	2500 ksi	0.3	1250 ksi
3.111	25 ksi (C)	5 ksi (C)	–8 ksi	25,000 ksi	2000 ksi	0.32	1500 ksi
3.112	200 MPa (C)	80 MPa (C)	–54 MPa	53 GPa	18 GPa	0.25	9 GPa
3.113	300 MPa (T)	50 MPa (T)	60 MPa	180 GPa	15 GPa	0.28	11 GPa

The strains at a point on a free surface of an orthotropic material are given in Problems 3.114 through 3.117 . Also given are the material constants. Using Equation (3.23) solve for the stresses σ_{xx}, σ_{yy}, and τ_{xy}.

Problem	ε_{xx}	ε_{yy}	γ_{xy}	E_x	E_y	ν_{xy}	G_{xy}
3.114	–1000 μ	500 μ	–250 μ	7500 ksi	2500 ksi	0.3	1250 ksi
3.115	–750 μ	–250 μ	400 μ	25,000 ksi	2000 ksi	0.32	1500 ksi
3.116	1500 μ	800 μ	600 μ	53 GPa	18 GPa	0.25	9 GPa
3.117	1500 μ	–750 μ	–450 μ	180 GPa	15 GPa	0.28	11 GPa

3.118 Using Equation (3.23), show that on a free surface of an orthotropic material

$$\sigma_{xx} = \frac{E_x(\varepsilon_{xx} + \nu_{yx}\varepsilon_{yy})}{1 - \nu_{yx}\nu_{xy}} \qquad \sigma_{yy} = \frac{E_y(\varepsilon_{yy} + \nu_{xy}\varepsilon_{xx})}{1 - \nu_{yx}\nu_{xy}} \qquad (3.24)$$

3.7* STRESS CONCENTRATION

Large stress gradients in a small region are called **stress concentration**. These large gradients could be due to sudden changes in geometry, material properties, or loading. We can use our theoretical models to calculate stress away from the regions of large stress concentration according to Saint-Venant's principle, which will be discussed in the next section. These stress values predicted by the theoretical models away from regions of stress concentration are called **nominal stresses**. Figure 3.29 shows photoelastic pictures (see Section 8.4.1) of two structural members under uniaxial tension. Large stress gradients near the circular cutout boundaries cause fringes to be formed. Each color boundary represents a fringe order that can be used in the calculation of the stresses.

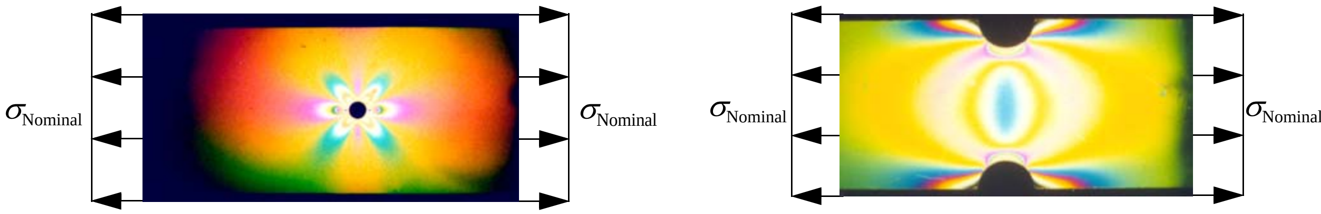


Figure 3.29 Photoelastic pictures showing stress concentration. (Courtesy Professor I. Miskioglu.)

Stress concentration factor is an engineering concept that permits us to extrapolate the results of our elementary theory into the region of large stress concentration where the assumptions on which the theory is based are violated. The stress concentration factor K_{conc} is defined as

$$\boxed{K_{\text{conc}} = \frac{\text{maximum stress}}{\text{nominal stress}}} \tag{3.25}$$

The stress concentration factor K_{conc} is found from charts, tables, or formulas that have been determined experimentally, numerically, analytically, or from a combination of the three. Section C.4 in Appendix shows several graphs that can be used in the calculation of stress concentration factors for problems in this book. Additional graphs can be found in handbooks describing different situations. Knowing the nominal stress and the stress concentration factor, the maximum stress can be estimated and used in design or to estimate the factor of safety. Example 3.9 demonstrates the use of the stress concentration factor.

3.8* SAINT-VENANT'S PRINCIPLE

Theories in mechanics of materials are constructed by making assumptions regarding load, geometry, and material variations. These assumptions are usually not valid near concentrated forces or moments, near supports, near corners or holes, near interfaces of two materials, and in flaws such as cracks. Fortunately, however, disturbance in the stress and displacement fields dissipates rapidly as one moves away from the regions where the assumptions of the theory are violated. Saint-Venant's principle states

> Two statically equivalent load systems produce nearly the same stress in regions at a distance that is at least equal to the largest dimension in the loaded region.

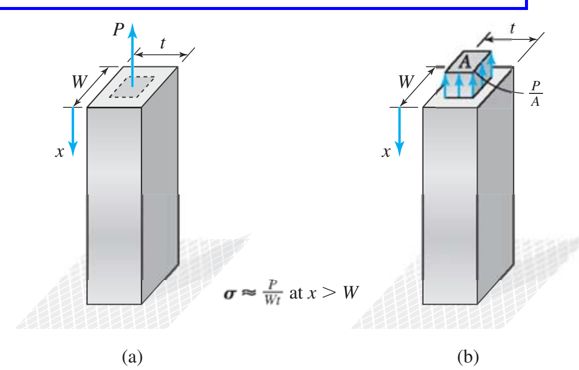


Figure 3.30 Stress due to two statically equivalent load systems.

Consider the two statically equivalent load systems shown in Figure 3.30. By Saint-Venant's principle the stress at a distance W away from the loads will be nearly uniform. In the region at a distance less than W the stress distribution will be different, and it is possible that there are also shear stress components present. In a similar manner, changes in geometry and materials have local effects that can be ignored at distances. We have considered the effect of changes in geometry and an engineering solution to the problem in Section 3.7 on stress concentration.

The importance of Saint-Venant's principle is that we can develop our theories with reasonable confidence away from the regions of stress concentration. These theories provide us with formulas for the calculation of nominal stress. We can then use the stress concentration factor to obtain maximum stress in regions of stress concentration where our theories are not valid.

EXAMPLE 3.9

Finite-element analysis (see Section 4.8) shows that a long structural component in Figure 3.31 carries a uniform axial stress of $\sigma_{nominal} = 35$ MPa (T). A hole in the center needs to be drilled for passing cables through the structural component. The yield stress of the material is $\sigma_{yield} = 200$ MPa. If failure due to yielding is to be avoided, determine the maximum diameter of the hole that can be drilled using a factor of safety of $K_{safety} = 1.6$.

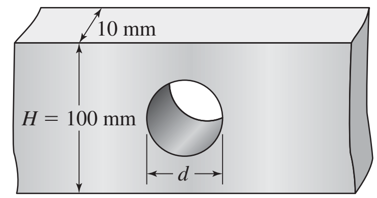


Figure 3.31 Component geometry in Example 3.9.

PLAN

We can compute the allowable (maximum) stress for factor of safety of 1.6 from Equation (3.10). From Equation (3.25) we can find the permissible stress concentration factor. From the plot of K_{gross} in Figure A.13 of Appendix C we can estimate the ratio of d/H. Knowing that $H = 100$ mm, we can find the maximum diameter d of the hole.

SOLUTION

From Equation (3.10) we obtain the allowable stress:

$$\sigma_{allow} = \frac{\sigma_{yield}}{K_{safety}} = \frac{200\text{ MPa}}{1.6} = 125\text{ MPa} \tag{E1}$$

From Equation (3.25) we calculate the permissible stress concentration factor:

$$K_{\text{conc}} \le \frac{\sigma_{allow}}{\sigma_{nominal}} = \frac{125\text{ MPa}}{35\text{ MPa}} = 3.57 \tag{E2}$$

From Figure A.13 of Appendix C we estimate the ratio of d/H as 0.367. Substituting $H = 100$ mm we obtain

$$\frac{d}{100\text{ mm}} \le 0.367 \qquad \text{or} \qquad d \le 36.7\text{ mm} \tag{E3}$$

For the maximum permissible diameter to the nearest millimeter we round downward.

ANS. $d_{max} = 36\text{ mm}$

COMMENTS

1. The value of $d/H = 0.367$ was found from linear interpolation between the value of $d/H = 0.34$, where the stress concentration factor is 0.35, and the value of $d/H = 0.4$, where the stress concentration factor is 0.375. These points were used as they are easily read from the graph. Because we are rounding downward in Equation (E3), any value between 0.36 and 0.37 is acceptable. In other words, the third place of the decimal value is immaterial.
2. As we used the maximum diameter of 36 mm instead of 36.7 mm, the effective factor of safety will be slightly higher than the specified value of 1.6, which makes this design a conservative design.
3. Creating the hole will change the stress around its. By per Saint-Venant's principle, the stress field far from the hole will not be significantly affected. This justifies the use of nominal stress without the hole in our calculation.

3.9* THE EFFECT OF TEMPERATURE

A material expands with an increase in temperature and contracts with a decrease in temperature. If the change in temperature is uniform, and if the material is isotropic and homogeneous, then all lines on the material will change dimensions by equal amounts. This will result in a normal strain, but there will be no change in the angles between any two lines, and hence there will be no shear strain produced. Experimental observations confirm this deduction. Experiments also show that the change in temperature ΔT is related to the thermal normal strain ε_T,

$$\varepsilon_T = \alpha \Delta T \tag{3.26}$$

where the Greek letter alpha α is the linear coefficient of thermal expansion. The linear relationship given by Equation (3.26) is valid for metals at temperatures well below the melting point. In this linear region the strains for most metals are small and the usual units for α are $\mu/^{\circ}$F or $\mu/^{\circ}$C, where $\mu = 10^{-6}$. Throughout the discussion in this section it is assumed that the material is in the linear region.

The tension test described in Section 3.1 is conducted at some ambient temperature. We expect the stress–strain curve to have the same character at two different ambient temperatures. If we raise the temperature by a small amount before we start the tension test then the expansion of specimen will result in a thermal strain, but there will be no stresses shifting the stress–strain curve from point O to point O_1, as shown in Figure 3.32. The total strain at any point is the sum of mechanical strain and thermal strains:

$$\boxed{\varepsilon = \frac{\sigma}{E} + \alpha \Delta T} \tag{3.27}$$

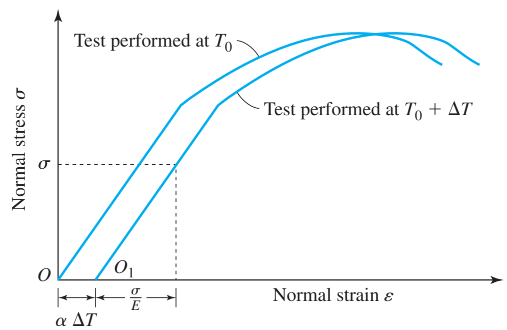


Figure 3.32 Effect of temperature on stress–strain curve.

Equation (3.27) and Figure 3.32 are valid only for small temperature changes well below the melting point. Material non-homogeneity, material anisotropy, nonuniform temperature distribution, or reaction forces from body constraints are the reasons for the generation of stresses from temperature changes. Alternatively, no thermal stresses are produced in a homogeneous, isotropic, unconstrained body due to uniform temperature changes.

The generalized Hooke's law relates mechanical strains to stresses. The total normal strain, as seen from Equation (3.27), is the sum of mechanical and thermal strains. For isotropic materials undergoing *small* changes in temperature, the generalized Hooke's law is written as shown in Equations (3.28a) through (3.28f).

$$\varepsilon_{xx} = \underbrace{\frac{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})}{E}}_{\text{Mechanical Strain}} + \underbrace{\alpha \Delta T}_{\text{Thermal Strain}} \tag{3.28a}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})}{E} + \alpha \Delta T \tag{3.28b}$$

$$\varepsilon_{zz} = \frac{\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})}{E} + \alpha \Delta T \tag{3.28c}$$

$$\gamma_{xy} = \tau_{xy}/G \tag{3.28d}$$

$$\gamma_{yz} = \tau_{yz}/G \tag{3.28e}$$

$$\gamma_{zx} = \tau_{zx}/G \tag{3.28f}$$

Homogeneity of the material or the uniformity of the temperature change are irrelevant as Hooke's law is written for a point and not for the whole body.

EXAMPLE 3.10

A circular bar ($E = 200$ GPa, $\nu = 0.32$, and $\alpha = 11.7\ \mu/°C$) has a diameter of 100 mm. The bar is built into a rigid wall on the left, and a gap of 0.5 mm exists between the right wall and the bar prior to an increase in temperature, as shown in Figure 3.33.The temperature of the bar is increased uniformly by 80°C. Determine the average axial stress and the change in the diameter of the bar.

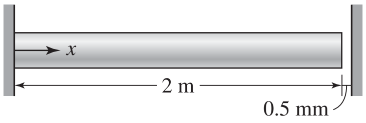


Figure 3.33 Bar in Example 3.10.

METHOD 1: PLAN

A reaction force in the axial direction will be generated to prevent an expansion greater than the gap. This would generate σ_{xx}. As there are no forces in the y or z direction, the other normal stresses σ_{yy} and σ_{zz} can be approximated to zero in Equation (3.28a). The total deformation is the gap, from which the total average axial strain for the bar can be found. The thermal strain can be calculated from the change in the given temperature. Thus in Equation (3.28a) the only unknown is σ_{xx}. Once σ_{xx} has been calculated, the strain ε_{yy} can be found from Equation (3.28b) and the change in diameter calculated.

SOLUTION

The total axial strain is the total deformation (gap) divided by the length of the bar,

$$\varepsilon_{xx} = \frac{0.5 \times 10^{-3}\ \text{m}}{2\ \text{m}} = 250 \times 10^{-6} \tag{E1}$$

$$\alpha\,\Delta T = 11.7 \times 10^{-6} \times 80 = 936 \times 10^{-6} \tag{E2}$$

Because σ_{yy} and σ_{zz} are zero, Equation (3.28a) can be written as $\varepsilon_{xx} = \sigma_{xx}/E + \alpha\,\Delta T$, from which we can obtain σ_{xx},

$$\sigma_{xx} = E(\varepsilon_{xx} - \alpha\,\Delta T) = (200 \times 10^{9}\ \text{N/m}^2)(250 - 936)10^{-6} = -137.2 \times 10^{6}\ \text{N/m}^2 \tag{E3}$$

ANS. $\sigma_{xx} = 137.2$ MPa (C)

From Equation (3.28b) we can obtain ε_{yy} and calculate the change in diameter,

$$\varepsilon_{yy} = -\nu\frac{\sigma_{xx}}{E} + \alpha\,\Delta T = -0.25\left(\frac{-137.2 \times 10^{6}\ \text{N/m}^2}{200 \times 10^{9}\ \text{N/m}^2}\right) + 936 \times 10^{-6} = 1.107 \times 10^{-3} \tag{E4}$$

$$\Delta D = \varepsilon_{yy} D = 1.107 \times 10^{-3} \times 100\ \text{mm} \tag{E5}$$

ANS. $\Delta D = 0.1107$ mm increase

COMMENTS

1. If $\alpha\,\Delta T$ were less than ε_{xx}, then σ_{xx} would come out as tension and our assumption that the gap closes would be invalid. In such a case there would be no stress σ_{xx} generated.
2. The increase in diameter is due partly to Poisson's effect and partly to thermal strain in the y direction.

METHOD 2: PLAN

We can think of the problem in two steps: (i) Find the thermal expansion δ_T initially ignoring the restraining effect of the right wall. (ii) Apply the force P to bring the bar back to the restraint position due to the right wall and compute the corresponding stress.

SOLUTION

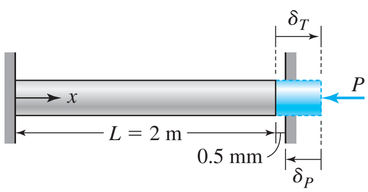


Figure 3.34 Approximate deformed shape of the bar in Example 3.10.

We draw an approximate deformed shape of the bar, assuming there is no right wall to restrain the deformation as shown in Figure 3.34.The thermal expansion δ_T is the thermal strain multiplied by the length of the bar,

$$\delta_T = (\alpha\,\Delta T)L = 11.7 \times 10^{-6} \times 80 \times 2\text{ m} = 1.872 \times 10^{-3}\text{ m} \quad \text{(E6)}$$

We obtain the contraction δ_P to satisfy the restraint imposed by the right wall by subtracting the gap from the thermal expansion.

$$\delta_P = \delta_T - 0.5 \times 10^{-3}\text{ m} = 1.372 \times 10^{-3}\text{ m} \quad \text{(E7)}$$

We can then find the mechanical strain and compute the corresponding stress:

$$\varepsilon_P = \frac{\delta_P}{L} = \frac{1.372 \times 10^{-3}\text{ m}}{2\text{ m}} = 0.686 \times 10^{-3} \quad \text{(E8)}$$

$$\sigma_P = E\varepsilon_P = (200 \times 10^{9}\text{ N/m}^2) \times 0.686 \times 10^{-3} = 137.2 \times 10^{6}\text{ N/m}^2 \quad \text{(E9)}$$

ANS. $\sigma_P = 137.2\text{ MPa }(C)$

The change in diameter can be found as in Method 1.

COMMENT

In Method 1 we ignored the intermediate steps and conducted the analysis at equilibrium. We implicitly recognized that for a linear system the process of reaching equilibrium is immaterial. In Method 2 we conducted the thermal and mechanical strain calculations separately. Method 1 is more procedural. Method 2 is more intuitive.

EXAMPLE 3.11

Solve Example 3.8 with a temperature increase of 20°C. Use $\alpha = 23\ \mu/^{\circ}$C.

PLAN

Shear stress is unaffected by temperature change and its value is the same as in Example 3.8. Hence $\tau_{xy} = 21$ MPa. In Equations 3.28a and 3.28b $\sigma_{zz} = 0$, $\varepsilon_{xx} = 650\ \mu$, and $\varepsilon_{yy} = 300\ \mu$ are known and $\alpha\Delta T$ can be found and substituted to generate two equations in the two unknown stresses σ_{xx} and σ_{yy}, which are found by solving the equations simultaneously. Then from Equation (3.28c), the normal strain ε_{zz} can be found.

SOLUTION

We can find the thermal strain as $\Delta T = 20$ and $\alpha\Delta T = 460 \times 10^{-6}$. Equations 3.28a and 3.28b and can be rewritten with $\sigma_{zz} = 0$,

$$\sigma_{xx} - \nu\sigma_{yy} = E(\varepsilon_{xx} - \alpha\Delta T) = (70 \times 10^{9}\text{ N/m}^2)(650 - 460)10^{-6} \quad \text{or}$$

$$\sigma_{xx} - 0.25\sigma_{yy} = 13.3\text{ MPa} \quad \text{(E1)}$$

$$\sigma_{yy} - \nu\sigma_{xx} = E(\varepsilon_{yy} - \alpha\Delta T) = (70 \times 10^{9}\text{ N/m}^2)(300 - 460)10^{-6} \quad \text{or}$$

$$\sigma_{yy} - 0.25\sigma_{xx} = -11.2\text{ MPa} \quad \text{(E2)}$$

By solving Equations (E1) and (E2) we obtain σ_{xx} and σ_{yy}.

ANS. $\sigma_{xx} = 11.2\text{ MPa(T)}$ $\quad \sigma_{yy} = 8.4\text{ MPa(C)}$

From Equation (3.28c) with $\sigma_{zz} = 0$ we obtain

$$\varepsilon_{zz} = \frac{-\nu(\sigma_{xx} + \sigma_{yy})}{E} + \alpha\,\Delta T = \frac{-0.25(11.2 - 8.4)(10^{6})\text{ N/m}^2}{(70 \times 10^{9}\text{ N/m}^2)} + 460 \times 10^{-6} \quad \text{(E3)}$$

ANS. $\varepsilon_{zz} = 460\ \mu$

COMMENT

1. Equations (E1) and (E2) once more have the same structure as in Example 3.8. The only difference is that in Example 3.8 we were given the mechanical strain and in this example we obtained the mechanical strain by subtracting the thermal strain from the total strain.

PROBLEM SET 3.3

Stress concentration

3.119 A steel bar is axially loaded, as shown in Figure P3.119. Determine the factor of safety for the bar if yielding is to be avoided. The normal yield stress for steel is 30 ksi. Use the stress concentration factor chart in Section C.4 in Appendix.

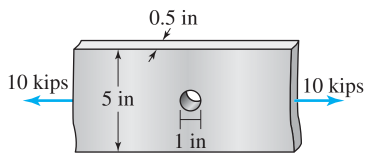


Figure P3.119

3.120 The stress concentration factor for a stepped flat tension bar with shoulder fillets shown in Figure P3.120 was determined as given by the equation below. The equation is valid only if $H/d > 1 + 2r/d$ and $L/H > 5.784 - 1.89r/d$. The nominal stress is P/dt. Make a chart for the stress concentration factor versus H/d for the following values of r/d: 0.2, 0.4, 0.6, 0.8, 1.0. (Use of a spreadsheet is recommended.)

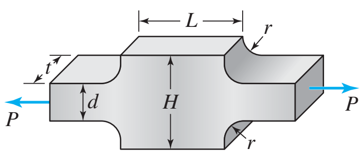


Figure P3.120

$$K_{conc} = 1.970 - 0.384\left(\frac{2r}{H}\right) - 1.018\left(\frac{2r}{H}\right)^2 + 0.430\left(\frac{2r}{H}\right)^3$$

3.121 Determine the maximum normal stress in the stepped flat tension bar shown in Figure P3.120 for the following data: $P = 9$ kips, $H = 8$ in, $d = 3$ in, $t = 0.125$ in, and $r = 0.625$ in.

3.122 An aluminum stepped tension bar is to carry a load $P = 56$ kN. The normal yield stress of aluminum is 160 MPa. The bar in Figure P3.120 has $H = 300$ mm, $d = 100$ mm, and $t = 10$ mm. For a factor of safety of 1.6, determine the minimum value r of the fillet radius if yielding is to be avoided.

3.123 The stress concentration factor for a flat tension bar with U-shaped notches shown in Figure P3.123 was determined as given by the equation below. The nominal stress is P/Ht. Make a chart for the stress concentration factor vs. r/d for the following values of H/d: 1.25, 1.50, 1.75, 2.0. (Use of a spreadsheet is recommended.)

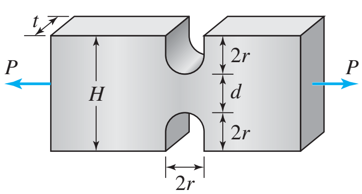


Figure P3.123

$$K_{conc} = 3.857 - 5.066\left(\frac{4r}{H}\right) + 2.469\left(\frac{4r}{H}\right)^2 - 0.258\left(\frac{4r}{H}\right)^3$$

3.124 Determine the maximum normal stress in the flat tension bar shown in Figure P3.123 for the following data: $P = 150$ kN, $H = 300$ mm, $r = 15$ mm, and $t = 5$ mm.

3.125 A steel tension bar with U-shaped notches of the type shown in Figure P3.123 is to carry a load $P = 18$ kips. The normal yield stress of steel is 30 ksi. The bar has $H = 9$ in., d = 6 in. and $t = 0.25$ in. For a factor of safety of 1.4, determine the value of r if yielding is to be avoided.

Temperature effects

3.126 An iron rim ($\alpha = 6.5\ \mu/°F$) of 35.98-in diameter is to be placed on a wooden cask of 36-in. diameter. Determine the minimum temperature increase needed to slip the rim onto the cask.

3.127 The temperature is increased by 60°C in both steel ($E_s = 200$ GPa, $\alpha_s = 12.0\ \mu/^\circ$C) and aluminum ($E = 72$ GPa, $\alpha = 23.0\ \mu/^\circ$C). Determine the angle by which the pointer rotates from the vertical position (Figure P3.127).

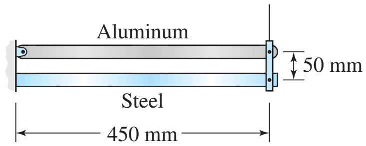


Figure P3.127

3.128 Solve Problem 3.73 if the temperature decrease is 25°C. Use $\alpha = 11.7\ \mu/°C$.

3.129 Solve Problem 3.74 if the temperature increase is 50°C. Use $\alpha = 23.6\ \mu/°C$.

3.130 Solve Problem 3.81 if the temperature increase is 40°F. Use $\alpha = 6.5\ \mu/°F$.

3.131 Solve Problem 3.82 if the temperature decrease is 100°F. Use $\alpha = 12.8\ \mu/°F$.

3.132 Solve Problem 3.83 if the temperature decrease is 75°C. Use $\alpha = 26.0\ \mu/°C$.

3.133 A plate ($E = 30{,}000$ ksi, $\nu = 0.25$, $\alpha = 6.5 \times 10^{-6}/°F$) cannot expand in the y direction and can expand at most by 0.005 in. in the x direction, as shown in Figure P3.133. Assuming plane stress, determine the average normal stresses in the x and y directions due to a uniform temperature increase of 100°F.

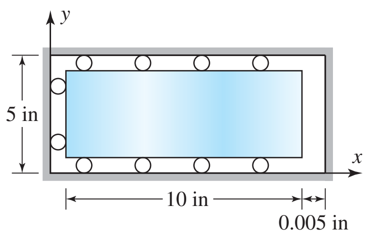


Figure P3.133

3.134 Derive the following relations of normal stresses in terms of normal strains from Equations (3.28a), (3.28b), and (3.28c):

$$\begin{aligned}
\sigma_{xx} &= [(1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}]\frac{E}{(1-2\nu)(1+\nu)} - \frac{E\alpha\,\Delta T}{1-2\nu}\\
\sigma_{yy} &= [(1-\nu)\varepsilon_{yy} + \nu\varepsilon_{zz} + \nu\varepsilon_{xx}]\frac{E}{(1-2\nu)(1+\nu)} - \frac{E\alpha\,\Delta T}{1-2\nu}\\
\sigma_{zz} &= [(1-\nu)\varepsilon_{zz} + \nu\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-2\nu)(1+\nu)} - \frac{E\alpha\,\Delta T}{1-2\nu}
\end{aligned} \tag{3.29}$$

3.135 For a point in plane stress show that

$$\sigma_{xx} = (\varepsilon_{xx} + \nu\varepsilon_{yy})\frac{E}{1-\nu^2} - \frac{E\alpha\,\Delta T}{1-\nu} \qquad \sigma_{yy} = (\varepsilon_{yy} + \nu\varepsilon_{xx})\frac{E}{1-\nu^2} - \frac{E\alpha\,\Delta T}{1-\nu} \tag{3.30}$$

3.136 For a point in plane stress show that

$$\varepsilon_{zz} = -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{xx} + \varepsilon_{yy}) + \left(\frac{1+\nu}{1-\nu}\right)\alpha\,\Delta T \tag{3.31}$$

3.10* FATIGUE

Try to break a piece of wire (such as a paper clip) by pulling on it by hand. You will not be able to break because you would need to exceed the ultimate stress of the material. Next take the same piece of wire and bend it one way and then the other a few times, and you will find that it breaks easily. The difference is the phenomena of fatigue.

All materials are *assumed* to have microcracks. These small crack length are not critical and are averaged as ultimate strength for the bulk material in a tension test. However, if the material is subjected to cyclic loading, these microcracks can grow until a crack reaches some critical length, at which time the remaining material breaks. The stress value at rupture in a cyclic loading is significantly lower than the ultimate stress of the material. Failure due to cyclic loading at stress levels significantly lower than the static ultimate stress is called **fatigue**.

Failure due to fatigue is like a brittle failure, irrespective of whether the material is brittle or ductile. There are two phases of failure. In the first phase the microcracks grow. These regions of crack growth can be identified by striation marks, also called beach marks, as shown in Figure 3.35. On examination of a fractured surface, this region of microcrack growth shows only small deformation. In phase 2, which is after the critical crack length has been reached, the failure surface of the region shows significant deformation.

Striation Marks

Figure 3.35 Failure of lead solder due to fatigue. (Courtesy Professor I. Miskioglu.)

The following strategy is used in design to account for fatigue failure. Experiments are conducted at different magnitude levels of cyclic stress, and the number of cycles at which the material fails is recorded. There is always significant scatter in the data. At low level of stress the failure may occur in millions and, at times, billions of cycles. To accommodate this large scale, a log scale is used for the number of cycles. A plot is made of stress versus the number of cycles to failure called the ***S–N* curve** as shown in Figure 3.36. Notice that the curve approaches a stress level asymtotically, implying that if stresses are kept below this level, then the material would not fail under cyclic loading. The highest stress level for which the material would not fail under cyclic loading is called **endurance limit** or **fatigue strength**.

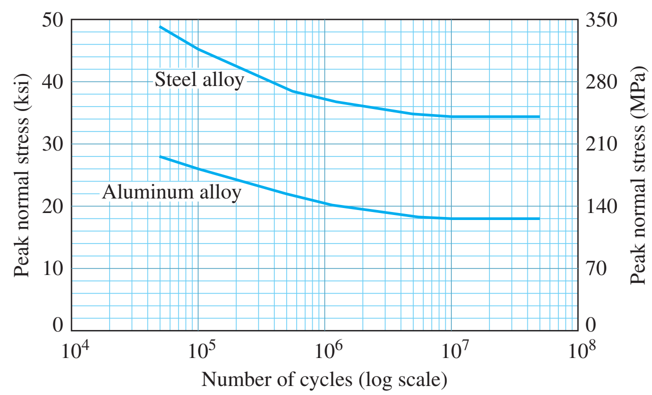


Figure 3.36 *S–N* curves.

It should be emphasized that a particular *S–N* curve for a material depends on many factors, such as manufacturing process, machining process, surface preparation, and operating environment. Thus two specimens made from the same steel alloys, but with a different history, will result in different *S–N* curves. Care must be taken to use an *S–N* curve that corresponds as closely as possible to the actual situation.

In a typical preliminary design, static stress analysis would be conducted using the peak load of the cyclic loading. Using an appropriate *S–N* curve, the number of cycles to failure for the peak stress value is calculated. This number of cycles to failure is the predicted life of the structural component. If the predicted life is unacceptable, then the component will be redesigned to lower the peak stress level and hence increase the number of cycles to failure.

EXAMPLE 3.12

The steel plate shown in Figure 3.37 has the *S–N* curve given in Figure 3.36. (a) Determine the maximum diameter of the hole to the nearest millimeter if the predicted life of one-half million cycles is desired for a uniform far-field stress $\sigma = 75$ MPa. (b) For the hole radius in part (a), what percentage reduction in far-field stress must occur if the predicted life is to increase to 1 million cycles?

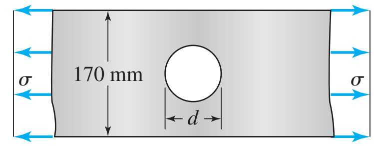


Figure 3.37 Uniaxially loaded plate with a hole in Example 3.12.

PLAN

(a) From Figure 3.36 we can find the maximum stress that the material can carry for one-half million cycles. From Equation (3.25) the gross stress concentration factor K_{gross} can be found. From the plot of K_{gross} in Figure A.13 of Appendix C we can estimate the ratio d/H and find the diameter d of the hole. (b) The percentage reduction in the gross nominal stress σ is the same as that in the maximum stress values in Figure 3.36, from one million cycles to one-half million cycles.

SOLUTION

(a) From Figure 3.36 the maximum allowable stress for one-half million cycles is estimated as 273 MPa. From Equation (3.25) the gross stress concentration factor is

$$K_{gross} = \frac{\sigma_{max}}{\sigma_{nominal}} = \frac{273 \text{ MPa}}{75 \text{ MPa}} = 3.64 \tag{E1}$$

From Figure A.13 of Appendix C the value of the ratio d/H corresponding to $K_{gross} = 3.64$ is 0.374.

$$d = 0.374 \times H = 0.374 \times 170 \text{ mm} = 63.58 \text{ mm} \tag{E2}$$

The maximum permissible diameter to the nearest millimeter can be obtained by rounding downward.

ANS. $d_{max} = 63$ mm

(b) From Figure 3.36 the maximum allowable stress for one million cycles is estimated as 259 MPa. Thus the percentage reduction in maximum allowable stress is [(273MPa – 259 MPa)/273 MPa](100) = 5.13%. As the geometry is the same as in part (a), the percentage reduction in far-field stress should be the same as in the maximum allowable stress.

ANS. The percentage reduction required is 5.13%.

COMMENT

1. A 5.13% reduction in peak stress value causes the predicted life cycle to double. Many factors can cause small changes in stress values, resulting in a very wide range of predictive life cycles. Examples include our estimates of the allowable stress in Figure 3.36, of the ratio d/H from Figure A.13 of Appendix C, of the far-field stress σ, and the tolerances of drilling the hole. Each is factor that can significantly affect our life prediction of the component. This emphasizes that the data used in predicting life cycles and failure due to fatigue must be of much higher accuracy than in traditional engineering analysis.

MoM in Action: The Comet / High Speed Train Accident

On January 10, 1954, the *de Havilland Comet* failed in midair near the Italian island Elba, killing all 35 people on board. On June 3, 1998, near the village of Eschede in Germany, a high-speed train traveling at nearly 200 km/h derailed, killing 101 people and injuring another 88. The events are a cautionary tale about the inherent dangers of new technologies and the high price of knowledge.

The *Comet* represented state-of-the-art technology. Passengers had a pressurized air cabin and slightly rounded square windows (Figure 1.39a) to look outside. The world's first commercial jet airliner flew 50% faster than the piston-engine aircrafts of that time, reducing flight times. It also flew higher, above adverse weather, for greater fuel efficiency and fewer vibrations. Its advanced aluminum alloy was postcard thin, to reduce weight, and adhesively bonded, lowering the risk of cracks spreading from rivets. Stress cycling due to pressurizing and depressurizing on plane that flies to 36,000 feet and returns to ground was simulated on a design prototype using a water tank. The plane was deemed safe for at least 16,000 flights.

On January 22, 1952, the *Comet* received a certificate of airworthiness. It crashed less than two years later after only 1290 flights, and the initial investigation failed to determine why. Flights resumed March 23, 1954, but on April 8, a second *Comet* crashed near Naples on its way from Rome to Cairo – after only 900 flights. Once more flights were grounded, while pressurizing and depressurizing testing was conducted on a plane that had gone through 1221 flights. It failed the tests after 1836 additional simulations.

Why did the initial testing on the prototype give such misleading results? Stresses near the window corners were far in excess of expectations, resulting in shorter fatigue life. Unlike static, fatigue test results should be used with great caution in extrapolating to field conditions. Passenger windows were made elliptical in shape, for a lower stress concentration. With this and other design improvements, *Comets* were used by many airlines for the next 30 years.

(a)

(b)

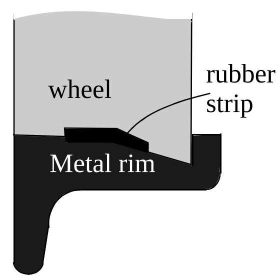


Figure 3.38 (a) de Havilland Comet 1 (b) Cross-section of high speed train wheel.

The high-speed intercity express (ICE) was the pride of the German railways. The first generation of these trains had single-cast wheels. At cruising speed, wheels deformation was causing vibrations. The wheels were redesigned with a rubber damping strip with a metal rim, as shown in Figure 1.39b. This design, already in use in streetcars, resolved the vibrations. However, the metal rims were failing earlier then predicted by design. The railway authority had noticed the problem long before the accident, but decided to merely replace the wheels more often. The decision proved disastrous.

Six kilometers from Eschede, the wheel rim from one axle peeled and punctured the floor. The train derailed in minutes. And investigation established that the rims become thinner owing to wear, and fatigue-induced cracks can cause failure earlier than the design prediction. The wheel design is now once more single cast, and alternative solutions to the vibration problems were found. Today the high-speed ICE is used for much of Germany.

No laboratory test can accurately predict fatigue life cycles under field conditions. Regular inspection of planes and high-speed train wheels for fatigue cracks is now standard practice.

3.11* NONLINEAR MATERIAL MODELS

Rubber, plastics, muscles, and other organic tissues exhibit nonlinearity in the stress–strain relationship, even at small strains. Metals also exhibit nonlinearity after yield stress. In this section we consider various nonlinear material models— the equations that represent the stress–strain nonlinear relationship. The material constants in the equations are found by least-squares fit of the stress–strain equation to the experimental data. For the sake of simplicity we shall assume that the material behavior is the same in tension and in compression.

We will consider three material models that are used in analytical and numerical analysis:

1. The *elastic–perfectly plastic* model, in which the nonlinearity is approximated by a constant.
2. The *linear strain-hardening* (or *bilinear*) model, in which the nonlinearity is approximated by a linear function.
3. The *power law* model, in which the nonlinearity is approximated by a one-term nonlinear function.

Other material models are described in the problems. The choice of material model depends not only on the material stress–strain curve, but also on the need for accuracy and the resulting complexity of analysis.

3.11.1 Elastic–Perfectly Plastic Material Model

Figure 3.39 shows the stress–strain curves describing an elastic–perfectly plastic behavior of a material. It is assumed that the material has the same behavior in tension and in compression. Similarly, for shear stress–strain, the material behavior is the same for positive and negative stresses and strains.

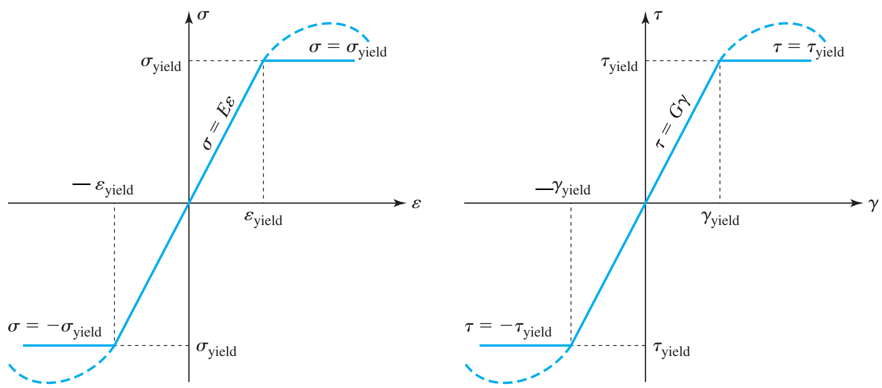


Figure 3.39 Elastic–perfectly plastic material behavior.

Before yield stress the stress–strain relationship is given by Hooke's law. After yield stress the stress is a constant. The elastic–perfectly plastic material behavior is a simplifying approximation[2] used to conduct an elastic–plastic analysis. The approximation is conservative in that it ignores the material capacity to carry higher stresses than the yield stress. The stress–strain curve are given by

$$\sigma = \begin{cases} \sigma_{yield}, & \varepsilon \geq \varepsilon_{yield} \\ E\varepsilon, & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield}, & \varepsilon \leq -\varepsilon_{yield} \end{cases} \tag{3.32}$$

$$\tau = \begin{cases} \tau_{yield}, & \gamma \geq \gamma_{yield} \\ G\gamma, & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield}, & \gamma \leq -\gamma_{yield} \end{cases} \tag{3.33}$$

The set of points forming the boundary between the elastic and plastic regions on a body is called **elastic–plastic boundary.** Determining the location of the elastic–plastic boundary is one of the critical issues in elastic–plastic analysis. The examples will show, the location of the elastic–plastic boundary is determined using two observations:

[2]Limit analysis is a technique based on elastic-plastic material behavior. It can be used to predict the maximum load a complex structure like a truss can support.

1. On the elastic–plastic boundary, the strain must be equal to the yield strain, and stress equal to yield stress. Deformations and strains are continuous at all points, including points at the elastic–plastic boundary.
2. If deformation is not continuous, then it is implied that holes or cracks are being formed in the material. If strains, which are derivative displacements, are not continuous, then corners are being formed during deformation.

3.11.2 Linear Strain-Hardening Material Model

Figure 3.40 shows the stress–strain curve for a linear strain-hardening model, also referred to as *bilinear* material[3] model. It is assumed that the material has the same behavior in tension and in compression. Similarly, for shear stress and strain, the material behavior is the same for positive and negative stresses and strains.

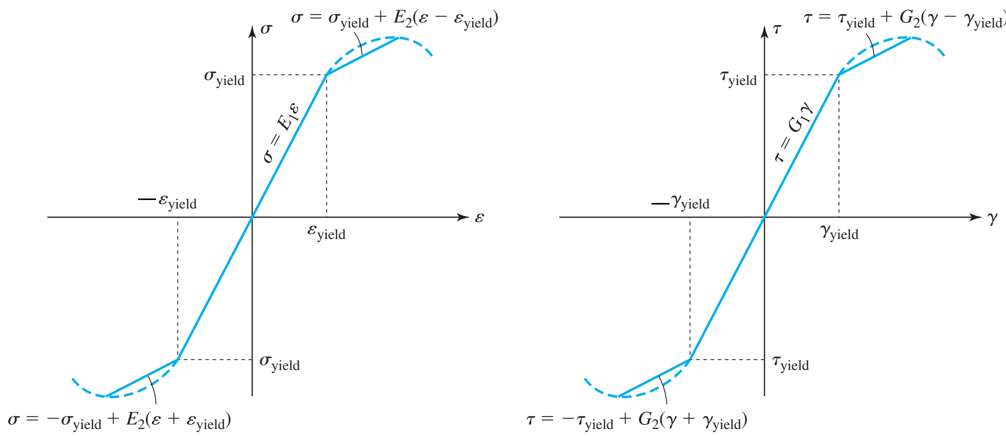


Figure 3.40 Linear strain-hardening model.

This is another conservative, simplifying approximation of material behavior: we once more ignore the material ability to carry higher stresses than shown by straight lines. The location of the elastic–plastic boundary is once more a critical issue in the analysis, and it is determined as in the previous section.

The stress–strain curves are given by

$$\sigma = \begin{cases} \sigma_{yield} + E_2(\varepsilon - \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \\ E_1\varepsilon & -\varepsilon_{yield} \leq \varepsilon \leq \varepsilon_{yield} \\ -\sigma_{yield} + E_2(\varepsilon + \varepsilon_{yield}) & \varepsilon \geq \varepsilon_{yield} \end{cases} \tag{3.34}$$

$$\tau = \begin{cases} \tau_{yield} + G_2(\gamma - \gamma_{yield}), & \gamma \geq \gamma_{yield} \\ G_1\gamma, & -\gamma_{yield} \leq \gamma \leq \gamma_{yield} \\ -\tau_{yield} + G_2(\gamma + \gamma_{yield}), & \gamma \leq \gamma_{yield} \end{cases} \tag{3.35}$$

3.11.3 Power-Law Model

Figure 3.41 shows a power-law representation of a nonlinear stress–strain curve. It is assumed that the material has the same behavior in tension and in compression. Similarly for shear stress and strain; the material behavior is the same for positive and negative stresses and strains. The stress–strain curve are given by

$$\sigma = \begin{cases} E\varepsilon^n, & \varepsilon \geq 0 \\ -E(-\varepsilon)^n, & \varepsilon < 0 \end{cases} \qquad \tau = \begin{cases} G\gamma^n, & \gamma \geq 0 \\ -G(-\gamma)^n, & \gamma < 0 \end{cases} \tag{3.36}$$

[3]Incremental plasticity is a numerical technique that approximated the non-linear stress-strain curve by series of straight lines over small intervals.

The constants E and G are the strength coefficients, and n is the strain-hardening coefficient. They are determined by least-squares fit to the experimental stress–strain curve. Materials such as most metals in plastic region or most plastics are represented by the solid curve with the strain-hardening coefficient less than one. Materials like soft rubber, muscles and other organic materials are represented by the dashed line with a strain-hardening coefficient greater than one.

From Equation (3.36) we note that when strain is negative, the term in parentheses becomes positive, permitting evaluation of the number to fractional powers. Furthermore with negative strain we obtain negative stress, as we should.

In Section 3.11.2 we saw that the stress–strain relationship could be written using different equations for different stress levels. We could, in a similar manner, combine a linear equation for the linear part and a nonlinear equation for the nonlinear part, or we could combine two nonlinear equations, thus creating additional material models. Other material models are considered in the problems.

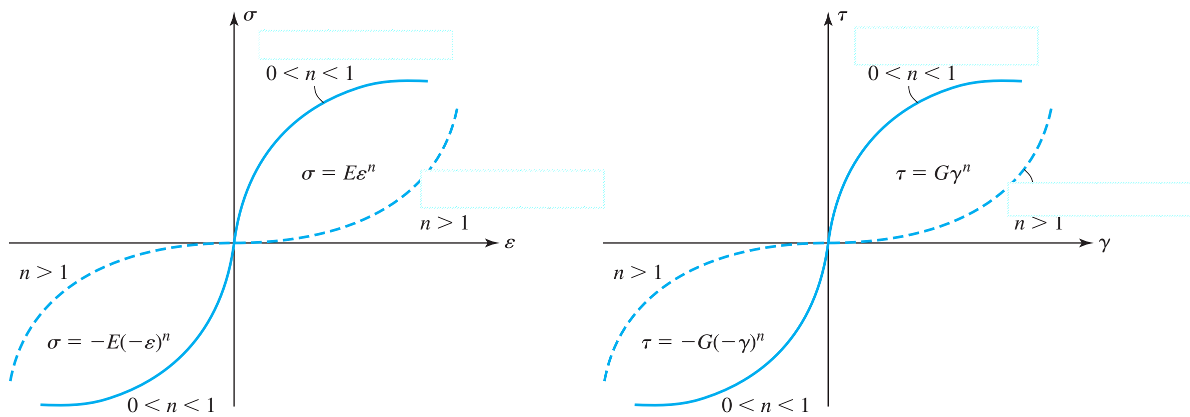


Figure 3.41 Nonlinear stress–strain curves.

EXAMPLE 3.13

Aluminum has a yield stress $\sigma_{yield} = 40$ ksi in tension, a yield strain $\varepsilon_{yield} = 0.004$, an ultimate stress $\sigma_{ult} = 45$ ksi, and the corresponding ultimate strain $\varepsilon_{ult} = 0.17$. Determine the material constants and plot the corresponding stress–strain curves for the following models: (a) the elastic–perfectly plastic model. (b) the linear strain-hardening model. (c) the nonlinear power-law model.

PLAN

We have coordinates of three points on the curve: P_0 ($\sigma_0 = 0.00$, $\varepsilon_0 = 0.000$), P_1 ($\sigma_1 = 40.0$, $\varepsilon_1 = 0.004$), and P_2 ($\sigma_2 = 45.0$, $\varepsilon_2 = 0.017$). Using these data we can find the various constants in the material models.

SOLUTION

(a) The modulus of elasticity E is the slope between points P_0 and P_1. After yield stress, the stress is a constant. The stress–strain behavior can be written as

$$E_1 = \frac{\sigma_1 - \sigma_0}{\varepsilon_1 - \varepsilon_0} = \frac{40\text{ ksi}}{0.004} = 10{,}000\text{ ksi} \tag{E1}$$

$$\textbf{ANS.} \quad \sigma = \begin{cases} 10{,}000\varepsilon\text{ ksi}, & |\varepsilon| \le 0.004 \\ 40\text{ ksi}, & |\varepsilon| \ge 0.004 \end{cases} \tag{E2}$$

(b) In the linear strain-hardening model the slope of the straight line before yield stress is as calculated in part (a). After the yield stress, the slope of the line can be found from the coordinates of points P_1 and P_2. The stress–strain behavior can be written as

$$E_2 = \frac{\sigma_2 - \sigma_1}{\varepsilon_2 - \varepsilon_1} = \frac{5\text{ ksi}}{0.013} = 384.6\text{ ksi} \tag{E3}$$

$$\textbf{ANS.} \quad \sigma = \begin{cases} 10{,}000\varepsilon\text{ ksi}, & |\varepsilon| \le 0.004 \\ 40 + 384.6(\varepsilon - 0.004)\text{ ksi}, & |\varepsilon| \ge 0.004 \end{cases} \tag{E4}$$

(c) The two constants E and n in $\sigma = E\varepsilon^n$ can be found by substituting the coordinates of the two point P_1 and P_2, to generate

$$40 = E(0.004)^n \tag{E5}$$

$$45 = E(0.017)^n \tag{E6}$$

Dividing Equation (E6) by Equation (E5) and taking the logarithm of both sides, we solve for n:

$$\ln\left(\frac{0.017}{0.004}\right)^n = \ln\left(\frac{45}{40}\right) \quad or \quad n\ \ln(4.25) = \ln(1.125) \quad \text{or} \quad n = 0.0814 \tag{E7}$$

Substituting Equation (E7) into Equation (E5), we obtain the value of E:

$$E = (40\text{ ksi})/(0.004)^{0.0814} = 62.7\ \text{ksi} \tag{E8}$$

We can now write the stress-strain equations for the power law model.

$$\textbf{ANS.} \qquad \sigma = \begin{cases} 62.7\varepsilon^{0.0814}\ \text{ksi} & \varepsilon \geq 0 \\ -62.7(-\varepsilon)^{0.0814}\ \text{ksi} & \varepsilon < 0 \end{cases} \tag{E9}$$

Stresses at different strains can be found using Equations (E2), (E4), and (E9) and plotted as shown in Figure 3.42.

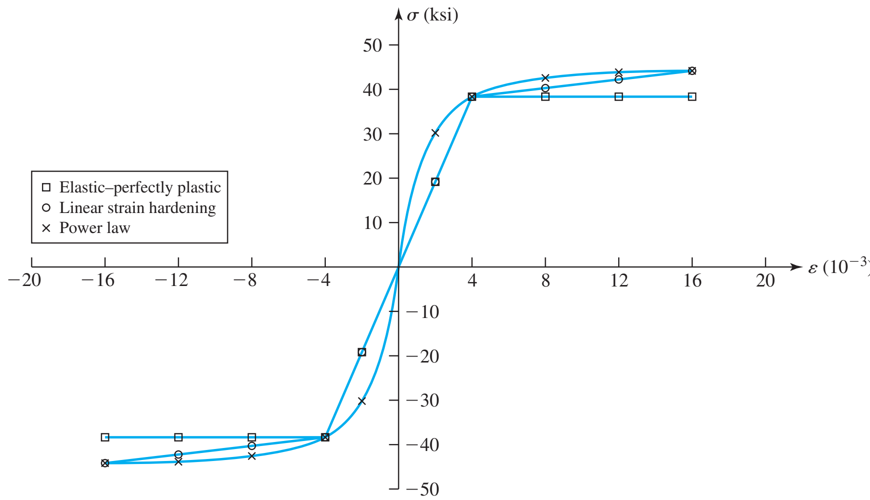


Figure 3.42 Stress–strain curves for different models in Example 3.13.

EXAMPLE 3.14

On a cross-section of a hollow circular shaft shown in Figure 3.43, the shear strain in polar coordinates was found to be $\gamma_{x\theta} = 3\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches. Write expressions for $\tau_{x\theta}$ as a function of ρ, and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section. Assume the shaft is made from elastic–perfectly plastic material that has a yield stress $\tau_{yield} = 24$ ksi and a shear modulus $G = 6000$ ksi.

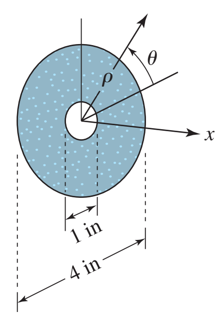


Figure 3.43 Hollow shaft in Example 3.14.

PLAN

The yield strain in shear γ_{yield} can be found from the yield stress τ_{yield} and the shear modulus G. The location of the elastic–plastic boundary ρ_y can be determined at which the given shear strain reaches the value of γ_{yield}. The shear stress at points before ρ_y can be found from Hooke's law, and after ρ_y it will be the yield stress.

SOLUTION

The location of the elastic–plastic boundary can be found as

$$\gamma_{\text{yield}} = \frac{\tau_{\text{yield}}}{G} = \frac{24 \times 10^3 \text{ ksi}}{6000 \times 10^3 \text{ ksi}} = 0.004 = 0.003\rho_y \qquad or \qquad \rho_y = \frac{0.004}{0.003} = 1.33 \text{ in.} \tag{E1}$$

Up to ρ_y stress and strain are related by Hooke's law, and hence

$$\tau_{x\theta} = G\gamma_{x\theta} = (6 \times 10^6 \text{ psi}) \times 3\rho \times 10^{-3} = 18\rho \times 10^3 \text{ psi} \tag{E2}$$

After ρ_y the stress is equal to τ_y, and the shear stress can be written as

$$\textbf{ANS.} \qquad \tau_{x\theta} = \begin{cases} 18\rho \text{ ksi}, & 0.5 \text{ in.} \le \rho \le 1.333 \text{ in.} \\ 24 \text{ ksi}, & 1.333 \text{ in.} \le \rho \le 2.0 \text{ in.} \end{cases} \tag{E3}$$

The shear strain and shear stress distributions across the cross section are shown in Figure 3.44.

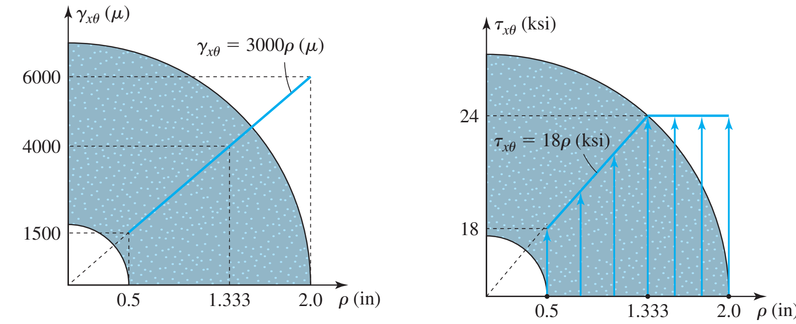


Figure 3.44 Strain and stress distributions.

COMMENT

1. In this problem we knew the strain distribution and hence could locate the elastic–plastic boundary easily. In most problems we do not know the strains due to a load, and finding the elastic–plastic boundary is significantly more difficult.

EXAMPLE 3.15

Resolve Example 3.14 assuming the shaft material has a stress–strain relationship given by $\tau = 450\gamma^{0.75}$ ksi.

PLAN

Substituting the strain expression into the stress–strain equation we can obtain stress as a function of ρ and plot it.

SOLUTION

Substituting the strain distribution into the stress–strain relation we obtain

$$\tau_{x\theta} = 450 \times 0.003^{0.75}\rho^{0.75} \tag{E1}$$

$$\textbf{ANS.} \qquad \tau_{x\theta} = 5.768\rho^{0.75} \text{ ksi}$$

The shear stress can be found at several points and plotted as shown in Figure 3.45.

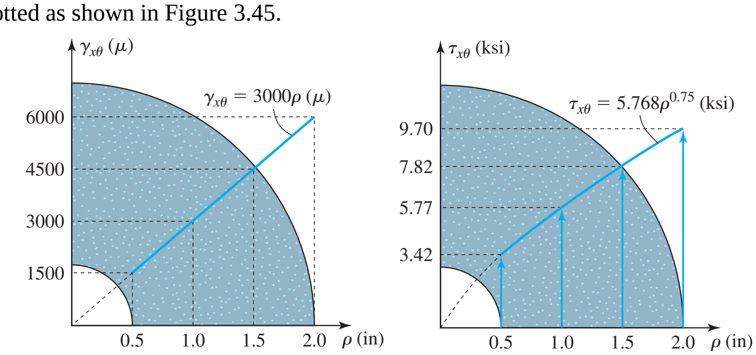


Figure 3.45 Strain and stress distributions in Example 3.15

COMMENT

1. We see that although the strain distribution is linear across the cross section, the stress distribution is nonlinear due to material nonlinearity. Deducing the stress distribution across the cross section would be difficult, but deducing a linear strain distribution is possible from geometric considerations, as will be seen in Chapter 5 for the torsion of circular shafts.

EXAMPLE 3.16

At a cross section of a beam shown in Figure 3.46, the normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.0125y$, with y measured in meters. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume the beam is made from elastic–perfectly plastic material that has a yield stress $\sigma_{\text{yield}} = 250$ MPa and a modulus of elasticity $E = 200$ GPa.

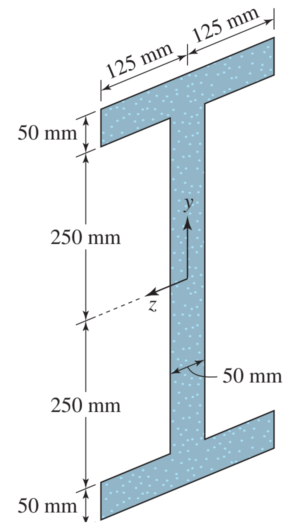


Figure 3.46 Beam cross section in Example 3.16.

PLAN

Points furthest from the origin will be the most strained, and the plastic zone will start from the top and bottom and move inward symmetrically. We can determine the yield strain $\varepsilon_{\text{yield}}$ from the given yield stress σ_{yield} and the modulus of elasticity E. We can then find the location of the elastic–plastic boundary by finding y_y at which the normal strain reaches the value of $\varepsilon_{\text{yield}}$. The normal stress before y_y can be found from Hooke's law, and after y_y it will be the yield stress.

SOLUTION

The location of the elastic–plastic boundary is given by

$$\varepsilon_{\text{yield}} = \frac{\sigma_{\text{yield}}}{E} = \frac{(\pm 250 \times 10^6)\ \text{N/m}^2}{200 \times 10^9\ \text{N/m}^2} = \pm 1.25 \times 10^{-3} = -0.0125 y_y \qquad \text{or}$$

$$y_y = \frac{\pm 1.25 \times 10^{-3}}{-0.0125} = \mp 0.1\ \text{m} \tag{E1}$$

Up to the elastic–plastic boundary, i.e., y_y, the material is in the linear range and Hooke's law applies. Thus,

$$\sigma_{xx} = (200 \times 10^9\ \text{N/m}^2)(-0.0125y) = -2500y\ \text{MPa} \qquad |y| \le 0.1\ \text{m} \tag{E2}$$

The normal stress as a function of y can be written as

ANS.
$$\sigma_{xx} = \begin{cases} -250\ \text{MPa}, & 0.1\ \text{m} \le y \le 0.3\ \text{m} \\ -2500y\ \text{MPa}, & -0.1\ \text{m} \le y \le 0.1\ \text{m} \\ 250\ \text{MPa}, & -0.3\ \text{m} \le y \le -0.1\ \text{m} \end{cases} \tag{E3}$$

The normal strain and stress as a function of y can be plotted as shown in Figure 3.47

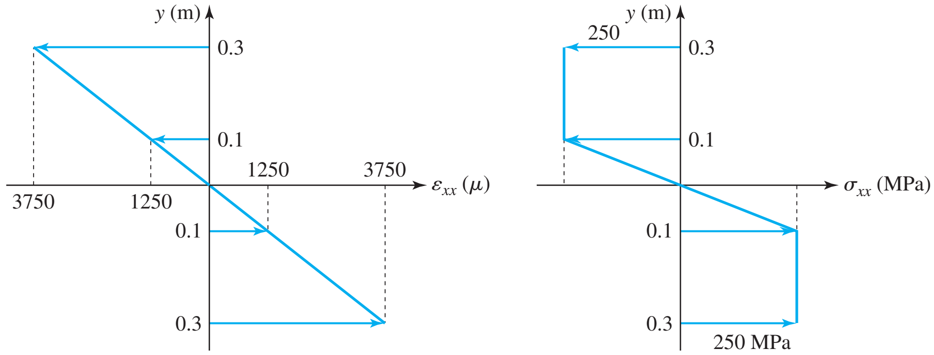


Figure 3.47 Strain and stress distributions in Example 3.16

COMMENTS

1. To better appreciate the stress distribution we can plot it across the entire cross section, as shown in Figure 3.48.

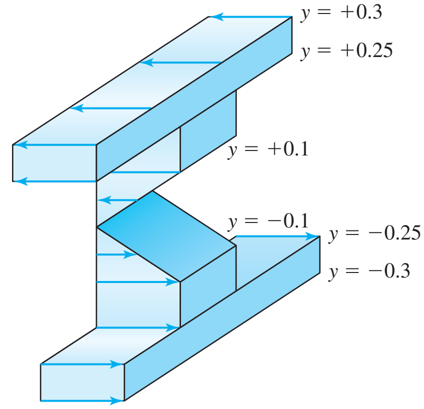


Figure 3.48 Stress distribution across cross section in Example 3.16

2. Once more we see that the stress distribution across the cross section will be difficult to deduce, but as will be seen in Chapter 6 for the symmetric bending of beams, we can deduce the approximate strain distribution from geometric considerations.

EXAMPLE 3.17

Resolve Example 3.16 assuming that the stress–strain relationship is given by $\sigma = 9000\varepsilon^{0.6}$ MPa in tension and in compression.

PLAN

We substitute the strain value in Equations (3.36) and obtain the equation for stress in terms of y.

SOLUTION

Substituting the strains in the stress–strain relation in Equations (3.36), we obtain

$$\sigma_{xx} = \begin{cases} 9000\varepsilon_{xx}^{0.6} \text{ MPa} & \varepsilon_{xx} \geq 0 \\ -9000(-\varepsilon_{xx})^{0.6} \text{ MPa} & \varepsilon_{xx} \leq 0 \end{cases} \quad \text{or}$$

$$\sigma_{xx} = \begin{cases} 9000(-0.0125y)^{0.6} \text{ MPa} & y \leq 0 \\ (-9000(0.0125y)^{0.6}) \text{ MPa} & y \geq 0 \end{cases}$$

$$\textbf{ANS.} \quad \sigma_{xx} = \begin{cases} 649.2(-y)^{0.6} \text{ MPa} & y \leq 0 \\ -649.2(y)^{0.6} \text{ MPa} & y \geq 0 \end{cases} \tag{E4}$$

The strains and stresses can be found at different values of y and plotted as shown in Figure 3.49.

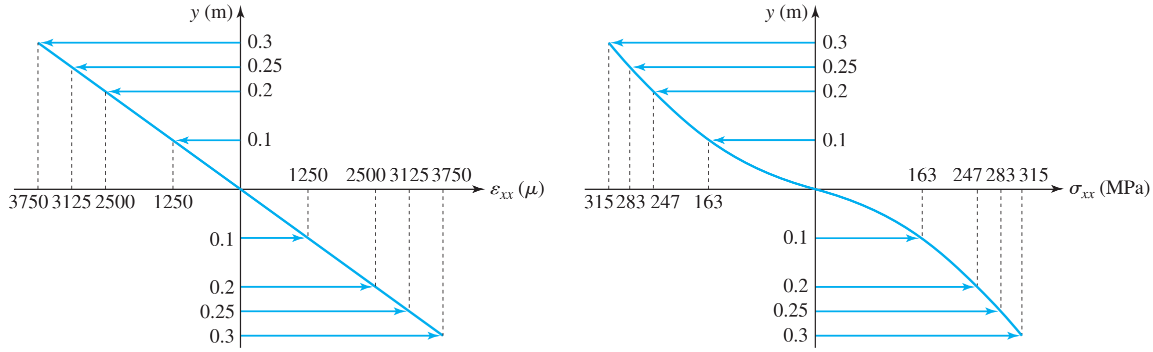


Figure 3.49 Strain and stress distributions in Example 3.17.

COMMENT

1. To better appreciate the stress distribution we can plot it across the entire cross section, as shown in Figure 3.50.

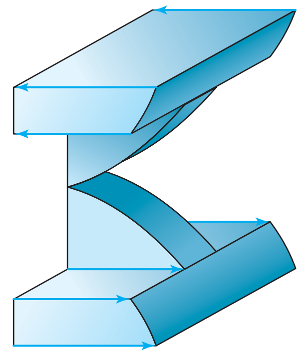


Figure 3.50 Stress distribution across cross section in Example 3.17.

PROBLEM SET 3.4

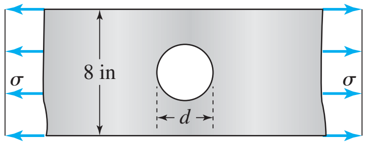
Fatigue

3.137 A machine component is made from a steel alloy that has an *S–N* curve as shown in Figure 3.36. Estimate the service life of the component if the peak stress is reversed at the rates shown. (a) 40 ksi at 200 cycles per minute. (b) 36 ksi at 250 cycles per minute. (c) 32 ksi at 300 cycles per minute.

3.138 A machine component is made from an aluminum alloy that has an *S–N* curve as shown in Figure 3.36. What should be the maximum permissible peak stress in MPa for the following situations: (a) 17 hours of service at 100 cycles per minute. (b) 40 hours of service at 50 cycles per minute. (c) 80 hours of service at 20 cycles per minute.

3.139 A uniaxial stress acts on an aluminum plate with a hole is shown in Figure P3.139. The aluminum has an S–N curve as shown in Figure 3.36. Predict the number of cycles the plate could be used if d = 3.2 in. and the far-field stress $\sigma = 6$ ksi.

Figure P3.139



3.140 A uniaxial stress acts on an aluminum plate with a hole is shown in Figure P3.139. The aluminum has an S–N curve as shown in Figure 3.36. Determine the maximum diameter of the hole to the nearest $\frac{1}{8}$ in., if the predicted service life of one-half million cycles is desired for a uniform far-field stress $\sigma = 6$ ksi.

3.141 A uniaxial stress acts on an aluminum plate with a hole is shown in Figure P3.139. The aluminum has an S–N curve as shown in Figure 3.36. Determine the maximum far-field stress σ if the diameter of the hole is 2.4 in. and a predicted service life of three-quarters of a million cycles is desired.

Nonlinear material models

3.142 Bronze has a yield stress σ_{yield} = 18 ksi in tension, a yield strain $\varepsilon_{\text{yield}}$ = 0.0012, ultimate stress σ_{ult} = 50 ksi, and the corresponding ultimate strain ε_{ult} = 0.50. Determine the material constants and plot the resulting stress–strain curve for (a) the elastic–perfectly plastic model. (b) the linear strain-hardening model. (c) the nonlinear power-law model.

3.143 Cast iron has a yield stress σ_{yield} = 220 MPa in tension, a yield strain $\varepsilon_{\text{yield}}$ = 0.00125, ultimate stress σ_{ult} = 340 MPa, and the corresponding ultimate strain ε_{ult} = 0.20. Determine the material constants and plot the resulting stress–strain curve for: (a) the elastic–perfectly plastic model. (b) the linear strain-hardening model. (c) the nonlinear power-law model.

3.144 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches. The shaft is made from an elastic–perfectly plastic material, which has a yield stress τ_{yield} = 18 ksi and a shear modulus G = 12,000 ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress $\tau_{x\theta}$ distributions across the cross section.

3.145 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches.The shaft is made form a bilinear material as shown in Figure 3.40. The material has a yield stress τ_{yield} = 18 ksi and shear moduli G_1 = 12,000 ksi and G_2 = 4800 ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress $\tau_{x\theta}$ distributions across the cross section.

3.146 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches.The shaft material has a stress–strain relationship given by $\tau = 243\gamma^{0.4}$ ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress $\tau_{x\theta}$ distributions across the cross section.

3.147 A solid circular shaft of 3-in diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho \times 10^{-3}$, where ρ is the radial coordinate measured in inches. The shaft material has a stress–strain relationship given by $\tau = 12{,}000\gamma - 120{,}000\gamma^2$ ksi. Write the expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain $\gamma_{x\theta}$ and shear stress τ_{xq} distributions across the cross section.

3.148 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters.The shaft is made from an elastic–perfectly plastic material that has a shear yield stress τ_{yield} = 175 MPa and a shear modulus G = 26 GPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.149 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters. The shaft is made from a bilinear material as shown in Figure 3.40. The material has a shear yield stress τ_{yield} = 175 MPa and shear moduli G_1 = 26 GPa and G_2 = 14 GPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.150 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters. The shaft material has a stress–strain relationship given by $\tau = 3435\gamma^{0.6}$ MPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.151 A hollow circular shaft has an inner diameter of 50 mm and an outside diameter of 100 mm. The shear strain at a section in polar coordinates was found to be $\gamma_{x\theta} = 0.2\rho$, where ρ is the radial coordinate measured in meters. The shaft material has a stress–strain relationship given by $\tau = 26{,}000\gamma - 208{,}000\gamma^2$ MPa. Write the expressions for τ_{xq} as a function of ρ and plot the shear strain γ_{xq} and shear stress τ_{xq} distributions across the cross section.

3.152 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam is made from an elastic–perfectly plastic material that has a yield stress $\sigma_{yield} = 250$ MPa and a modulus of elasticity $E = 200$ GPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression

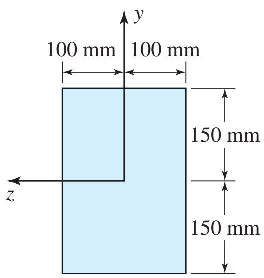


Figure P3.152

3.153 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam is made from a bilinear material as shown in Figure 3.40. The material has a yield stress $\sigma_{yield} = 250$ MPa and moduli of elasticity $E_1 = 200$ GPa and $E_2 = 80$ GPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression.

3.154 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam material has a stress–strain relationship given by $\sigma = 952\varepsilon^{0.2}$ MPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression.

3.155 A rectangular beam has the dimensions shown in Figure P3.152. The normal strain due to bending about the z axis was found to vary as $\varepsilon_{xx} = -0.01y$, with y measured in meters. The beam material has a stress–strain relationship given by $\sigma = 200\varepsilon - 2000\varepsilon^2$ MPa. Write the expressions for normal stress σ_{xx} as a function of y and plot the σ_{xx} distribution across the cross section. Assume similar material behavior in tension and compression.

3.12* CONCEPT CONNECTOR

Several pioneers concluded that formulas in the mechanics of materials depend on quantities that must be measured experimentally yet Thomas Young is given credit for discovering the modulus of elasticity. History also shows that there was a great controversy over the minimum number of independent constants needed to describe the linear relationship between stress and strain. The controversy took 80 years to resolve. Experimental data were repeatedly explained away when they did not support the prevalent theories at that time. Section 3.12.1 describes the vagaries of history in giving credit and the controversy over material constants.

Composite structural members are a growing area of application of mechanics of materials. Fishing rods, bicycle frames, the wings and control surfaces of aircrafts, tennis racquets, boat hulls, storage tanks, reinforced concrete bars, wooden beams stiffened with steel, laminated shafts, and fiberglass automobile bodies are just some examples. Section 3.12.2 discusses material grouping as a prelude to the discussion of composites in Section 3.12.3

3.12.1 History: Material Constants

Even as a child, Robert Hooke (1635-1703) took an interest in mechanical toys and drawings. In 1662 he was appointed curator of experiments for the Royal Society in England, thanks to his inventive abilities and his willingness to design apparatus in pursuit of ideas from other Society fellow as well as his own. A skilled architect and surveyor, he assisted Christopher Wren on the city that rose up again after the Great Fire of London. He left his mark as well in optics, astronomy, and biology, and indeed he coined the word *cell*. Among his many works are experiments on springs and elastic bodies. In 1678 he published his results, including the linear relationship between force and deformation now known as Hooke's law. In his words, "*Ut tensio sic vis*": as the extension, so is the force. (In 1680 in France, while conducting experiments with beams, Edme Mariotte arrived at the same linear relationship independently.) In acknowledgment of his work on elastic bodies, the stress–strain relation of Equation (3.1) is also called Hooke's law.

Leonard Euler (1707-1783), in his mathematical studies on beam buckling published in 1757, also used Hooke's law and introduced what he called the *moment of stiffness*. He suggested that this moment of stiffness could be determined experimentally. This moment of stiffness is the bending rigidity, which we will study in Chapter 6 on beam bending. It was Giordano Ricardi who proposed the idea that material constant must be determined experimentally. In 1782 he described the first six modes of vibration for chimes of brass and steel, giving values for the modulus of elasticity. Credit for defining and measuring the modulus of elasticity, however, is given to Thomas Young (1773-1829), and is often referred to as Young's modulus.

After Young resigned in 1803 from the Royal Institute in England, where he had been professor of natural philosophy, he published his course material. Here he defines the modulus of elasticity in terms of the pressure produced at the base of a column of given cross-sectional area due to its own weight. This definition includes the area of cross section, which is like the axial rigidity we will study in chapter 4 on axial members. The definition of the modulus of elasticity as purely a material property—independent of geometry—is a later development.

Thomas Young

Simeon Poisson

Figure 3.51 Constants named for these pioneers.

Poisson's ratio is part of a controversy that raged over most of the 19th century. The molecular theory of stress, initiated by the French engineer and physicist Claude-Louis Navier (1785-1836), is based on the central-force concept described in the Section 1.5. Navier himself derived the equilibrium equations, in terms of displacement, using this theory, but with only one independent material constant for isotropic bodies. In 1839 George Green started from the alternative viewpoint that at equilibrium the potential energy must be minimum. Green came to the conclusion that there must be two independent constants for the isotropic stress–strain relationship. From his own independent analysis, also using Navier's molecular theory of stress, Poisson had concluded that the Poisson's ratio must be 1/4. With this value of Poisson's ratio, the equilibrium equations of Green and Navier become identical. While Guillaume Wertheim's experiments on glass and brass in 1848 did not support this value, Wertheim continued to believe that only one independent material constant was needed.

Believers in the existence of one independent material constant dismissed experimental results on the basis that the materials on which the experiment was conducted were not truly isotropic. In the case of anisotropic material, Cauchy and Poisson (using Navier's molecular theory of stress) concluded that there were fifteen independent material constants, whereas Green's analysis showed that twenty-one independent material constants relate stress to strain. The two viewpoints could be resolved if there were six relationships between the material constants. Woldemar Voigt's experiments between 1887 and 1889 on single crystals with known anisotropic properties showed that the six relationships between the material constants were untenable. Nearly half a century after the deaths of Navier, Poisson, Cauchy, and Green, his experimental results finally resolved the controversy. Today we accept that isotropic materials have two independent constants in the general linear stress–strain relationship, whereas anisotropic materials have twenty-one.

3.12.2 Material Groups

There are thirty-one types of crystals. Bodies made up of these crystals can be grouped into classes, based on the independent material constants in the linear stress–strain relationship. The most general anisotropic material, which requires twenty-one independent constants, is also called *triclinic* material. Three other important nonisotropic material groups are monoclinic, orthotropic, and transversely isotropic materials.

- *Monoclinic* materials require thirteen independent material constants. Here the z plane is the plane of symmetry. This implies that the stress–strain relationships are the same in the positive and negative z directions.
- *Orthotropic* materials require nine independent constants. Orthotropic materials have two orthogonal planes of symmetry. In other words, if we rotate the material by 90° about the x or the y axis, we obtain the same stress–strain relationships.
- *Transversely isotropic* materials require five independent material constants. Transversely isotropic materials are isotropic in a plane. In other words, rotation by an arbitrary angle about the z axis does not change the stress–strain relationship, and the material is isotropic in the xy plane.
- *Isotropic* materials require only two independent material constants. Rotation about the x, y, or z axis by any arbitrary angle results in the same stress–strain relationship.

3.12.3 Composite Materials

A body made from more than one material can be called a *composite*. The ancient Egyptians made composite bricks for building the pyramids by mixing straw and mud. The resulting brick was stronger than the brick made from mud alone. Modern polymer composites rely on the same phenomenological effect in mixing fibers and epoxies to get high strength and high stiffness per unit weight.

Fibers are inherently stiffer and stronger than bulk material. Bulk glass such as in window panes has a breaking strength of a few thousand psi. Glass fibers, however, have a breaking strength on the order of one-half million psi. The increase in strength and stiffness is due to a reduction of defects and the alignment of crystals along the fiber axis. The plastic epoxy holds these high-strength and high-stiffness fibers together.

In long or continuous-fiber composites, a *lamina* is constructed by laying the fibers in a given direction and pouring epoxy on top. Clearly each lamina will have different mechanical properties in the direction of the fibers and in the direction perpendicular to the fibers. If the properties of the fibers and the epoxies are averaged (or *homogenized*), then each lamina can be regarded as an orthotropic material. Laminae with different fiber orientations are then put together to create a *laminate*, as shown in Figure 3.52. The overall properties of the laminate can be controlled by the orientation of the fibers and the stacking sequence of the laminae. The designer thus has additional design variables, and material properties can thus be tailored to the design requirements. Continuous-fiber composite technology is still very expensive compared to that of metals, but a significant weight reduction justifies its use in the aerospace industry and in specialty sports equipment.

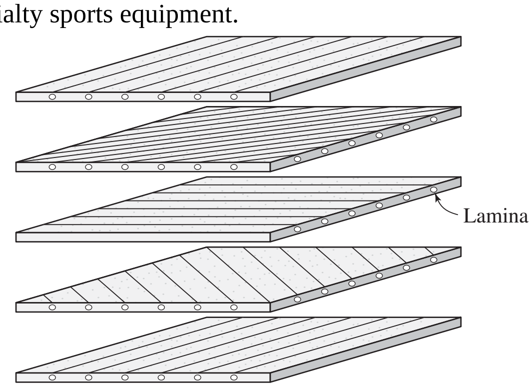


Figure 3.52 Laminate construction.

One way of producing short-fiber composites is to spray fibers onto epoxy and cure the mixture. The random orientation of the fibers results in an overall transversely isotropic material whose properties depend on the ratio of the volume of fibers to the volume of epoxy. Chopped fibers are cheaper to produce than the continuous-fiber composites and are finding increasing use in automobile and marine industries for designing secondary structures, such as body panels.

3.13 CHAPTER CONNECTOR

In this chapter we studied the many ways to describe material behavior and established the empirical relationships between stress and strain. We saw that the number of material constants needed depends on the material model we wish to incorporate into our analysis. The simplest material model is the linear, elastic, isotropic material that requires only two material constants.

We also studied how material models can be integrated into a logic by which we can relate displacements to external forces. A more complex material model changes the stress distribution across the cross section, but not the key equations—the relationships between displacements and strains or between stresses and internal forces and moments. Similarly, we can add complexity to the relationship between displacements and strains without changing the material model. Thus the modular structure of the logic permits us to add complexities at several points, then carry the complexity forward into the equations that are otherwise unchanged.

Chapters 4 through 7 will extend the logic shown in Figure 3.15 to axial members, torsion of circular shafts, and symmetric bending of beams. The idea is to develop the simplest possible theories for these structural members. To do so, we shall impose limitations and make assumptions regarding loading, material behavior, and geometry. The difference between limitations and assumptions is in the degree to which the theory must be modified when a limitation or assumption is not valid. An entire theory must be redeveloped if a limitation is to be overcome. In contrast, assumptions are points where complexities can be added, and the derivation path that was established for the simplified theory can then be repeated. Examples, problems, and optional sections will demonstrate the addition of complexities to the simplified theories.

All these theories will have certain limitations in common:

1. The length of the member is significantly (approximately 10 times) greater than the greatest dimension in the cross section. Approximations across the cross section are now possible, as the region of approximation is small. We will deduce constant or linear approximations of deformation across the cross section and confirm the validity of an approximation from photographs of deformed shapes.

2. We are away from regions of stress concentration, where displacements and stresses can be three-dimensional. The results from the simplified theories can be extrapolated into the region of stress concentration, as described in Section 3.7.

3. The variation of external loads or change in cross-sectional area is gradual, except in regions of stress concentration. The theory of elasticity shows that this limitation is necessary; otherwise the approximations across the cross section would be untenable.

4. The external loads are such that axial, torsion, and bending can be studied individually. This requires not only that the applied loads be in a given direction, but also that the loads pass through a specific point on the cross section.

Often reality is more complex than even the most sophisticated theory can explain, and the relationship between variables must be modified empirically. These empirically modified formulas of mechanics of materials form the basis of most structural and machine design.

POINTS AND FORMULAS TO REMEMBER

- The point up to which stress and strain are linearly related is called proportional limit.
- The largest stress in the stress–strain curve is called ultimate stress.
- The sudden decrease in the cross-sectional area after ultimate stress is called necking.
- The stress at the point of rupture is called fracture or rupture stress.
- The region of the stress–strain curve in which the material returns to the undeformed state when applied forces are removed is called elastic region.
- The region in which the material deforms permanently is called plastic region.
- The point demarcating the elastic from the plastic region is called yield point.
- The stress at yield point is called yield stress.
- The permanent strain when stresses are zero is called plastic strain.
- The offset yield stress is a stress that would produce a plastic strain corresponding to the specified offset strain.
- A material that can undergo large plastic deformation before fracture is called ductile material.
- A material that exhibits little or no plastic deformation at failure is called brittle material.
- Hardness is the resistance to indentation.
- Raising the yield point with increasing strain is called strain hardening.
- A ductile material usually yields when the maximum shear stress exceeds the yield shear stress of the material.
- A brittle material usually ruptures when the maximum tensile normal stress exceeds the ultimate tensile stress of the material.
- $\sigma = E\varepsilon$ (3.1) $\quad \nu = -\left(\dfrac{\varepsilon_{\text{lateral}}}{\varepsilon_{\text{longitudinal}}}\right)$ (3.2) $\quad \tau = G\gamma$ (3.3)
- where E is the modulus of elasticity, ν is Poisson’s ratio, and G is the shear modulus of elasticity.
- The slope of the tangent to the stress–strain curve at a given stress value is called tangent modulus.
- The slope of the line that joins the origin to the given stress value is called secant modulus.
- Failure implies that a component or a structure does not perform the function for which it was designed.
- Failure could be due to too little or too much deformation or strength.
- Factor of safety: $K_{\text{safety}} = \dfrac{\text{failure-producing value}}{\text{computed (allowable) value}}$ (3.10)
- The factor of safety must always be greater than 1.
- The failure-producing value could be the value of deformation, yield stress, ultimate stress, or loads on a structure.
- An isotropic material has a stress–strain relation that is independent of the orientation of the coordinate system.
- In a homogeneous material the material constants do not change with the coordinates x, y, or z of a point.
- There are only two independent material constants in a linear stress–strain relationship for an isotropic material, but there can be 21 independent material constants in an anisotropic material.
- Generalized Hooke’s law for isotropic materials:
- $$\begin{aligned} \varepsilon_{xx} &= [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E & \gamma_{xy} &= \tau_{xy}/G \\ \varepsilon_{yy} &= [\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})]/E & \gamma_{yz} &= \tau_{yz}/G \\ \varepsilon_{zz} &= [\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})]/E & \gamma_{zx} &= \tau_{zx}/G \end{aligned} \qquad G = \frac{E}{2(1+\nu)}$$ (3.14a) through (3.14f)

CHAPTER FOUR

AXIAL MEMBERS

Learning objectives

1. Understand the theory, its limitations, and its applications for the design and analysis of axial members.
2. Develop the discipline to draw free-body diagrams and approximate deformed shapes in the design and analysis of structures.

The tensile forces supporting the weight of the Mackinaw bridge (Figure 4.1a) act along the longitudinal axis of each cable. The compressive forces raising the weight of the dump on a truck act along the axis of the hydraulic cylinders. The cables and hydraulic cylinders are **axial members**, long straight bodies on which the forces are applied along the longitudinal axis. Connecting rods in an engine, struts in aircraft engine mounts, members of a truss representing a bridge or a building, spokes in bicycle wheels, columns in a building—all are examples of axial members

(a) (b)

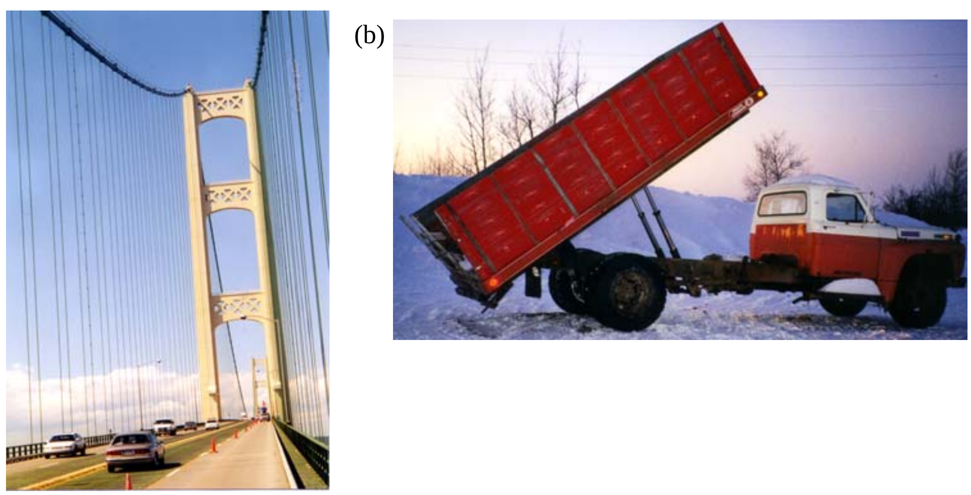


Figure 4.1 Axial members: (a) Cables of Mackinaw bridge. (b) Hydraulic cylinders in a dump truck.

This chapter develops the simplest theory for axial members, following the logic shown in Figure 3.15 but subject to the limitations described in Section 3,13. We can then apply the formulas to statically determinate and indeterminate structures. The two most important tools in our analysis will be free-body diagrams and approximate deformed shapes.

4.1 PRELUDE TO THEORY

As a prelude to theory, we consider two numerical examples solved using the logic discussed in Section 3.2. Their solution will highlight conclusions and observations that will be formalized in the development of the theory in Section 4.2.

EXAMPLE 4.1

Two thin bars are securely attached to a rigid plate, as shown in Figure 4.2. The cross-sectional area of each bar is 20 mm^2. The force F is to be placed such that the rigid plate moves only horizontally by 0.05 mm without rotating. Determine the force F and its location h for the following two cases: (a) Both bars are made from steel with a modulus of elasticity $E = 200$ GPa. (b) Bar 1 is made of steel ($E = 200$ GPa) and bar 2 is made of aluminum ($E = 70$ GPa).

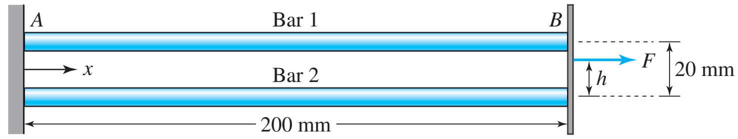


Figure 4.2 Axial bars in Example 4.1.

PLAN

The relative displacement of point B with respect to A is 0.05 mm, from which we can find the axial strain. By multiplying the axial strain by the modulus of elasticity, we can obtain the axial stress. By multiplying the axial stress by the cross-sectional area, we can obtain the internal axial force in each bar. We can draw the free-body diagram of the rigid plate and by equilibrium obtain the force F and its location h.

SOLUTION

1. *Strain calculations:* The displacement of B is $u_B = 0.05$ mm. Point A is built into the wall and hence has zero displacement. The normal strain is the same in both rods:

$$\varepsilon_1 = \varepsilon_2 = \frac{u_B - u_A}{x_B - x_A} = \frac{0.05 \text{ mm}}{200 \text{ mm}} = 250 \ \mu\text{mm/mm} \qquad \text{(E1)}$$

2. *Stress calculations:* From Hooke's law $\sigma = E\varepsilon$, we can find the normal stress in each bar for the two cases.

Case (a): Because E and ε_1 are the same for both bars, the stress is the same in both bars. We obtain

$$\sigma_1 = \sigma_2 = (200 \times 10^9 \text{ N/m}^2) \times 250 \times 10^{-6} = 50 \times 10^6 \text{ N/m}^2 (\text{T}) \qquad \text{(E2)}$$

Case (b): Because E is different for the two bars, the stress is different in each bar

$$\sigma_1 = E_1\varepsilon_1 = (200 \times 10^9 \text{ N/m}^2) \times 250 \times 10^{-6} = 50 \times 10^6 \text{ N/m}^2 (\text{T}) \qquad \text{(E3)}$$

$$\sigma_2 = E_2\varepsilon_2 = 70 \times 10^9 \times 250 \times 10^{-6} = 17.5 \times 10^6 \text{ N/m}^2 (\text{T}) \qquad \text{(E4)}$$

3. *Internal forces:* Assuming that the normal stress is uniform in each bar, we can find the internal normal force from $\boldsymbol{N} = \sigma A$, where $A = 20 \text{ mm}^2 = 20 \times 10^{-6} \text{ m}^2$.

Case (a): Both bars have the same internal force since stress and cross-sectional area are the same,

$$\boldsymbol{N}_1 = \boldsymbol{N}_2 = (50 \times 10^6 \text{ N/m}^2)(20 \times 10^{-6} \text{ m}^2) = 1000 \text{ N (T)} \qquad \text{(E5)}$$

Case (b): The equivalent internal force is different for each bar as stresses are different.

$$N_1 = \sigma_1 A_1 = (50 \times 10^6 \text{ N/m}^2)(20 \times 10^{-6} \text{ m}^2) = 1000 \text{ N (T)} \qquad \text{(E6)}$$

$$\boldsymbol{N}_2 = \sigma_2 A_2 = (17.5 \times 10^6 \text{ N/m}^2)(20 \times 10^{-6} \text{ m}^2) = 350 \text{ N (T)} \qquad \text{(E7)}$$

4. *External force:* We make an imaginary cut through the bars, show the internal axial forces as tensile, and obtain free-body diagram shown in Figure 4.3. By equilibrium of forces in x direction we obtain

$$F = \boldsymbol{N}_1 + \boldsymbol{N}_2 \qquad \text{(E8)}$$

By equilibrium of moment point O in Figure 4.3, we obtain

$$\boldsymbol{N}_1(20 - h) - \boldsymbol{N}_2 h = 0 \qquad \text{(E9)}$$

$$h = \frac{20\boldsymbol{N}_1}{\boldsymbol{N}_1 + \boldsymbol{N}_2} \qquad \text{(E10)}$$

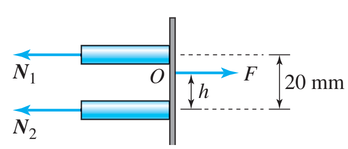


Figure 4.3 Free-body diagram in Example 4.1.

Case (a): Substituting Equation (E5) into Equations (E8) and (E10), we obtain F and h:

$$F = 1000 \text{ N} + 1000 \text{ N} = 2000 \text{ N} \qquad h = \frac{20 \text{ mm} \times 1000 \text{ N}}{(1000 \text{ N} + 1000 \text{ N})} = 10 \text{ mm}$$

ANS. $F = 2000$ N $\quad h = 10$ mm

Case (b): Substituting Equations (E6) and (E7) into Equations (E8) and (E10), we obtain F and h:

$$F = 1000\text{ N} + 350\text{ N} = 1350\text{ N} \qquad h = \frac{20\text{ mm} \times 1000\text{ N}}{(1000\text{ N} + 350\text{ N})} = 14.81\text{ mm}$$

ANS. $F = 1350$ N $h = 14.81$ mm

COMMENTS

1. Both bars, irrespective of the material, were subjected to the same axial strain. This is the fundamental kinematic assumption in the development of the theory for axial members, discussed in Section 4.2.
2. The sum on the right in Equation (E8) can be written $\sum_{i=1}^{n=2} \sigma_i \, \Delta A_i$, where σ_i is the normal stress in the ith bar, ΔA_i is the cross-sectional area of the ith bar, and $n = 2$ reflects that we have two bars in this problem. If we had n bars attached to the rigid plate, then the total axial force would be given by summation over n bars. As we increase the number of bars n to infinity, the cross-sectional area ΔA_i tends to zero (or infinitesimal area dA) as we try to fit an infinite number of bars on the same plate, resulting in a continuous body. The sum then becomes an integral, as discussed in Section 4.1.1.
3. If the external force were located at any point other than that given by the value of h, then the plate would rotate. Thus, for pure axial problems with no bending, a point on the cross section must be found such that the internal moment from the axial stress distribution is zero. To emphasize this, consider the left side of Equation (E9), which can be written as $\sum_{i=1}^{n} y_i \sigma_i \, \Delta A_i$, where y_i is the coordinate of the ith rod's centroid. The summation is an expression of the internal moment that is needed for static equivalency. This internal moment must equal zero if the problem is of pure axial deformation, as discussed in Section 4.1.1.
4. Even though the strains in both bars were the same in both cases, the stresses were different when E changed. Case (a) corresponds to a homogeneous cross section, whereas case (b) is analogous to a laminated bar in which the non-homogeneity affects the stress distribution.

4.1.1 Internal Axial Force

In this section we formalize the key observation made in Example 4.1: the normal stress σ_{xx} can be replaced by an equivalent internal axial force using an integral over the cross-sectional area. Figure 4.4 shows the statically equivalent systems. The axial force on a differential area $\sigma_{xx}\, dA$ can be integrated over the entire cross section to obtain

$$\boxed{\boldsymbol{N} = \int_A \sigma_{xx}\, dA} \tag{4.1}$$

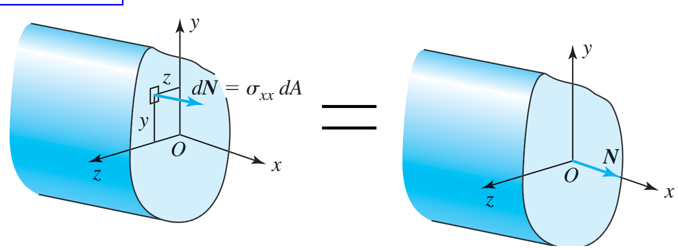


Figure 4.4 Statically equivalent internal axial force.

If the normal stress distribution σ_{xx} is to be replaced by only an axial force at the origin, then the internal moments $\boldsymbol{M}_y$ and $\boldsymbol{M}_z$ must be zero at the origin, and from Figure 4.4 we obtain

$$\int_A y\sigma_{xx}\, dA = 0 \tag{4.2a}$$

$$\int_A z\sigma_{xx}\, dA = 0 \tag{4.2b}$$

Equations (4.1), (4.2a), and (4.2b) are independent of the material models because they represent static equivalency between the normal stress on the cross section and internal axial force. If we were to consider a laminated cross section or nonlinear material, then it would affect the value and distribution of σ_{xx} across the cross section, but Equation (4.1) relating σ_{xx} and $\boldsymbol{N}$ would remain unchanged, and so would the zero moment condition of Equations (4.2a) and (4.2b). Equations (4.2a) and (4.2b) are used to determine the location at which the internal and external forces have to act for pure axial problem without bending, as discussed in Section 4.2.6.

EXAMPLE 4.2

Figure 4.5 shows a homogeneous wooden cross section and a cross section in which the wood is reinforced with steel. The normal strain for both cross sections is uniform, $\varepsilon_{xx} = -200\ \mu$. The moduli of elasticity for steel and wood are $E_{steel} = 30{,}000$ ksi and $E_{wood} = 8000$ ksi. (a) Plot the σ_{xx} distribution for each of the two cross sections shown. (b) Calculate the equivalent internal axial force $\boldsymbol{N}$ for each cross section using Equation (4.1).

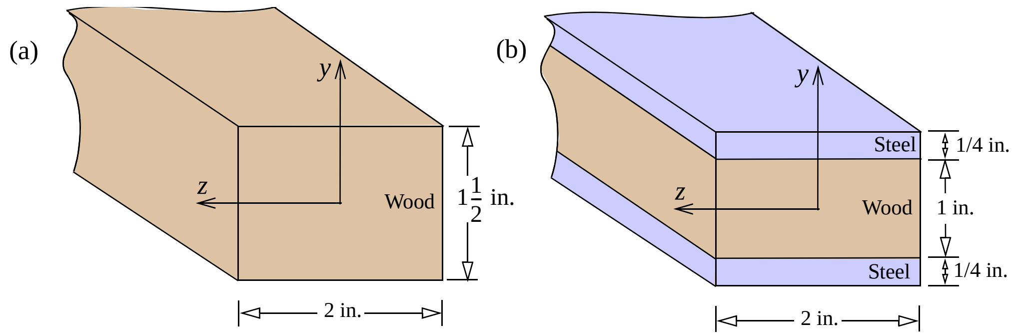


Figure 4.5 Cross sections in Example 4.2. (*a*) Homogeneous. (*b*) Laminated.

PLAN

(a) Using Hooke's law we can find the stress values in each material. Noting that the stress is uniform in each material, we can plot it across the cross section. (b) For the homogeneous cross section we can perform the integration in Equation (4.1) directly. For the nonhomogeneous cross section we can write the integral in Equation (4.1) as the sum of the integrals over steel and wood and then perform the integration to find $\boldsymbol{N}$.

SOLUTION

(a) From Hooke's law we can write

$$(\sigma_{xx})_{wood} = (8000\text{ ksi})(-200)10^{-6} = -1.6\text{ ksi} \tag{E1}$$

$$(\sigma_{xx})_{steel} = (30000\text{ ksi})(-200)10^{-6} = -6\text{ ksi} \tag{E2}$$

For the homogeneous cross section the stress distribution is as given in Equation (E1), but for the laminated case it switches to Equation (E2), depending on the location of the point where the stress is being evaluated, as shown in Figure 4.6.

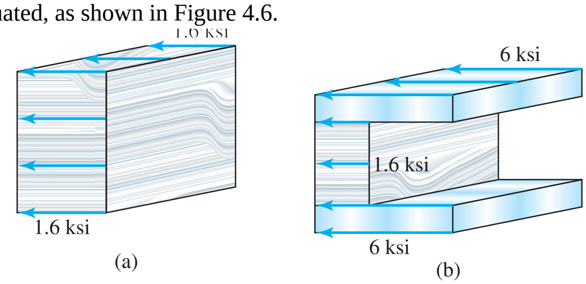


Figure 4.6 Stress distributions in Example 4.2. (*a*) Homogeneous cross section. (*b*) Laminated cross section.

(b) **Homogeneous cross section:** Substituting the stress distribution for the homogeneous cross section in Equation (4.1) and integrating, we obtain the equivalent internal axial force,

$$\boldsymbol{N} = \int_A (\sigma_{xx})_{wood}\, dA = (\sigma_{xx})_{wood} A = (-1.6\text{ ksi})(2\text{ in.})(1.5\text{ in.}) = -4.8\text{ kips} \tag{E3}$$

ANS. $\boldsymbol{N} = 4.8$ kips (C)

Laminated cross section: The stress value changes as we move across the cross section. Let A_{sb} and A_{st} represent the cross-sectional areas of steel at the bottom and the top. Let A_w represent the cross-sectional area of wood. We can write the integral in Equation (4.1) as the sum of three integrals, substitute the stress values of Equations (E1) and (E2), and perform the integration:

$$\boldsymbol{N} = \int_{A_{sb}} \sigma_{xx}\, dA + \int_{A_w} \sigma_{xx}\, dA + \int_{A_{st}} \sigma_{xx}\, dA = \int_{A_{sb}} (\sigma_{xx})_{steel}\, dA + \int_{A_w} (\sigma_{xx})_{wood}\, dA + \int_{A_{st}} (\sigma_{xx})_{steel}\, dA \quad \text{or} \tag{E4}$$

$$\boldsymbol{N} = (\sigma_{xx})_{steel} A_{sb} + (\sigma_{xx})_{wood} A_w + (\sigma_{xx})_{steel} A_{st} \quad \text{or} \tag{E5}$$

$$\boldsymbol{N} = (-6\text{ ksi})(2\text{ in.})\left(\frac{1}{4}\text{ in.}\right) + (-1.6\text{ ksi})(1\text{ in.})(2\text{ in.}) + (-6\text{ ksi})(2\text{ in.})\left(\frac{1}{4}\text{ in.}\right) = -9.2\text{ kips} \tag{E6}$$

ANS. $\boldsymbol{N} = 9.2$ kips (C)

COMMENTS

1. Writing the integral in the internal axial force as the sum of integrals over each material, as in Equation (E4), is equivalent to calculating the internal force carried by each material and then summing, as shown in Figure 4.7.

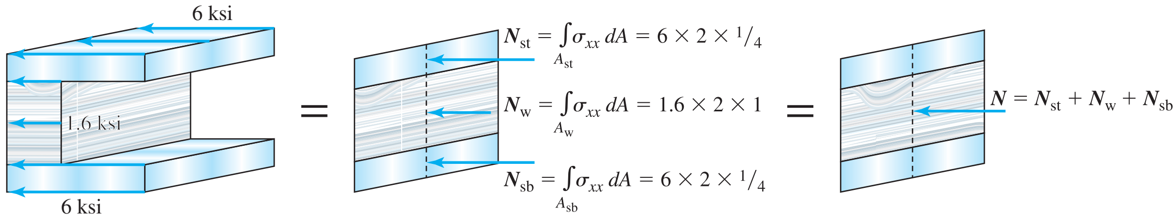


Figure 4.7 Statically equivalent internal force in Example 4.2 for laminated cross section.

2. The cross section is geometrically as well as materially symmetric. Thus we can locate the origin on the line of symmetry. If the lower steel strip is not present, then we will have to determine the location of the equivalent force.
3. The example demonstrates that although the strain is uniform across the cross section, the stress is not. We considered material non-homogeneity in this example. In a similar manner we can consider other models, such as elastic–perfectly plastic or material models that have nonlinear stress–strain curves.

PROBLEM SET 4.1

4.1 Aluminum bars ($E = 30{,}000$ ksi) are welded to rigid plates, as shown in Figure P4.1. All bars have a cross-sectional area of 0.5 in^2. Due to the applied forces the rigid plates at A, B, C, and D are displaced in x direction without rotating by the following amounts: $u_A = -0.0100$ in., $u_B = 0.0080$ in., $u_C = -0.0045$ in., and $u_D = 0.0075$ in. Determine the applied forces F_1, F_2, F_3, and F_4.

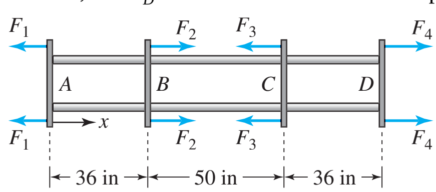


Figure P4.1

4.2 Brass bars between sections A and B, aluminum bars between sections B and C, and steel bars between sections C and D are welded to rigid plates, as shown in Figure P4.2. The rigid plates are displaced in the x direction without rotating by the following amounts: $u_B = -1.8$ mm, $u_C = 0.7$ mm, and $u_D = 3.7$ mm. Determine the external forces F_1, F_2, and F_3 using the properties given in Table P4.2

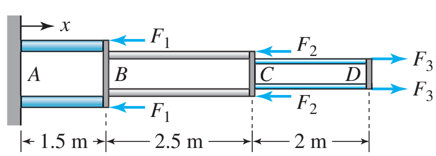


Figure P4.2

TABLE P4.2

	Brass	Aluminum	Steel
Modulus of elasticity	70 GPa	100 GPa	200 GPa
Diameter	30 mm	25 mm	20 mm

4.3 The ends of four circular steel bars ($E = 200$ GPa) are welded to a rigid plate, as shown in Figure P4.3. The other ends of the bars are built into walls. Owing to the action of the external force F, the rigid plate moves to the right by 0.1 mm without rotating. If the bars have a diameter of 10 mm, determine the applied force F.

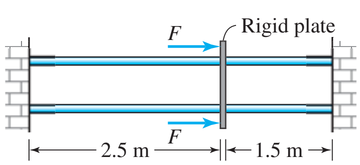


Figure P4.3

4.4 Rigid plates are securely fastened to bars A and B, as shown in Figure P4.4. A gap of 0.02 in. exists between the rigid plates before the forces are applied. After application of the forces the normal strain in bar A was found to be 500 μ. The cross-sectional area and the modulus of elasticity for each bar are as follows: $A_A = 1$ in.^{2}, $E_A = 10{,}000$ ksi, $A_B = 0.5$ in.^{2}, and $E_B = 30{,}000$ ksi. Determine the applied forces F, assuming that the rigid plates do not rotate.

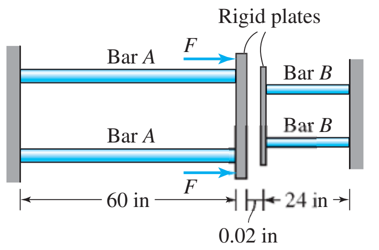


Figure P4.4

4.5 The strain at a cross section shown in Figure P4.5 of an axial rod is assumed to have the uniform value $\varepsilon_{xx} = 200\ \mu$. (a) Plot the stress distribution across the laminated cross section. (b) Determine the equivalent internal axial force $\boldsymbol{N}$ and its location from the bottom of the cross section. Use $E_{\text{alu}} = 100$ GPa, $E_{\text{wood}} = 10$ GPa, and $E_{\text{steel}} = 200$ GPa.

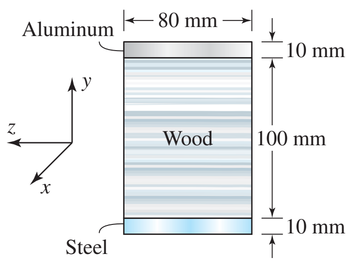


Figure P4.5

4.6 A reinforced concrete bar shown in Figure P4.6 is constructed by embedding 2-in. × 2-in. square iron rods. Assuming a uniform strain $\varepsilon_{xx} = -1500\ \mu$ in the cross section, (a) plot the stress distribution across the cross section; (b) determine the equivalent internal axial force $\boldsymbol{N}$. Use $E_{\text{iron}} = 25{,}000$ ksi and $E_{\text{conc}} = 3000$ ksi.

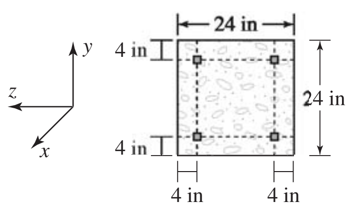


Figure P4.6

4.2 THEORY OF AXIAL MEMBERS

In this section we will follow the procedure in Section 4.1 with variables in place of numbers to develop formulas for axial deformation and stress. The theory will be developed subject to the following limitations:

1. The length of the member is significantly greater than the greatest dimension in the cross section.
2. We are away from the regions of stress concentration.
3. The variation of external loads or changes in the cross-sectional areas is gradual, except in regions of stress concentration.
4. The axial load is applied such that there is no bending.
5. The external forces are not functions of time that is, we have a static problem. (See Problems 4.37, 4.38, and 4.39 for dynamic problems.)

Figure 4.8 shows an externally distributed force per unit length $p(x)$ and external forces F_1 and F_2 acting at each end of an axial bar. The cross-sectional area $A(x)$ can be of any shape and could be a function of x.

Sign convention: The displacement u is considered positive in the positive x direction. The internal axial force $\boldsymbol{N}$ is considered positive in tension negative in compression.

The theory has two objectives:

1. To obtain a formula for the relative displacements $u_2 - u_1$ in terms of the internal axial force $\boldsymbol{N}$.
2. To obtain a formula for the axial stress σ_{xx} in terms of the internal axial force $\boldsymbol{N}$.

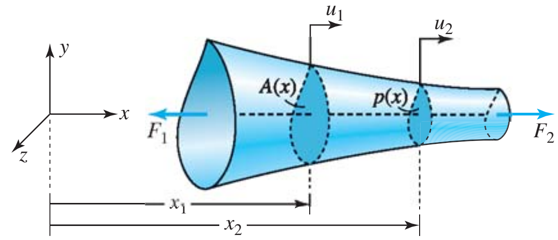


Figure 4.8 Segment of an axial bar.

We will take $\Delta x = x_2 - x_1$ as an infinitesimal distance so that the gradually varying distributed load $p(x)$ and the cross-sectional area $A(x)$ can be treated as constants. We then approximate the deformation across the cross section and apply the logic shown in Figure 4.9. The assumptions identified as we move from each step are also points at which complexities can later be added, as discussed in examples and "Stretch Yourself" problems.

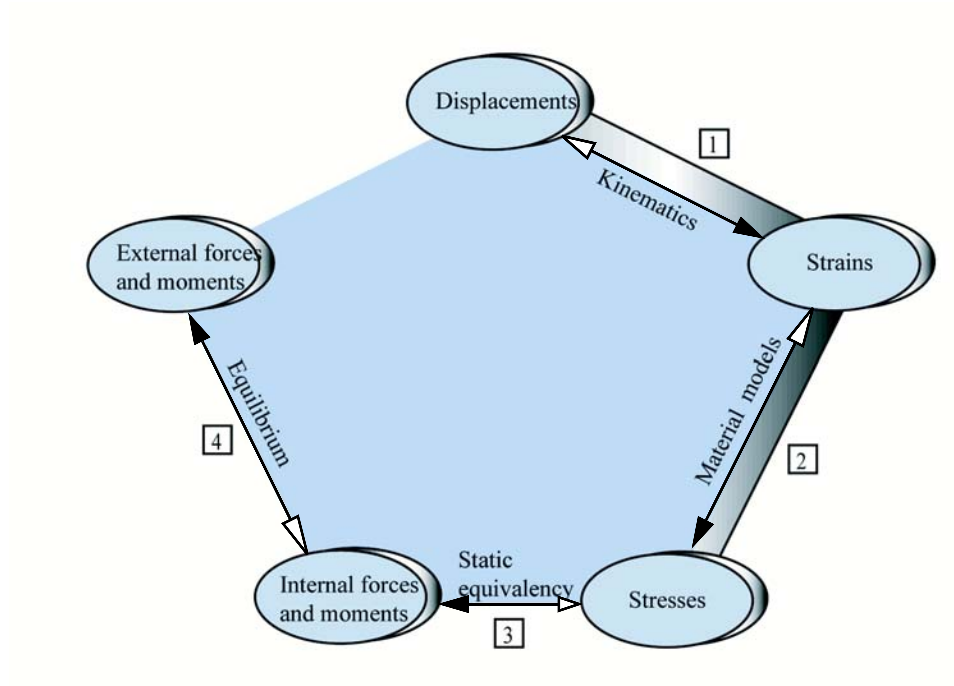


Figure 4.9 Logic in mechanics of materials.

4.2.1 Kinematics

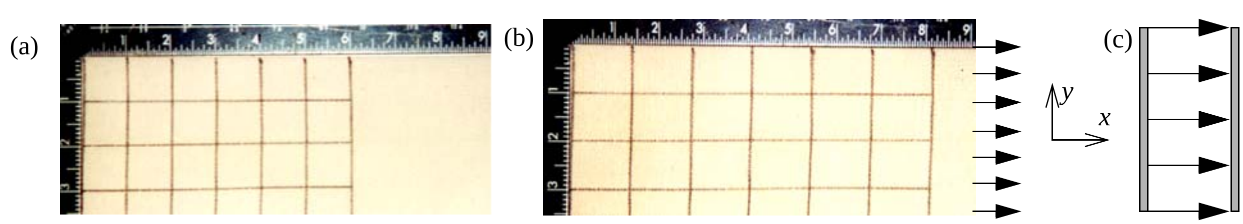


Figure 4.10 Axial deformation: (*a*) original grid; (*b*) deformed grid. (c) u is constant in y direction.

Figure 4.10 shows a grid on an elastic band that is pulled in the axial direction. The vertical lines remain approximately vertical, but the horizontal distance between the vertical lines changes. Thus all points on a vertical line are displaced by equal amounts. If this surface observation is also true in the interior of an axial member, then all points on a cross-section displace by equal amounts, but each cross-section can displace in the x direction by a different amount, leading to Assumption 1.

Assumption 1 Plane sections remain plane and parallel.

Assumption 1 implies that u cannot be a function of y but can be a function of x

$$u = u(x) \tag{4.3}$$

As an alternative perspective, because the cross section is significantly smaller than the length, we can approximate a function such as u by a constant treating it as uniform over a cross section. In Chapter 6, on beam bending, we shall approximate u as a linear function of y.

4.2.2 Strain Distribution

Assumption 2 Strains are small.[1]

If points x_2 and x_1 are close in Figure 4.8, then the strain at any point x can be calculated as

$$\varepsilon_{xx} = \lim_{\Delta x \to 0}\left(\frac{u_2 - u_1}{x_2 - x_1}\right) = \lim_{\Delta x \to 0}\left(\frac{\Delta u}{\Delta x}\right) \quad \text{or}$$

$$\boxed{\varepsilon_{xx} = \frac{du}{dx}(x)} \tag{4.4}$$

Equation (4.4) emphasizes that the axial strain is uniform across the cross section and is only a function of x. In deriving Equation (4.4) we made no statement regarding material behavior. In other words, Equation (4.4) does not depend on the material model if Assumptions 1 and 2 are valid. But clearly if the material or loading is such that Assumptions 1 and 2 are not tenable, then Equation (4.4) will not be valid.

4.2.3 Material Model

Our motivation is to develop a simple theory for axial deformation. Thus we make assumptions regarding material behavior that will permit us to use the simplest material model given by Hooke's law.

Assumption 3 Material is isotropic.

Assumption 4 Material is linearly elastic.[2]

Assumption 5 There are no inelastic strains.[3]

Substituting Equation (4.4) into Hooke's law, that is, $\sigma_{xx} = E\,\varepsilon_{xx}$, we obtain

$$\sigma_{xx} = E\frac{du}{dx} \tag{4.5}$$

Though the strain does not depend on y or z, we cannot say the same for the stress in Equation (4.5) since E could change across the cross section, as in laminated or composite bars.

4.2.4 Formulas for Axial Members

Substituting σ_{xx} from Equation (4.5) into Equation (4.1) and noting that du/dx is a function of x only, whereas the integration is with respect to y and z ($dA = dy\,dz$), we obtain

$$\mathbf{N} = \int_A E\frac{du}{dx}\,dA = \frac{du}{dx}\int_A E\,dA \tag{4.6}$$

[1]See Problem 4.40 for large strains.
[2]See Problem 4.36 for nonlinear material behavior.
[3]Inelastic strains could be due to temperature, humidity, plasticity, viscoelasticity, and so on. We shall consider inelastic strains due to temperature in Section 4.5.

Consistent with the motivation for the simplest possible formulas, E should not change across the cross section as implied in Assumption 6. We can take E outside the integral.

Assumption 6 Material is homogeneous across the cross section.

With material homogeneity, we then obtain

$$N = E\frac{\mathrm{d}u}{\mathrm{d}x}\int_A dA = EA\frac{\mathrm{d}u}{\mathrm{d}x} \text{ or}$$

$$\boxed{\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{N}{EA}} \tag{4.7}$$

The higher the value of EA, the smaller will be the deformation for a given value of the internal force. Thus the rigidity of the bar increases with the increase in EA. This implies that an axial bar can be made more rigid by either choosing a stiffer material (a higher value of E) or increasing the cross-sectional area, or both. Example 4.5 brings out the importance of axial rigidity in design. The quantity EA is called **axial rigidity**.

Substituting Equation (4.7) into Equation (4.5), we obtain

$$\boxed{\sigma_{xx} = \frac{N}{A}} \tag{4.8}$$

In Equation (4.8)**,** N and A do not change across the cross section and hence axial stress is uniform across the cross section. We have used Equation (4.8) in Chapters 1 and 3, but this equation is valid only if all the limitations are imposed, and if Assumptions 1 through 6 are valid.

We can integrate Equation (4.7) to obtain the deformation between two points:

$$u_2 - u_1 = \int_{u_1}^{u_2} du = \int_{x_1}^{x_2} \frac{N}{EA}\,dx \tag{4.9}$$

where u_1 and u_2 are the displacements of sections at x_1 and x_2, respectively. To obtain a simple formula we would like to take the three quantities N, E, and A outside the integral, which means these quantities should not change with x. To achieve this simplicity, we make the following assumptions:

Assumption 7 The material is homogeneous between x_1 and x_2. (E is constant)

Assumption 8 The bar is not tapered between x_1 and x_2. (A is constant)

Assumption 9 The external (hence internal) axial force does not change with x between x_1 and x_2. (N is constant)

If Assumptions 7 through 9 are valid, then N, E, and A are constant between x_1 and x_2, and we obtain

$$\boxed{u_2 - u_1 = \frac{N(x_2 - x_1)}{EA}} \tag{4.10}$$

In Equation (4.10), points x_1 and x_2 must be chosen such that neither N, E, nor A changes between these points.

4.2.5 Sign Convention for Internal Axial Force

The axial stress σ_{xx} was replaced by a statically equivalent internal axial force N. Figure 4.11 shows the sign convention for the positive axial force as tension.

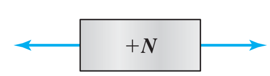


Figure 4.11 Sign convention for positive internal axial force.

N is an internal axial force that has to be determined by making an imaginary cut and drawing a free-body diagram. In what direction should N be drawn on the free-body diagram? There are two possibilities:

1. N is always drawn in tension on the imaginary cut as per our sign convention. The equilibrium equation then gives a positive or a negative value for N. A positive value of σ_{xx} obtained from Equation (4.8) is tensile and a negative value is

compressive. Similarly, the relative deformation obtained from Equation (4.10) is extension for positive values and contraction for negative values. The displacement *u* will be positive in the positive *x* direction.

2. ***N*** is drawn on the imaginary cut in a direction to equilibrate the external forces. Since inspection is being used in determining the direction of ***N***, tensile and compressive σ_{xx} and extension or contraction for the relative deformation must also be determined by inspection.

4.2.6 Location of Axial Force on the Cross Section

For pure axial deformation the internal bending moments must be zero. Equations (4.2a) and (4.2b) can then be used to determine the location of the point where the internal axial force and hence the external forces must pass for pure axial problems. Substituting Equation (4.5) into Equations (4.2a) and (4.2b) and noting that du/dx is a function of *x* only, whereas the integration is with respect to *y* and *z* ($dA = dy\,dz$), we obtain

$$\int_A y\sigma_{xx}\,dA = \int_A yE\frac{du}{dx}\,dA = \frac{du}{dx}\int_A yE\,dA = 0 \text{ or}$$

$$\int_A yE\,dA = 0 \tag{4.11a}$$

$$\int_A z\sigma_{xx}\,dA = \int_A zE\frac{du}{dx}\,dA = \frac{du}{dx}\int_A zE\,dA = 0 \text{ or}$$

$$\int_A zE\,dA = 0 \tag{4.11b}$$

Equations (4.11a) and (4.11b) can be used to determine the location of internal axial force for composite materials. If the cross section is homogenous (Assumption 6), then *E* is constant across the cross section and can be taken out side the integral:

$$\int_A y\,dA = 0 \tag{4.12a}$$

$$\int_A z\,dA = 0 \tag{4.12b}$$

Equations (4.12a) and (4.12b) are satisfied if *y* and *z* are measured from the centroid. (See Appendix A.4.) We will have *pure axial deformation if the external and internal forces are colinear and passing through the centroid of a homogenous cross section*. This assumes implicitly that the centroids of all cross sections must lie on a straight line. This eliminates curved but not tapered bars.

4.2.7 Axial Stresses and Strains

In the Cartesian coordinate system all stress components except σ_{xx} are assumed zero. From the generalized Hooke's law for isotropic materials, given by Equations (3.14a) through (3.14c), we obtain the normal strains for axial members:

$$\boxed{\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx}} \tag{4.13}$$

where ν is the Poisson's ratio. In Equation (4.13), the normal strains in *y* and *z* directions are due to Poisson's effect. Assumption 1, that plane sections remain plane and parallel implies that no right angle would change during deformation, and hence the assumed deformation implies that shear strains in axial members are zero. Alternatively, if shear stresses are zero, then by Hooke's law shear strains are zero.

EXAMPLE 4.3

Solid circular bars of brass (E_{br} = 100 GPa, ν_{br} = 0.34) and aluminum (E_{al} = 70 GPa, ν_{al} = 0.33) having 200 mm diameter are attached to a steel tube (E_{st} = 210 GPa, ν_{st} = 0.3) of the same outer diameter, as shown in Figure 4.12. For the loading shown determine: (a) The movement of the plate at *C* with respect to the plate at *A*. (b) The change in diameter of the brass cylinder. (c) The maximum inner diameter to the nearest millimeter in the steel tube if the factor of safety with respect to failure due to yielding is to be at least 1.2. The yield stress for steel is 250 MPa in tension.

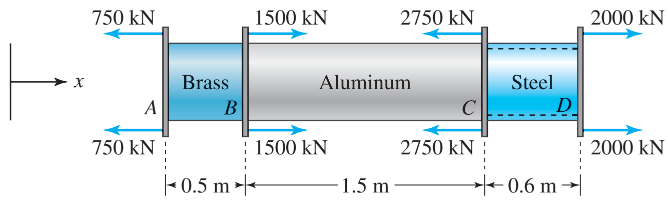


Figure 4.12 Axial member in Example 4.3.

PLAN

(a) We make imaginary cuts in each segment and determine the internal axial forces by equilibrium. Using Equation (4.10) we can find the relative movements of the cross sections at *B* with respect to *A* and at *C* with respect to *B* and add these two relative displacements to obtain the relative movement of the cross section at *C* with respect to the section at *A*. (b) The normal stress σ_{xx} in *AB* can be obtained from Equation (4.8) and the strain ε_{yy} found using Equation (4.13). Multiplying the strain by the diameter we obtain the change in diameter. (c) We can calculate the allowable axial stress in steel from the given failure values and factor of safety. Knowing the internal force in *CD* we can find the cross-sectional area from which we can calculate the internal diameter.

SOLUTION

(a) The cross-sectional areas of segment *AB* and *BC* are

$$A_{AB} = A_{BC} = \frac{\pi}{4}(0.2\text{ m})^2 = 31.41\times10^{-3}\text{ m}^2 \tag{E1}$$

We make imaginary cuts in segments *AB*, *BC*, and *CD* and draw the free-body diagrams as shown in Figure 4.13. By equilibrium of forces we obtain the internal axial forces

$$N_{AB} = 1500\text{ kN} \qquad N_{BC} = 1500\text{ kN} - 3000\text{ kN} = -1500\text{ kN} \qquad N_{CD} = \quad = 4000\text{ kN} \tag{E2}$$

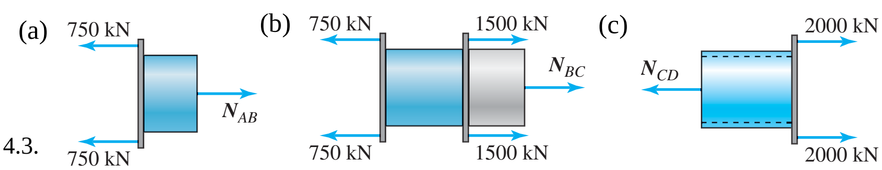


Figure 4.13 Free body diagrams in Example 4.3.

We can find the relative movement of point *B* with respect to point *A*, and *C* with respect to *B* using Equation (4.10):

$$u_B - u_A = \frac{N_{AB}(x_B - x_A)}{E_{AB}A_{AB}} = \frac{(1500\times10^3\text{ N})(0.5\text{ m})}{(100\times10^9\text{ N/m}^2)(31.41\times10^{-3}\text{ m}^2)} = 0.2388\times10^{-3}\text{ m} \tag{E3}$$

$$u_C - u_B = \frac{N_{BC}(x_C - x_B)}{E_{BC}A_{BC}} = \frac{(-1500\times10^3\text{ N})(1.5\text{ m})}{(70\times10^9\text{ N/m}^2)(31.41\times10^{-3}\text{ m}^2)} = -1.0233\times10^{-3}\text{ m} \tag{E4}$$

Adding Equations (E3) and (E4) we obtain the relative movement of point C with respect to A:

$$u_C - u_A = (u_C - u_B) + (u_B - u_A) = (0.2388\text{ m} - 1.0233\text{ m})10^{-3} = -0.7845\times10^{-3}\text{ m} \tag{E5}$$

ANS. $u_C - u_A = 0.7845$ mm contraction

(b) We can find the axial stress σ_{xx} in *AB* using Equation (4.8):

$$\sigma_{xx} = \frac{N_{AB}}{A_{AB}} = \frac{1500\times10^3\text{ N}}{(31.41\times10^{-3}\text{ m}^2)} = 47.8\times10^6\text{ N/m}^2 \tag{E6}$$

Substituting σ_{xx}, E_{br} = 100 GPa, ν_{br} = 0.34 in Equation (4.13), we can find ε_{yy}. Multiplying ε_{yy} by the diameter of 200 mm, we then obtain the change in diameter Δd,

$$\varepsilon_{yy} = -\frac{\nu_{br}\sigma_{xx}}{E_{br}} = -\frac{0.34(47.8\times10^6\text{ N/m}^2)}{100\times10^9\text{ N/m}^2} = -0.162\times10^{-3} = \frac{\Delta d}{200\text{ mm}} \tag{E7}$$

ANS. $\Delta d = -0.032$ mm

(c) The axial stress in segment *CD* is

$$\sigma_{CD} = \frac{N_{CD}}{A_{CD}} = \frac{4000 \times 10^3 \text{ N}}{(\pi/4)[(0.2 \text{ m})^2 - D_i^2]} = \frac{16{,}000 \times 10^3}{[\pi(0.2^2 - D_i^2)]} \text{ N/m}^2 \tag{E8}$$

Using the given factor of safety, we determine the value of D_i:

$$K = \frac{\sigma_{\text{yield}}}{\sigma_{CD}} = \frac{250 \times 10^6 \times [\pi(0.2^2 - D_i^2)]}{16{,}000 \times 10^3} = 49.09(0.2^2 - D_i^2) \geq 1.2 \quad \text{or}$$

$$D_i^2 \leq 0.2^2 - 24.45 \times 10^{-3} \qquad \text{or} \qquad D_i \leq 124.7 \times 10^{-3} \text{ m} \tag{E9}$$

To the nearest millimeter, the diameter that satisfies the inequality in Equation (E9) is 124 mm.

ANS. $D_i = 124$ mm

COMMENTS

1. On a free-body diagram some may prefer to show $\boldsymbol{N}$ in a direction that counterbalances the external forces, as shown in Figure 4.14. In such cases the sign convention is not being followed.

We note that $u_B - u_A = 0.2388\ (10^{-3})$ m is extension and $u_C - u_B = 1.0233\ (10^{-3})$ m is contraction. To calculate $u_C - u_A$ we must now manually subtract $u_C - u_B$ from $u_B - u_A$.

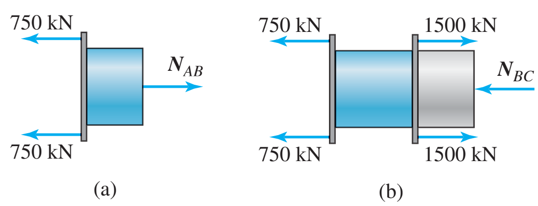


Figure 4.14 Alternative free body diagrams in Example 4.3.

2. An alternative way of calculating of $u_C - u_A$ is

$$u_C - u_A = \int_{x_A}^{x_C} \frac{\boldsymbol{N}}{EA} dx = \underbrace{\int_{x_A}^{x_B} \frac{\boldsymbol{N}_{AB}}{E_{AB}A_{AB}} dx}_{u_C - u_B} + \underbrace{\int_{x_B}^{x_C} \frac{\boldsymbol{N}_{BC}}{E_{BC}A_{BC}} dx}_{u_B - u_A}$$

or, written more compactly,

$$\Delta u = \sum_{i=1}^{n} \frac{\boldsymbol{N}_i \Delta x_i}{E_i A_i} \tag{4.14}$$

where n is the number of segments on which the summation is performed, which in our case is 2. Equation (4.14) can be used only if the sign convention for the internal force $\boldsymbol{N}$ is followed.

3. Note that $\boldsymbol{N}_{BC} - \boldsymbol{N}_{AB} = -3000$ kN and the magnitude of the applied external force at the section at B is 3000 kN. Similarly, $\boldsymbol{N}_{CD} - \boldsymbol{N}_{BC}$ = 5500 kN, which is the magnitude of the applied external force at the section at C. In other words, the internal axial force jumps by the value of the external force as one crosses the external force from left to right. We will make use of this observation in the next section, when we develop a graphical technique for finding the internal axial force.

4.2.8 Axial Force Diagram

In Example 4.3 we constructed several free-body diagrams to determine the internal axial force in different segments of the axial member. An axial force diagram is a graphical technique for determining internal axial forces, which avoids the repetition of drawing free-body diagrams.

An **axial force diagram** is a plot of the internal axial force $\boldsymbol{N}$ versus x. To construct an axial force diagram we create a small template to guide us in which direction the internal axial force will jump, as shown in Figure 4.15*a* and Figure 4.15*b*. An **axial template** is a free-body diagram of a small segment of an axial bar created by making an imaginary cut just before and just after the section where the external force is applied.

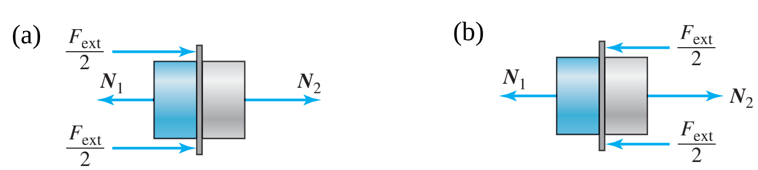


Figure 4.15 Axial bar templates. Template Equations $\boldsymbol{N}_2 = \boldsymbol{N}_1 - F_{\text{ext}}$ $\boldsymbol{N}_2 = \boldsymbol{N}_1 + F_{\text{extt}}$

The external force F_{ext} on the template can be drawn either to the left or to the right. The ends represent the imaginary cut just to the left and just to the right of the applied external force. On these cuts the internal axial forces are drawn in tension. An equilibrium equation—that is, the template equation—is written as shown in Figure 4.15 If the external force on the axial bar is in the direction of the assumed external force on the template, then the value of $\boldsymbol{N}_2$ is calculated according to the template equation. If the external force on the axial bar is opposite to the direction shown on the template, then $\boldsymbol{N}_2$ is calculated by changing the sign of F_{ext} in the template equation. Example 4.4 demonstrates the use of templates in constructing axial force diagrams.

EXAMPLE 4.4

Draw the axial force diagram for the axial member shown in Example 4.3 and calculate the movement of the section at *C* with respect to the section at *A*.

PLAN

We can start the process by considering an imaginary extension on the left. In the imaginary extension the internal axial force is zero. Using the template in Figure 4.15*a* to guide us, we can draw the axial force diagram. Using Equation (4.14), we can find the relative displacement of the section at *C* with respect to the section at *A*.

SOLUTION

Let *LA* be an imaginary extension on the left of the shaft, as shown in Figure 4.16*a*. Clearly the internal axial force in the imaginary segment *LA* is zero. As one crosses the section at *A*, the internal force must jump by the applied axial force of 1500 kN. Because the forces at *A* are in the opposite direction to the force F_{ext} shown on the template in Figure 4.15*a*, we must use opposite signs in the template equation. The internal force just after the section at *A* will be +1500 kN. This is the starting value in the internal axial force diagram.

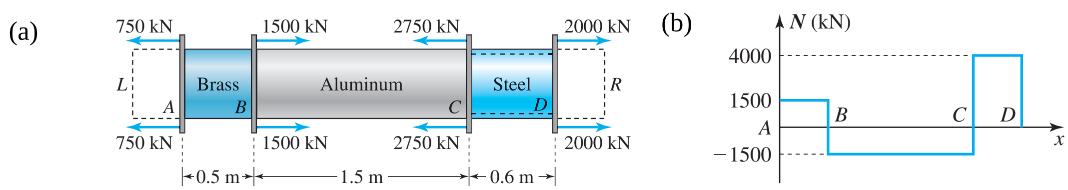


Figure 4.16 (a) Extending the axial bar for an axial force diagram. (b) Axial force diagram.

We approach the section at *B* with an internal force value of +1500 kN. The force at *B* is in the same direction as the force shown on the template in Figure 4.15*a*. Hence we subtract 3000 as per the template equation, to obtain a value of –1500 kN, as shown in Figure 4.16*b*.

We now approach the section at *C* with an internal force value of –1500 kN and note that the forces at *C* are opposite to those on the template in Figure 4.15*a*. Hence we add 5500 to obtain +4000 kN.

The force at *D* is in the same direction as that on the template in Figure 4.15*a*, and after subtracting we obtain a zero value in the imaginary extended bar *DR*. The return to zero value must always occur because the bar is in equilibrium.

From Figure 4.16*b* the internal axial forces in segments *AB* and *BC* are $\boldsymbol{N}_{AB} = 1500$ kN and $\boldsymbol{N}_{BC} = -1500$ kN. The crosssectional areas as calculated in Example 4.3 are $A_{AB} = A_{BC} = 31.41 \times 10^{-3}\ \text{m}^2$ and modulas of elasticity for the two sections are $E_{AB} = 100$ GPa and $E_{BC} = 70$ GPa. Substituting these values into Equation (4.14) we obtain the relative deformation of the section at *C* with respect to the section at *A*,

$$\Delta u = u_C - u_A = \frac{\boldsymbol{N}_{AB}(x_B - x_A)}{E_{AB}A_{AB}} + \frac{\boldsymbol{N}_{BC}(x_C - x_B)}{E_{BC}A_{BC}} \tag{E1}$$

$$u_C - u_A = \frac{(1500 \times 10^3\ \text{N})(0.5\ \text{m})}{(100 \times 10^9\ \text{N/m}^2)(31.41 \times 10^{-3}\ \text{m}^2)} + \frac{(-1500 \times 10^3\ \text{N})(1.5\ \text{m})}{(70 \times 10^9\ \text{N/m}^2)(31.41 \times 10^{-3}\ \text{m}^2)} = -0.7845 \times 10^{-3}\ \text{m} \quad \text{or} \tag{E2}$$

ANS. $u_C - u_A = 0.7845$ mm contraction

COMMENT

1. We could have used the template in Figure 4.15*b* to create the axial force diagram.We approach the section at *A* and note that the +1500 kN is in the same direction as that shown on the template of Figure 4.15*b*. As per the template equation we add. Thus our starting value is +1500 kN, as shown in Figure 4.16. As we approach the section at *B*, the internal force $\boldsymbol{N}_1$ is +1500 kN, and the applied force of 3000 kN is in the opposite direction to the template of Figure 4.15*b*, so we subtract to obtain $\boldsymbol{N}_2$ as –1500 kN. We approach the section at *C* and note that the applied force is in the same direction as the applied force on the template of Figure 4.15*b*. Hence we add 5500 kN to obtain +4000 kN. The force at section *D* is opposite to that shown on the template of Figure 4.15*b*, so we subtract 4000 to get a zero value in the extended portion *DR*. The example shows that the direction of the external force F_{ext} on the template is immaterial.

EXAMPLE 4.5

A 1-m-long hollow rod is to transmit an axial force of 60 kN. Figure 4.17 shows that the inner diameter of the rod must be 15 mm to fit existing attachments. The elongation of the rod is limited to 2.0 mm. The shaft can be made of titanium alloy or aluminum. The modulus of elasticity E, the allowable normal stress σ_{allow}, and the density γ for the two materials is given in Table 4.1. Determine the minimum outer diameter to the nearest millimeter of the lightest rod that can be used for transmitting the axial force.

TABLE 4.1 **Material properties in Example 4.4**

Material	E (GPa)	σ_{allow} (MPa)	γ (mg/m^3)
Titanium alloy	96	400	4.4
Aluminum	70	200	2.8

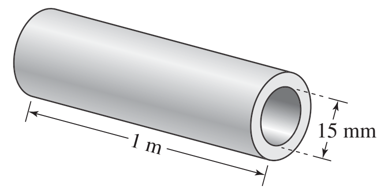


Figure 4.17 Cylindrical rod in Example 4.5.

PLAN

The change in radius affects only the cross-sectional area A and no other quantity in Equations (4.8) and (4.10). For each material we can find the minimum cross-sectional area A needed to satisfy the stiffness and strength requirements. Knowing the minimum A for each material, we can find the minimum outer radius. We can then find the volume and hence the mass of each material and make our decision on the lighter bar.

SOLUTION

We note that for both materials $x_2 - x_1 = 1$ m. From Equations (4.8) and (4.10) we obtain for titanium alloy the following limits on A_{Ti}:

$$(\Delta u)_{Ti} = \frac{(60 \times 10^3 \text{ N})(1 \text{ m})}{(96 \times 10^9 \text{ N/m}^2)A_{Ti}} \leq 2 \times 10^{-3} \text{ m} \quad \text{or} \quad A_{Ti} \geq 0.313 \times 10^{-3} \text{ m}^2 \tag{E1}$$

$$(\sigma_{max})_{Ti} = \frac{(60 \times 10^3 \text{ N})}{A_{Ti}} \leq 400 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad A_{Ti} \geq 0.150 \times 10^{-3} \text{ m}^2 \tag{E2}$$

Using similar calculations for the aluminum shaft, we obtain the following limits on A_{Al}:

$$(\Delta u)_{Al} = \frac{(60 \times 10^3 \text{ N}) \times 1}{(70 \times 10^9 \text{ N/m}^2)A_{Al}} \leq 2 \times 10^{-3} \text{ m} \quad \text{or} \quad A_{Al} \geq 1.071 \times 10^{-3} \text{ m}^2 \tag{E3}$$

$$(\sigma_{max})_{Al} = \frac{(60 \times 10^3 \text{ N})}{A_{Al}} \leq 200 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad A_{Al} \geq 0.300 \times 10^{-3} \text{ m}^2 \tag{E4}$$

Thus if $A_{Ti} \geq 0.313 \times 10^{-3} \text{ m}^2$, it will meet both conditions in Equations (E1) and (E2). Similarly if $A_{Al} \geq 1.071 \times 10^{-3} \text{ m}^2$, it will meet both conditions in Equations (E3) and (E4). The external diameters D_{Ti} and D_{Al} are then

$$A_{Ti} = \frac{\pi}{4}(D_{Ti}^2 - 0.015^2) \geq 0.313 \times 10^{-3} \qquad D_{Ti} \leq 24.97 \times 10^{-3} \text{ m} \tag{E5}$$

$$A_{Al} = \frac{\pi}{4}(D_{Al}^2 - 0.015^2) \geq 1.071 \times 10^{-3} \qquad D_{Al} \leq 39.86 \times 10^{-3} \text{ m} \tag{E6}$$

Rounding upward to the closest millimeter, we obtain

$$D_{Ti} = 25(10^{-3}) \text{ m} \qquad D_{Al} = 40(10^{-3}) \text{ m} \tag{E7}$$

We can find the mass of each material by taking the product of the material density and the volume of a hollow cylinder,

$$m_{Ti} = \left[(4.4 \times 10^6 \text{ g/m}^3)\left\{\frac{\pi}{4}(0.025^2 - 0.015^2) \text{ m}^2\right\}\right](1 \text{ m}) = 1382 \ g \tag{E8}$$

$$m_{Al} = \left[(2.8 \times 10^6 \text{ g/m}^3)\left\{\frac{\pi}{4}(0.040^2 - 0.015^2) \text{ m}^2\right\}\right](1 \text{ m}) = 3024 \ g \tag{E9}$$

From Equations (E8) and (E9) we see that the titanium alloy shaft is lighter.

ANS. A titanium alloy shaft with an outside diameter of 25 mm should be used.

COMMENTS

1. For both materials the stiffness limitation dictated the calculation of the external diameter, as can be seen from Equations (E1) and (E3).

2. Even though the density of aluminum is lower than that of titanium alloy, the mass of titanium is less. Because of the higher modulus of elasticity of titanium alloy, we can meet the stiffness requirement using less material than with aluminum.
3. The answer may change if cost is a consideration. The cost of titanium per kilogram is significantly higher than that of aluminum. Thus based on material cost we may choose aluminum. However, if the weight affects the running cost, then economic analysis is needed to determine whether the material cost or the running cost is higher.
4. If in Equation (E5) we had 24.05×10^{-3} m on the right-hand side, our answer for D_{Ti} would still be 25 mm because we have to round upward to ensure meeting the greater-than sign requirement in Equation (E5).

EXAMPLE 4.6

A rectangular aluminum bar ($E_{al} = 10{,}000$ ksi, $\nu = 0.25$) of $\frac{3}{4}$-in. thickness consists of a uniform and tapered cross section, as shown in Figure 4.18. The depth in the tapered section varies as $h(x) = (2 - 0.02x)$ in. Determine: (a) The elongation of the bar under the applied loads. (b) The change in dimension in the y direction in section BC.

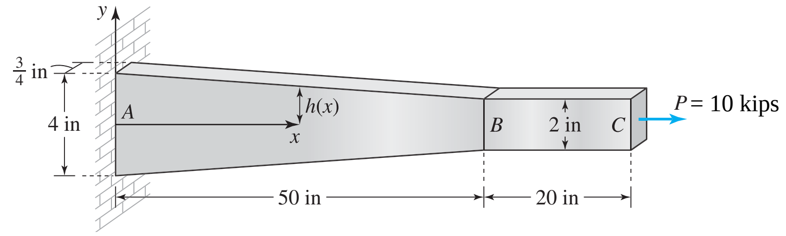


Figure 4.18 Axial member in Example 4.6.

PLAN

(a) We can use Equation (4.10) to find $u_C - u_B$. Noting that cross-sectional area is changing with x in segment AB, we integrate Equation (4.7) to obtain $u_B - u_A$. We add the two relative displacements to obtain $u_C - u_A$ and noting that $u_A = 0$ we obtain the extension as u_C. (b) Once the axial stress in BC is found, the normal strain in the y direction can be found using Equation (4.13). Multiplying by 2 in., the original length in the y direction, we then find the change in depth.

SOLUTION

The cross-sectional areas of AB and BC are

$$A_{BC} = \left(\frac{3}{4}\text{ in.}\right)(2\text{ in.}) = 1.5\text{ in.}^2 \qquad A_{AB} = \left(\frac{3}{4}\text{ in.}\right)(2h\text{ in.}) = 1.5(2 - 0.02x)\text{ in.}^2 \tag{E1}$$

Figure 4.19 Free-body diagrams in Example 4.6.

(a) We can make an imaginary cuts in segment AB and BC, to obtain the free-body diagrams in Figure 4.19. By force equilibrium we obtain the internal forces,

$$N_{AB} = 10\text{ kips} \qquad N_{BC} = 10\text{ kips} \tag{E2}$$

The relative movement of point C with respect to point B is

$$u_C - u_B = \frac{N_{BC}(x_C - x_B)}{E_{BC}A_{BC}} = \frac{(10\text{ kips})(20\text{ in.})}{(10{,}000\text{ ksi})(1.5\text{ in.}^2)} = 13.33 \times 10^{-3}\text{ in.} \tag{E3}$$

Equation (4.7) for segment AB can be written as

$$\left(\frac{du}{dx}\right)_{AB} = \frac{N_{AB}}{E_{AB}A_{AB}} = \frac{10\text{ kips}}{(10{,}000\text{ ksi})[1.5(2 - 0.02x)\text{ in.}^2]} \tag{E4}$$

Integrating Equation (E4), we obtain the relative displacement of B with respect to A:

$$\int_{u_A}^{u_B} du = \left[\int_{x_A=0}^{x_B=50} \frac{10^{-3}}{1.5(2 - 0.02x)}\,dx\right]\text{in. or}$$

$$u_B - u_A = \frac{10^{-3}}{1.5}\frac{1}{(-0.02)}\ln(2 - 0.02x)\Big|_0^{50} = -\frac{10^{-3}}{0.03}[\ln(1) - \ln(2)]\text{ in.} = 23.1 \times 10^{-3}\text{ in.} \tag{E5}$$

We obtain the relative displacement of C with respect to A by adding Equations (E3) and (E5):

$$u_C - u_A = 13.33 \times 10^{-3} + 23.1 \times 10^{-3} = 36.43 \times 10^{-3}\text{ in.} \tag{E6}$$

We note that point A is fixed to the wall, and thus $u_A = 0$.

ANS. $u_C = 0.036$ in. elongation

(b) The axial stress in BC is $\sigma_{AB} = N_{BC}/A_{BC} = 10/1.5 = 6.667$ ksi. From Equation (4.13) the normal strain in y direction can be found,

$$\varepsilon_{yy} = -\frac{\nu_{AB}\sigma_{AB}}{E_{AB}} = -\frac{0.25 \times (6.667 \text{ ksi})}{(10{,}000 \text{ ksi})} = -0.1667 \times 10^{-3} \quad \text{(E7)}$$

The change in dimension in the y direction Δv can be found as

$$\Delta v = \varepsilon_{yy}(2 \text{ in.}) = -0.3333 \times 10^{-3} \text{ in.} \quad \text{(E8)}$$

ANS. $\Delta v = 0.3333 \times 10^{-3}$ in. contraction

COMMENT

1. An alternative approach is to integrate Equation (E4):

$$u(x) = -\frac{10^{-3}}{0.03}\ln(2 - 0.02x) + c \quad \text{(E9)}$$

To find constant of integration c, we note that at $x = 0$ the displacement $u = 0$. Hence, $c = (10^{-3}/0.03)\ln(2)$. Substituting the value, we obtain

$$u(x) = -\frac{10^{-3}}{0.03}\ln\left(\frac{2 - 0.02x}{2}\right) \quad \text{(E10)}$$

Knowing u at all x, we can obtain the extension by substituting $x = 50$ to get the displacement at C.

EXAMPLE 4.7

The radius of a circular truncated cone in Figure 4.20 varies with x as $R(x) = (r/L)(5L - 4x)$. Determine the extension of the truncated cone due to its own weight in terms of E, L, r, and γ, where E and γ are the modulus of elasticity and the specific weight of the material, respectively.

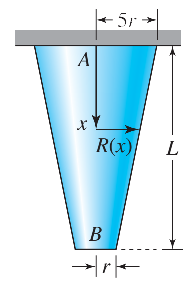


Figure 4.20 Truncated cone in Example 4.7.

PLAN

We make an imaginary cut at location x and take the lower part of the truncated cone as the free-body diagram. In the free-body diagram we can find the volume of the truncated cone as a function of x. Multiplying the volume by the specific weight, we can obtain the weight of the truncated cone and equate it to the internal axial force, thus obtaining the internal force as a function of x. We then integrate Equation (4.7) to obtain the relative displacement of B with respect to A.

SOLUTION

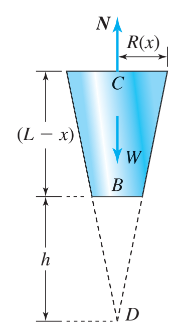


Figure 4.21 Free-body diagram of truncated cone in Example 4.7.

Figure 4.21 shows the free-body diagram after making a cut at some location x. We can find the volume V of the truncated cone by subtracting the volumes of two complete cones between C and D and between B and D. We obtain the location of point D,

$$R(x = L + h) = \frac{r}{L}[5L - 4(L + h)] = 0 \qquad \text{or} \qquad h = L/4 \quad \text{(E1)}$$

The volume of the truncated cone is

$$V = \frac{1}{3}\pi R^2\left(L - x + \frac{L}{4}\right) - \frac{1}{3}\pi r^2\frac{L}{4} = \frac{\pi}{12}\left[\frac{r^2}{L^2}(5L - 4x)^3 - r^2 L\right] \quad \text{(E2)}$$

By equilibrium of forces in Figure 4.21 we obtain the internal axial force:

$$N = W = \gamma V = \frac{\gamma\pi r^2}{12L^2}[(5L-4x)^3 - L^3] \tag{E3}$$

The cross-sectional area at location x (point C) is

$$A = \pi R^2 = \pi\frac{r^2}{L^2}(5L-4x)^2 \tag{E4}$$

Equation (4.7) can be written as

$$\frac{du}{dx} = \frac{N}{EA} = \frac{\dfrac{\gamma\pi r^2}{12L^2}[(5L-4x)^3 - L^3]}{E\pi\dfrac{r^2}{L^2}(5L-4x)^2} \tag{E5}$$

Integrating Equation (E5) from point A to point B, we obtain the relative movement of point B with respect to point A:

$$\int_{u_A}^{u_B} du = \int_{x_A=0}^{x_B=L} \frac{\gamma}{12E}\left[(5L-4x) - \frac{L^3}{(5L-4x)^2}\right]dx \quad \text{or}$$

$$u_B - u_A = \frac{\gamma}{12E}\left[5Lx - 2x^2 - \frac{L^3}{4(5L-4x)}\right]\Bigg|_0^L = \frac{\gamma L^2}{12E}\left(5 - 2 - \frac{1}{4} + \frac{1}{20}\right) = \frac{7\gamma L^2}{30E} \tag{E6}$$

Point A is built into the wall, hence $u_A = 0$. We obtain the extension of the bar as displacement of point B.

ANS. $u_B = \left(\dfrac{7\gamma L^2}{30E}\right)$ downward

COMMENTS

1. *Dimension check:* We write $O(\ )$ to represent the dimension of a quantity. F has dimensions of force and L of length. Thus, the modulus of elasticity E, which has dimensions of force per unit area, is represented as $O(F/L^2)$. The dimensional consistency of our answer is then checked as

$$\gamma \to O\left(\frac{F}{L^3}\right) \qquad L \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad u \to O(L) \qquad \frac{\gamma L^2}{E} \to O\left(\frac{(F/L^3)L^2}{F/L^2}\right) \to O(L) \to \text{checks}$$

2. An alternative approach to determining the volume of the truncated cone in Figure 4.21 is to find first the volume of the infinitesimal disc shown in Figure 4.22. We then integrate from point C to point B:

$$V = \int_x^L dV = \int_x^L \pi R^2 dx = \int_x^L \pi\frac{r^2}{L^2}(5L-4x)^2\,dx = -\pi\frac{r^2}{L^2}\frac{(5L-4x)^3}{3(-4)}\Bigg|_x^L \tag{E7}$$

3. On substituting the limits we obtain the volume given by (E2), as before:

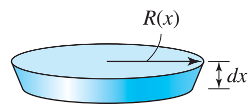


Figure 4.22 Alternative approach to finding volume of truncated cone.

4. The advantage of the approach in comment 2 is that it can be used for any complex function representation of $R(x)$, such as given in Problems 4.27 and 4.28, whereas the approach used in solving the example problem is only valid for a linear representation of $R(x)$.

4.2.9* General Approach to Distributed Axial Forces

Distributed axial forces are usually due to inertial forces, gravitational forces, or frictional forces acting on the surface of the axial bar. The internal axial force $\boldsymbol{N}$ becomes a function of x when an axial bar is subjected to a distributed axial force $p(x)$, as seen in Example 4.7. If $p(x)$ is a simple function, then we can find $\boldsymbol{N}$ as a function of x by drawing a free-body diagram, as we did in Example 4.7. However, if the distributed force $p(x)$ is a complex function, it may be easier to use the alternative described in this section.

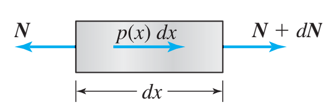


Figure 4.23 Equilibrium of an axial element.

Consider an infinitesimal axial element created by making two imaginary cuts at a distance dx from each other, as shown in Figure 4.23. By equilibrium of forces in the x direction we obtain: $(\boldsymbol{N} + d\boldsymbol{N}) + p(x)dx - \boldsymbol{N} = 0$ or

$$\boxed{\frac{d\boldsymbol{N}}{dx} + p(x) = 0} \tag{4.15}$$

Equation (4.15) assumes that $p(x)$ is positive in the positive x direction. If $p(x)$ is zero in a segment of the axial bar, then the internal force $\boldsymbol{N}$ is a constant in that segment.

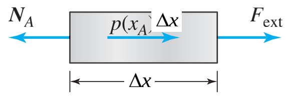


Figure 4.24 Boundary condition on internal axial force.

Equation (4.15) can be integrated to obtain the internal force $\boldsymbol{N}$. The integration constant can be found by knowing the value of the internal force $\boldsymbol{N}$ at either end of the bar. To obtain the value of $\boldsymbol{N}$ at the end of the shaft (say, point A), a free-body diagram is constructed after making an imaginary cut at an infinitesimal distance Δx from the end as shown in Figure 4.24) and writing the equilibrium equation as

$$\lim_{\Delta x \to 0}[F_{\text{ext}} - \boldsymbol{N}_A - p(x_A)\Delta x] = 0 \qquad \boldsymbol{N}_A = F_{\text{ext}}$$

This equation shows that the distributed axial force does not affect the boundary condition on the internal axial force. The value of the internal axial force $\boldsymbol{N}$ at the end of an axial bar is equal to the concentrated external axial force applied at the end.

Suppose the weight per unit volume, or, the specific weight of a bar, is γ. By multiplying the specific weight by the cross-sectional area A, we would obtain the weight per unit length. Thus $p(x)$ is equal to γA in magnitude. If x coordinate is chosen in the direction of gravity, then $p(x)$ is positive: $[p(x) = +\gamma A]$. If it is opposite to the direction of gravity, then $p(x)$ is negative:$[p(x) = -\gamma A]$.

EXAMPLE 4.8

Determine the internal force $\boldsymbol{N}$ in Example 4.7 using the approach outlined in Section 4.2.9.

PLAN

The distributed force $p(x)$ per unit length is the product of the specific weight times the area of cross section. We can integrate Equation (4.15) and use the condition that the value of the internal force at the free end is zero to obtain the internal force as a function of x.

SOLUTION

The distributed force $p(x)$ is the weight per unit length and is equal to the specific weight times the area of cross section $A = \pi R^2 = \pi(r^2/L^2)(5L - 4x)^2$:

$$p(x) = \gamma A = \gamma\pi\frac{r^2}{L^2}(5L-4x)^2 \tag{E1}$$

We note that point B ($x = L$) is on a free surface and hence the internal force at B is zero. We integrate Equation (4.15) from L to x after substituting $p(x)$ from Equation (E1) and obtain $\boldsymbol{N}$ as a function of x,

$$\int_{N_B=0}^{N} d\boldsymbol{N} = -\int_{x_B=L}^{x} p(x)\,dx = -\int_L^x \gamma\left[\pi\frac{r^2}{L^2}(5L-4x)^2\right]dx = -\left(\gamma\,\pi\frac{r^2}{L^2}\right)\left[\frac{(5L-4x)^3}{-4\times 3}\right]\Bigg|_L^x \tag{E2}$$

ANS. $$\boldsymbol{N} = \frac{\gamma\pi r^2}{12L^2}[(5L-4x)^3 - L^3]$$

COMMENT

1. An alternative approach is to substitute (E1) into Equation (4.15) and integrate to obtain

$$\boldsymbol{N}(x) = \frac{\gamma\pi r^2}{12L^2}(5L-4x)^3 + c_1 \tag{E3}$$

To determine the integration constant, we use the boundary condition that at $\boldsymbol{N}$ ($x = L$) = 0, which yields $c_1 = -(\gamma\pi r^2/12L^2)L^3$. Substituting this value into Equation (E3), we obtain $\boldsymbol{N}$ as before.

Consolidate your knowledge

1. Identity five examples of axial members from your daily life.
2. With the book closed, derive Equation (4.10), listing all the assumptions as you go along.

QUICK TEST 4.1

Time: 20 minutes/Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E.

1. Axial *strain* is uniform across a nonhomogeneous cross section.

2. Axial *stress* is uniform across a nonhomogeneous cross section.

3. The formula $\sigma_{xx} = N/A$ can be used for finding the stress on a cross section of a tapered axial member.

4. The formula $u_2 - u_1 = N(x_2 - x_1)/EA$ can be used for finding the deformation of a segment of a tapered axial member.

5. The formula $\sigma_{xx} = N/A$ can be used for finding the stress on a cross section of an axial member subjected to distributed forces.

6. The formula $u_2 - u_1 = N(x_2 - x_1)/EA$ can be used for finding the deformation of a segment of an axial member subjected to distributed forces.

7. The equation $N = \int_A \sigma_{xx}\, dA$ *cannot* be used for nonlinear materials.

8. The equation $N = \int_A \sigma_{xx}\, dA$ *can* be used for a nonhomogeneous cross section.

9. External axial forces must be collinear and pass through the centroid of a homogeneous cross section for no bending to occur.

10. Internal axial forces jump by the value of the concentrated external axial force at a section.

PROBLEM SET 4.2

4.7 A crane is lifting a mass of 1000-kg, as shown in Figure P4.7. The weight of the iron ball at *B* is 25 kg. A single cable having a diameter of 25 mm runs between *A* and *B*. Two cables run between *B* and *C*, each having a diameter of 10 mm. Determine the axial stresses in the cables.

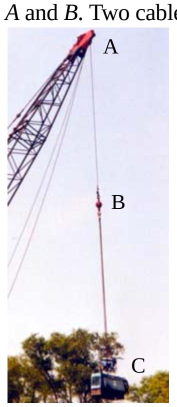


Figure P4.7

4.8 The counterweight in a lift bridge has 12 cables on the left and 12 cables on the right, as shown in Figure P4.8. Each cable has an effective diameter of 0.75 in, a length of 50 ft, a modulus of elasticity of 30,000 ksi, and an ultimate strength of 60 ksi. (a) If the counterweight is 100 kips, determine the factor of safety for the cable. (b) What is the extension of each cable when the bridge is being lifted?

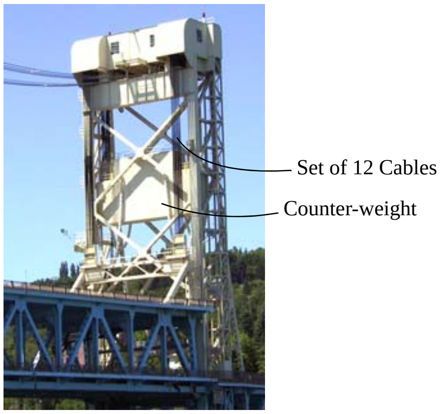


Figure P4.8

4.9 (a) Draw the axial force diagram for the axial member shown in Figure P4.9. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 8000$ kips. Determine the movement of the section at *D* with respect to the section at *A*.

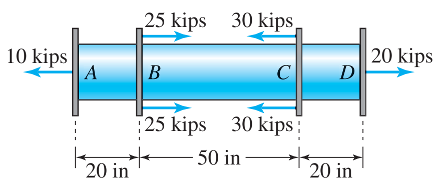


Figure P4.9

4.10 (a) Draw the axial force diagram for the axial member shown in Figure P4.10. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 80{,}000$ kN. Determine the movement of the section at *C*.

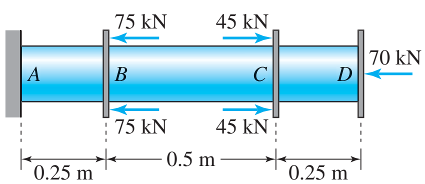


Figure P4.10

4.11 (a) Draw the axial force diagram for the axial member shown in Figure P4.11. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 2000$ kips. Determine the movement of the section at *B*.

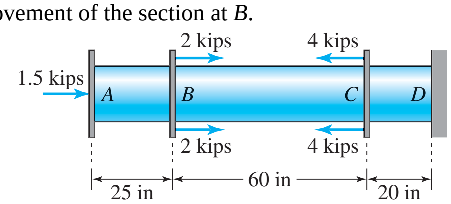


Figure P4.11

4.12 (a) Draw the axial force diagram for the axial member shown in Figure P4.12. (b) Check your results for part *a* by finding the internal forces in segments AB, BC, and CD by making imaginary cuts and drawing free-body diagrams. (c) The axial rigidity of the bar is $EA = 50{,}000$ kN. Determine the movement of the section at *D* with respect to the section at *A*.

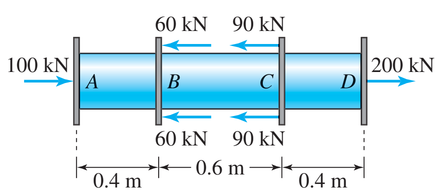


Figure P4.12

4.13 Three segments of 4-in. × 2-in. rectangular wooden bars ($E = 1600$ ksi) are secured together with rigid plates and subjected to axial forces, as shown in Figure P4.13. Determine: (a) the movement of the rigid plate at D with respect to the plate at A; (b) the maximum axial stress.

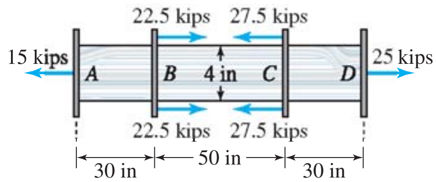


Figure P4.13

4.14 Aluminum bars ($E = 30{,}000$ ksi) are welded to rigid plates, as shown in Figure P4.1. All bars have a cross-sectional area of 0.5 in^2. The applied forces are $F_1 = 8$ kips, $F_2 = 12$ kips, and $F_3 = 9$ kips. Determine (a) the displacement of the rigid plate at D with respect to the rigid plate at A. (b) the maximum axial stress in the assembly.

4.15 Brass bars between sections A and B, aluminum bars between sections B and C, and steel bars between sections C and D are welded to rigid plates, as shown in Figure P4.2. The properties of the bars are given in Table 4.2 The applied forces are $F_1 = 90$ kN, $F_2 = 40$ kN, and $F_3 = 70$ kN. Determine (a) the displacement of the rigid plate at D.(b) the maximum axial stress in the assembly.

4.16 A solid circular steel ($E_s = 30{,}000$ ksi) rod BC is securely attached to two hollow steel rods AB and CD as shown. Determine (a) the angle of displacement of section at D with respect to section at A; (b) the maximum axial stress in the axial member.

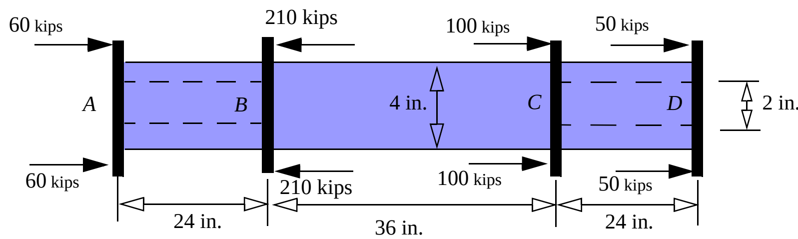


Figure P4.16

4.17 Two circular steel bars ($E_s = 30{,}000$ ksi, $\nu_s = 0.3$) of 2-in. diameter are securely connected to an aluminum bar ($E_{a1} = 10{,}000$ ksi, $\nu_{al} = 0.33$) of 1.5-in. diameter, as shown in Figure P4.17. Determine (a) the displacement of the section at C with respect to the wall; (b) the maximum change in the diameter of the bars.

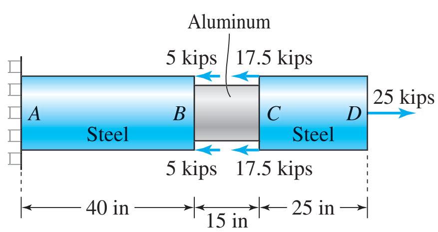


Figure P4.17

4.18 Two cast-iron pipes ($E = 100$ GPa) are adhesively bonded together, as shown in Figure P4.18. The outer diameters of the two pipes are 50 mm and 70 mm and the wall thickness of each pipe is 10 mm. Determine the displacement of end B with respect to end A.

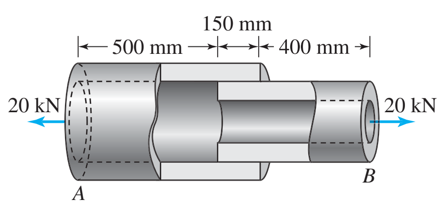


Figure P4.18

Tapered axial members

4.19 The tapered bar shown in Figure P4.19 has a cross-sectional area that varies as $A = K(2L - 0.25\,x)^2$. Determine the elongation of the bar in terms of P, L, E, and K.

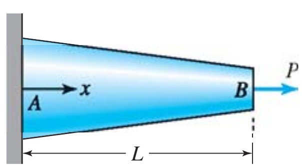


Figure P4.19

4.20 The tapered bar shown in Figure P4.19 has a cross-sectional area that varies as $A = K(4L - 3x)$. Determine the elongation of the bar in terms of *P*, *L*, *E*, and *K*.

4.21 A tapered and an untapered solid circular steel bar (E = 30,000 ksi) are securely fastened to a solid circular aluminum bar (E = 10,000 ksi), as shown in Figure P4.21. The untapered steel bar has a diameter of 2 in. The aluminum bar has a diameter of 1.5 in. The diameter of the tapered bars varies from 1.5 in to 2 in. Determine (a) the displacement of the section at *C* with respect to the section at *A*; (b) the maximum axial stress in the bar.

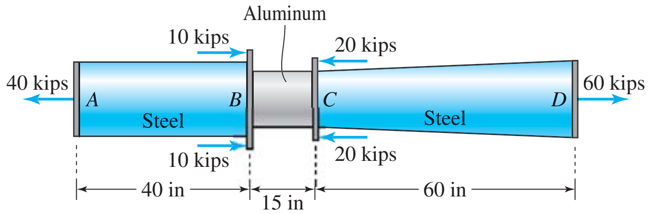


Figure P4.21

Distributed axial force

4.22 The column shown in Figure P4.22 has a length *L*, modulus of elasticity *E*, and specific weight γ. The cross section is a circle of radius *a*. Determine the contraction of each column in terms of *L*, *E*, γ, and *a*.

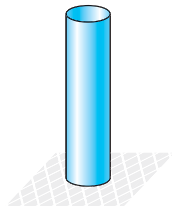


Figure P4.22

4.23 The column shown in Figure P4.23 has a length *L*, modulus of elasticity *E*, and specific weight γ. The cross section is an equilateral triangle of side *a*.Determine the contraction of each column in terms of *L*, *E*, γ, and *a*.

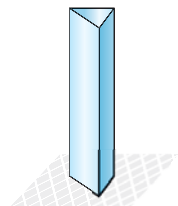


Figure P4.23

4.24 The column shown in Figure P4.24 has a length *L*, modulus of elasticity *E*, and specific weight γ. The cross-sectional area is *A*. Determine the contraction of each column in terms of *L*, *E*, γ, and *A*.

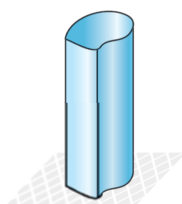


Figure P4.24

4.25 On the truncated cone of Example 4.7 a force $P = \gamma \pi r^2 L/5$ is also applied, as shown in Figure P4.25. Determine the total elongation of the cone due to its weight and the applied force. (*Hint*: Use superposition.)

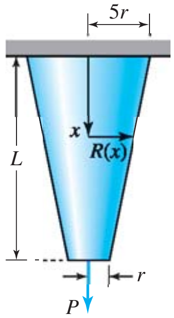


Figure P4.25

4.26 A 20-ft-tall thin, hollow tapered tube of a uniform wall thickness of $\frac{1}{8}$ in. is used for a light pole in a parking lot, as shown in Figure P4.26. The mean diameter at the bottom is 8 in., and at the top it is 2 in. The weight of the lights on top of the pole is 80 lb. The pole is made of aluminum alloy with a specific weight of 0.1 lb/in^3, a modulus of elasticity $E = 11{,}000$ ksi, and a shear modulus of rigidity $G = 4000$ ksi. Determine (a) the maximum axial stress; (b) the contraction of the pole. (*Hint*: Approximate the cross-sectional area of the thin-walled tube by the product of circumference and thickness.)

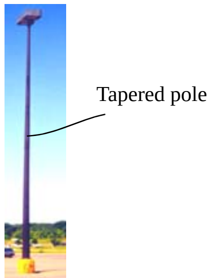


Figure P4.26

4.27 Determine the contraction of a column shown in Figure P4.27 due to its own weight. The specific weight is $\gamma = 0.28$ lb/in.^{3}, the modulus of elasticity is $E = 3600$ ksi, the length is $L = 120$ in., and the radius is $R = \sqrt{240 - x}$, where R and x are in inches.

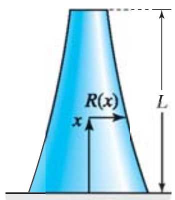


Figure P4.27

4.28 Determine the contraction of a column shown in Figure P4.27 due to its own weight. The specific weight is $\gamma = 24$ kN/m^3 the modulus of elasticity is $E = 25$ GPa, the length is $L = 10$ m and the radius is $R = 0.5e^{-0.07x}$, where R and x are in meters.

4.29 The frictional force per unit length on a cast-iron pipe being pulled from the ground varies as a quadratic function, as shown in Figure P4.29. Determine the force F needed to pull the pipe out of the ground and the elongation of the pipe before the pipe slips, in terms of the modulus of elasticity E, the cross-sectional area A, the length L, and the maximum value of the frictional force $f_{max.}$

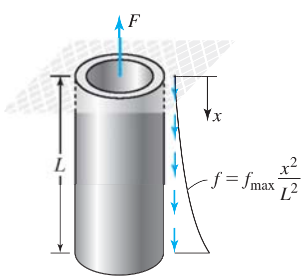


Figure P4.29

Design problems

4.30 The spare wheel in an automobile is stored under the vehicle and raised and lowered by a cable, as shown in Figure P4.30. The wheel has a mass of 25 kg. The ultimate strength of the cable is 300 MPa, and it has an effective modulus of elasticity $E = 180$ GPa. The cable length is 36 cm. (a) For a factor of safety of 4, determine to the nearest millimeter the minimum diameter of the cable if failure due to rupture is to be avoided. (b) What is the maximum extension of the cable for the answer in part (a)?

Figure P4.30

4.31 An adhesively bonded joint in wood (E = 1800 ksi) is fabricated as shown in Figure P4.31. If the total elongation of the joint between A and D is to be limited to 0.05 in., determine the maximum axial force F that can be applied.

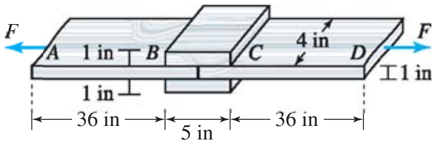


Figure P4.31

4.32 A 5-ft-long hollow rod is to transmit an axial force of 30 kips. The outer diameter of the rod must be 6 in. to fit existing attachments. The relative displacement of the two ends of the shaft is limited to 0.027 in. The axial rod can be made of steel or aluminum. The modulus of elasticity E, the allowable axial stress σ_{allow}, and the specific weight γ are given in Table 4.32. Determine the maximum inner diameter in increments of $\frac{1}{8}$ in. of the lightest rod that can be used for transmitting the axial force and the corresponding weight.

TABLE P4.32 **Material properties**

Material	E (ksi)	σ_{allow} (ksi)	γ (lb/in.^{3})
Steel	30,000	24	0.285
Aluminum	10,000	14	0.100

4.33 A hitch for an automobile is to be designed for pulling a maximum load of 3600 lb. A solid square bar fits into a square tube and is held in place by a pin, as shown in Figure P4.33. The allowable axial stress in the bar is 6 ksi, the allowable shear stress in the pin is 10 ksi, and the allowable axial stress in the steel tube is 12 ksi. To the nearest $\frac{1}{16}$ in., determine the minimum cross-sectional dimensions of the pin, the bar, and the tube. (*Hint:* The pin is in double shear.)

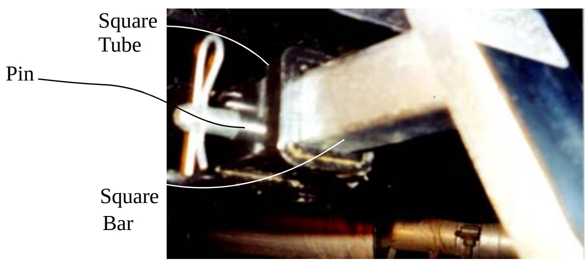


Figure P4.33

Stretch yourself

4.34 An axial rod has a constant axial rigidity EA and is acted upon by a distributed axial force $p(x)$. If at the section at A the internal axial force is zero, show that the relative displacement of the section at B with respect to the displacement of the section at A is given by

$$u_B - u_A = \frac{1}{EA}\left[\int_{x_A}^{x_B}(x - x_B)p(x)\,dx\right] \qquad (4.16)$$

4.35 A composite laminated bar made from n materials is shown in Figure P4.35. E_i and A_i are the modulus of elasticity and cross sectional area of the ith material. (a) If Assumptions from 1 through 5 are valid, show that the stress $(\sigma_{xx})_i$ in the ith material is given Equation (4.17a), where N is the total internal force at a cross section. (b) If Assumptions 7 through 9 are valid, show that relative deformation $u_2 - u_1$ is given by Equation (4.17b). (c) Show that for E_1=E_2=E_3....=E_n=E Equations (4.17a) and (4.17b) give the same results as Equations (4.8) and (4.10).

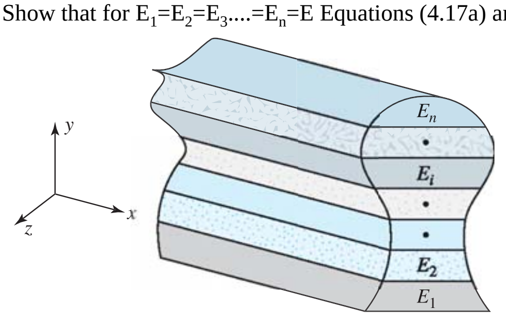


Figure P4.35

$$(\sigma_{xx})_i = \frac{NE_i}{\sum_{j=1}^{n} E_j A_j} \qquad (4.17a)$$

$$u_2 - u_1 = \frac{N(x_2 - x_1)}{\sum_{j=1}^{n} E_j A_j} \qquad (4.17b)$$

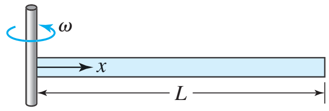
4.36 The stress–strain relationship for a nonlinear material is given by the power law $\sigma = E\varepsilon^n$. If all assumptions except Hooke's law are valid, show that

$$u_2 - u_1 = \left(\frac{N}{EA}\right)^{1/n}(x_2 - x_1) \tag{4.17}$$

and the axial stress σ_{xx} is given by (4.8).

4.37 Determine the elongation of a rotating bar in terms of the rotating speed ω, density γ, length L, modulus of elasticity E, and cross-sectional area A (Figure P4.37). (*Hint:* The body force per unit volume is $\rho\omega^2 x$.)

Figure P4.37



4.38 Consider the dynamic equilibrium of the differential elements shown in Figure P4.38, where N is the internal force, γ is the density, A is the cross-sectional area, and $\partial^2 u/\partial t^2$ is acceleration. By substituting for N from Equation (4.7) into the dynamic equilibrium equation, derive the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{where} \qquad c = \sqrt{\frac{E}{\gamma}} \tag{4.18}$$

The material constant c is the velocity of propagation of sound in the material.

4.39 Show by substitution that the functions $f(x - ct)$ and $g(x + ct)$ satisfy the wave equation, Equation (4.18).

4.40 The strain displacement relationship for large axial strain is given by

$$\varepsilon_{xx} = \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 \tag{4.19}$$

where we recognize that as u is only a function of x, the strain from (4.19) is uniform across the cross section. For a linear, elastic, homogeneous material show that

$$\frac{du}{dx} = \sqrt{1 + \frac{2N}{EA}} - 1 \tag{4.20}$$

The axial stress σ_{xx} is given by (4.8).

Computer problems

4.41 Table P4.41 gives the measured radii at several points along the axis of the solid tapered rod shown in Figure P4.41. The rod is made of aluminum (E = 100 GPa) and has a length of 1.5 m. Determine (a) the elongation of the rod using numerical integration; (b) the maximum axial stress in the rod.

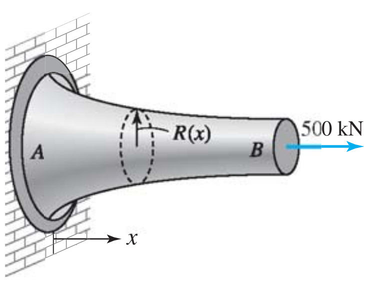


Figure P4.41

TABLE P4.41

x (m)	R(x) (mm)	x (m)	R(x) (mm)
0.0	100.6	0.8	60.1
0.1	92.7	0.9	60.3
0.2	82.6	1.0	59.1
0.3	79.6	1.1	54.0
0.4	75.9	1.2	54.8
0.5	68.8	1.3	54.1
0.6	68.0	1.4	49.4
0.7	65.9	1.5	50.6

4.42 Let the radius of the tapered rod in Problem 4.41 be represented by the equation $R(x) = a + bx$. Using the data in Table P4.41 determine constants a and b by the least-squares method and then find the elongation of the rod by analytical integration.

4.43 Table 4.43 shows the values of the distributed axial force at several points along the axis of the hollow steel rod (E = 30,000 ksi) shown in Figure P4.43. The rod has a length of 36 in., an outside diameter of 1 in., and an inside diameter of 0.875 in. Determine (a) the displacement of end A using numerical integration; (b) the maximum axial stress in the rod.

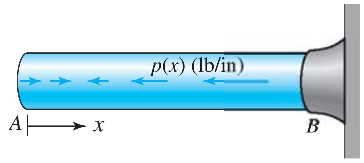


Figure P4.43

TABLE P4.43

x (inches)	$p(x)$ (lb/in.)	x (in.)	$p(x)$ (lb/in.)
0	260	21	–471
3	106	24	–598
6	32	27	–645
9	40	30	–880
12	–142	33	–1035
15	–243	36	–1108
18	–262		

4.44 Let the distributed force $p(x)$ in Problem 4.43 be represented by the equation $p(x) = cx^2 + bx + a$. Using the data in Table P4.43 determine constants a, b, and c by the least-squares method and then find the displacement of the section at A by analytical integration.

4.3 STRUCTURAL ANALYSIS

Structures are usually an assembly of axial bars in different orientations. Equation (4.10) assumes that the bar lies in x direction, and hence in structural analysis the form of Equation (4.21) is preferred over Equation (4.10).

$$\boxed{\delta = \frac{NL}{EA}} \tag{4.21}$$

where $L = x_2 - x_1$ and $\delta = u_2 - u_1$ in Equation (4.10). L represents the original length of the bar and δ represents deformation of the bar in the *original direction* irrespective of the movement of points on the bar. It should also be recognized that L, E, and A are positive. Hence the sign of δ is the same as that of $\boldsymbol{N}$:

- If $\boldsymbol{N}$ is a tensile force, then δ is elongation.
- If $\boldsymbol{N}$ is a compressive force, then δ is contraction.

4.3.1 Statically Indeterminate Structures

Statically indeterminate structures arise when there are more supports than needed to hold a structure in place. These extra supports are included for safety or to increase the stiffness of the structures. Each extra support introduces additional unknown reactions, and hence the total number of unknown reactions exceeds the number of static equilibrium equations. The **degree of static redundancy** is the number of unknown reactions minus the number of equilibrium equations. If the degree of static redundancy is zero, then we have a statically determinate structure and all unknowns can be found from equilibrium equations. If the degree of static redundancy is not zero, then we need additional equations to determine the unknown reactions. These additional equations are the relationships between the deformations of bars. **Compatibility equations** are geometric relationships between the deformations of bars that are derived from the deformed shapes of the structure. The number of compatibility equations needed is always equal to the degree of static redundancy.

Drawing the approximate deformed shape of a structure for obtaining compatibility equations is as important as drawing a free-body diagram for writing equilibrium equations. *The deformations shown in the deformed shape of the structure must be consistent with the direction of forces drawn on the free-body diagram.* Tensile (compressive) force on a bar on free body diagram must correspond to extension (contraction) of the bar shown in deformed shape.

In many structures there are gaps between structural members. These gaps may be by design to permit expansion due to temperature changes, or they may be inadvertent due to improper accounting for manufacturing tolerances. We shall make use of the observation that for a linear system it does not matter how we reach the final equilibrium state. We therefore shall start by assuming that at the final equilibrium state the gap is closed. At the end of analysis we will check if our assumption of gap closure is correct or incorrect and make corrections as needed.

Displacement, strain, stress, and internal force are all related as depicted by the logic shown in Figure 4.9 and incorporated in the formulas developed in Section 4.2. If one of these quantities is found, then the rest could be found for an axial member. Thus theoretically, in structural analysis, any of the four quantities could be treated as an unknown variable. Analysis however, is traditionally conducted using either forces (internal or reaction) or displacements as the unknown variables, as described in the two methods that follow.

4.3.2 Force Method, or Flexibility Method

In this method internal forces or reaction forces are treated as the unknowns. The coefficient L/EA, multiplying the internal unknown force in Equation (4.21), is called the **flexibility coefficient**. If the unknowns are internal forces (rather than reaction forces), as is usually the case in large structures, then the matrix in the simultaneous equations is called the **flexibility matrix**. Reaction forces are often preferred in hand calculations because the number of unknown reactions (degree of static redundancy) is either equal to or less than the total number of unknown internal forces.

4.3.3 Displacement Method, or Stiffness Method

In this method the displacements of points are treated as the unknowns. The minimum number of displacements that are necessary to describe the deformed geometry is called **degree of freedom**. The coefficient multiplying the deformation EA/L is called the **stiffness coefficient**. Using small-strain approximation, the relationship between the displacement of points and the deformation of the bars is found from the deformed shape and substituted in the compatibility equations. Using Equation (4.21) and equilibrium equations, the displacement and the external forces are related. The matrix multiplying the unknown displacements in a set of algebraic equations is called the **stiffness matrix**.

4.3.4 General Procedure for Indeterminate Structure

The procedure outlined can be used for solving statically indeterminate structure problems by either the force method or by the displacement method.

1. If there is a gap, assume it will close at equilibrium.

2. Draw free-body diagrams, noting the tensile and compressive nature of internal forces. Write equilibrium equations relating internal forces to each other.

or

3. Write equilibrium equations in which the internal forces are written in terms of reaction forces, if the force method is to be used.

4. Draw an exaggerated approximate deformed shape, ensuring that the deformation is consistent with the free body diagrams of step 2. Write compatibility equations relating deformation of the bars to each other.

or

5. Write compatibility equations in terms of unknown displacements of points on the structure, if displacement method is to be used.

6. Write internal forces in terms of deformations using Equation (4.21).

7. Solve the equations of steps 2, 3, and 4 simultaneously for the unknown forces (for force method) or for the unknown displacements (for displacement method).

8. Check whether the assumption of gap closure in step 1 is correct.

Both the force method and the displacement method are used in Examples 4.10 and 4.11 to demonstrate the similarities and differences in the two methods.

EXAMPLE 4.9

The three bars in Figure 4.25 are made of steel (E = 30,000 ksi) and have cross-sectional areas of 1 in^2. Determine the displacement of point D

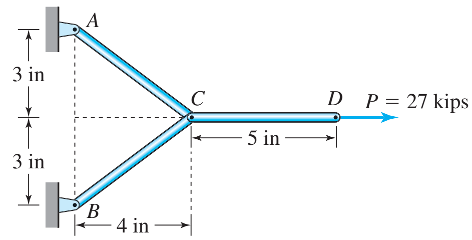


Figure 4.25 Geometry in Example 4.9

PLAN

The displacement of point D with respect to point C can be found using Equation (4.21). The deformation of rod AC or BC can also be found from Equation (4.21) and related to the displacement of point C using small-strain approximation.

SOLUTION

Figure 4.26 shows the free body diagrams. By equilibrium of forces in Figure 4.26a, we obtain

$$\boldsymbol{N}_{CD} = 27 \text{ kips} \tag{E1}$$

By equilibrium of forces in Figure 4.26b, we obtain

$$\boldsymbol{N}_{CA} = \boldsymbol{N}_{CB} \tag{E2}$$

$$N_{CA}\cos\theta + N_{CB}\cos\theta = 27 \text{ kips} \tag{E3}$$

Substituting for θ from Figure 4.26c and solving Equations (E2) and (E3), we obtain

$$2\boldsymbol{N}_{CA}\left(\frac{4}{5}\right) = 27 \text{ kips} \qquad \text{or} \qquad \boldsymbol{N}_{CA} = \boldsymbol{N}_{CB} = 16.875 \text{ kips} \tag{E4}$$

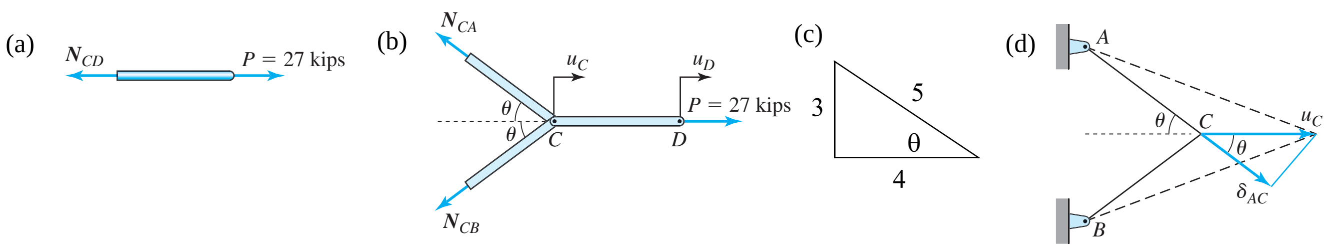


Figure 4.26 Free-body diagrams and deformed geometry in Example 4.9.

From Equation (4.21) we obtain the relative displacement of D with respect to C as shown in Equation (E5) and deformation of bar AC in Equation (E6).

$$\delta_{CD} = u_D - u_C = \frac{N_{CD}L_{CD}}{E_{CD}A_{CD}} = \frac{(27 \text{ kips})(5 \text{ in.})}{(30{,}000 \text{ ksi})(1 \text{ in.}^2)} = 4.5 \times 10^{-3} \text{ in.} \tag{E5}$$

$$\delta_{AC} = \frac{N_{CA}L_{CA}}{E_{CA}A_{CA}} = \frac{(16.875 \text{ kips})(5 \text{ in.})}{(30{,}000 \text{ ksi})(1 \text{ in.}^2)} = 2.8125 \times 10^{-3} \text{ in.} \tag{E6}$$

Figure 4.26d shows the exaggerated deformed geometry of the two bars AC and BC. The displacement of point C can be found by

$$u_C = \frac{\delta_{AC}}{\cos\theta} = 3.52 \times 10^{-3} \text{ in.} \tag{E7}$$

Adding Equations (E5) and (E7) we obtain the displacement of point D,

$$u_D = (4.5 + 3.52)10^{-3} \text{ in.}$$

ANS. $u_D = 0.008$ in

COMMENT

1. This was a statically determinate problem as we could find the internal forces in all members by static equilibrium.

EXAMPLE 4.10

An aluminum rod (E_{al} = 70 GPa) is securely fastened to a rigid plate that does not rotate during the application of load P is shown in Figure 4.27. A gap of 0.5 mm exists between the rigid plate and the steel rod (E_{st} = 210 GPa) before the load is applied. The aluminum rod has a diameter of 20 mm and the steel rod has a diameter of 10 mm. Determine (a) the movement of the rigid plate; (b) the axial stress in steel.

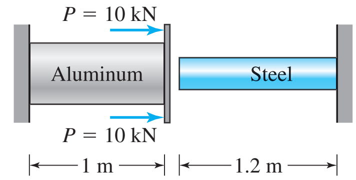


Figure 4.27 Geometry in Example 4.10.

FORCE METHOD: PLAN

We assume that the force P is sufficient to close the gap at equilibrium. The two unknown wall reactions minus one equilibrium equation results in 1 degree of static redundancy. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 Assume force P is sufficient to close the gap. If this assumption is correct, then steel will be in compression and aluminum will be in tension.

Step 2 The degree of static redundancy is 1. Thus we use one unknown reaction to formulate our equilibrium equations. We make imaginary cuts at the equilibrium position and obtain the free-body diagrams in Figure 4.28. By equilibrium of forces we can obtain the internal forces in terms of the wall reactions,

$$N_{al} = R_L \qquad N_S = 20(10^3) - R_L \tag{E1}$$

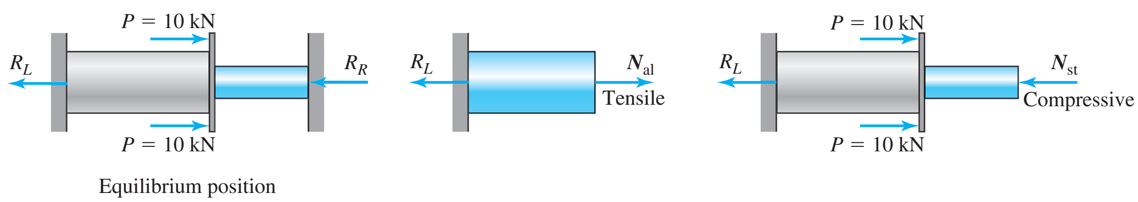


Figure 4.28 Free-body diagrams in Example 4.10.

Step 3 Figure 4.29 shows the exaggerated deformed shape. The deformation of aluminum is extension and steel in contraction, to ensure consistency with the tensile and compressive axial forces shown on the free-body diagrams in Figure 4.28. The compatibility equation can be written

$$\delta_{st} = (\delta_{al} - 0.0005)\ \text{m} \tag{E2}$$

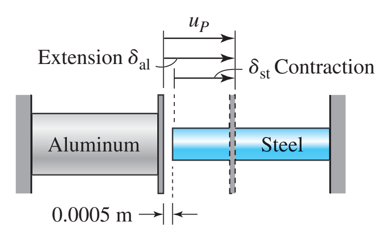


Figure 4.29 Approximate deformed shape in Example 4.10.

Step 4 The radius of the aluminum rod is 0.01 m, and the radius of the steel rod is 0.005 m. We can write the deformation of aluminum and steel in terms of the internal forces,

$$\delta_{al} = \frac{N_{al}L_{al}}{E_{al}A_{al}} = \frac{N_{al}(1\ \text{m})}{(70 \times 10^9\ \text{N/m}^2)[\pi(0.01\ \text{m})^2]} = 0.04547 N_{al} \times 10^{-6}\ \text{m} \tag{E3}$$

$$\delta_{st} = \frac{N_{st}L_{st}}{E_{st}A_{st}} = \frac{N_{st}(1.2\ \text{m})}{(210 \times 10^9\ \text{N/m}^2)[\pi(0.005\ \text{m})^2]} = 0.07277 N_{st} \times 10^{-6}\ \text{m} \tag{E4}$$

Step 5 Substituting Equation (E1) into Equations (E3) and (E4), we obtain deformation in terms of the unknown reactions,

$$\delta_{al} = 0.04547 R_L \times 10^{-6}\ \text{m} \tag{E5}$$

$$\delta_{st} = 0.07277(20 \times 10^3 - R_L)10^{-6}\ \text{m} = (1455.4 - 0.07277 R_L)10^{-6}\ \text{m} \tag{E6}$$

Substituting Equations (E5) and (E6) into Equation (E2), we can solve for R_L.

$$1455.4 - 0.07277 R_L = 0.04547 R_L - 500 \quad \text{or} \quad R_L = 16{,}538\ \text{N} \tag{E7}$$

Substituting Equation (E7) into Equations (E1) and (E1) we obtain the internal forces,

$$N_{al} = 16{,}538\ \text{N} \qquad N_{st} = 3462\ \text{N} \tag{E8}$$

Step 6 The positive value of the force in steel confirms that it is compressive and the assumption of the gap being closed is correct.

(a) Substituting Equation (E7) into Equation (E5), we obtain the deformation of aluminum, which is equal to the movement of the rigid plate u_P:

$$u_P = \delta_{al} = (0.04547)\ (16{,}538)\ 10^{-6} = 0.752\ (10^{-3})\ \text{m}$$

ANS. $u_P = 0.752$ mm

(b) The normal stress in steel can be found from Equation (4.8).

$$\sigma_{st} = \frac{N_{st}}{A_{st}} = \frac{3462\ \text{N}}{\pi(0.005\ \text{m})^2} = 44.1(10^6)\ \text{N/m}^2 \tag{E9}$$

ANS. $\sigma_{st} = 44.1$ MPa (C)

COMMENTS

1. The assumption about gap closure is correct because movement of plate $u_P = 0.752$ mm is greater than the gap.
2. An alternative approach is to use internal forces as the unknowns. We can make a cut on either side of the rigid plate at the equilibrium position and draw the free-body diagram, as shown in Figure 4.30*a*. We can then write the equilibrium equation,

$$N_{st} + N_{al} = 20 \times 10^3\ \text{N} \tag{E10}$$

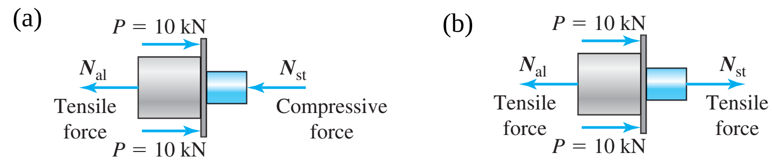


Figure 4.30 (a) Alternative free-body diagram in Example 4.10. (b) Tensile forces in free-body diagram in Example 4.10.

Substituting Equations (E3) and (E4) into (E2), we obtain

$$0.04547N_{al} - 0.07277N_{st} = 500\ \text{N} \tag{E11}$$

Equations (E10) and (E11) can be written in matrix form as

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0.04547 & -0.07277 \end{bmatrix}}_{[F]} \underbrace{\begin{Bmatrix} N_{at} \\ N_{st} \end{Bmatrix}}_{\{N\}} = \underbrace{\begin{Bmatrix} 20 \times 10^3 \\ 500 \end{Bmatrix}}_{\{P\}}$$

The matrix $[F]$ is called the *flexibility matrix*.

3. With internal forces as unknowns we had to solve two equations simultaneously, as elaborated in comment 2. With the reaction force as the unknown we had only one unknown, which is the number of degrees of static redundancy. Thus for hand calculations the reaction forces as unknowns are preferred when using the force method. But in computer programs the process of substitution in step 5 is difficult to implement compared to constructing the equilibrium and compatibility equations in terms of internal forces. Thus in computer methods internal forces are treated as unknowns in force methods.
4. Suppose we had started with the direction of the force in steel as tension as shown in Figure 4.30*b*. Then we would get the following equilibrium equation:

$$-N_{st} + N_{al} = 20 \times 10^3\ \text{N} \tag{E12}$$

Suppose we incorrectly do not make any changes in Equation (E2) or Equation (E6)—that is, we continue to use the deformation in steel as contraction even though the assumed force is tensile, we then solve Equations (E12) and (E11), we obtain $N_{al} = 34996$ N and $N_{st} =$ 14996 N. These answers demonstrate how a simple error in sign produces dramatically different results.

DISPLACEMENT METHOD: PLAN

Let the plate move to the right by the amount u_P and assume that the gap is closed. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 Assume the gap is closed.

Step 2 We can substitute (E1) into (E1) to eliminate R_L and obtain the equilibrium equation,

$$N_{st} + N_{al} = 20 \times 10^3\ \text{N} \tag{E13}$$

We could also obtain this equation from the free-body diagram shown in Figure 4.30.

Step 3 We draw the exaggerated deformed shape, as shown in Figure 4.29, and obtain the deformation of the bars in terms of the plate displacement u_P as

$$\delta_{al} = u_P \tag{E14}$$

$$\delta_{st} = u_P - 0.0005\ \text{m} \tag{E15}$$

Step 4 We can write the internal forces in terms of deformation,

$$N_{al} = \delta_{al}\left(\frac{E_{al}A_{al}}{L_{al}}\right) = 21.99(10^6)\delta_{al}\ \text{N} \tag{E16}$$

$$N_{st} = \delta_{st}\left(\frac{E_{st}A_{st}}{L_{st}}\right) = 13.74(10^6)\delta_{st} \text{ N} \tag{E17}$$

Step 5 We can substitute Equations (E14) and (E15) into Equations (E16) and (E17) to obtain the internal forces in terms of u_P.

$$N_{al} = 21.99\,(10^6)\,u_P \text{ N} \tag{E18}$$

$$N_{st} = 13.74\,(u_P - 0.0005)(10^6) \text{ N} \tag{E19}$$

Substituting Equations (E18) and (E19) into (E13) we obtain the displacement u_P.

$$21.99u_P + 13.74(u_P - 0.0005) = 20(10^{-3}) \qquad \text{or} \qquad u_P = 0.752(10^{-3}) \text{ m} \tag{E20}$$

ANS. $u_P = 0.752$ mm

Step 6 As $u_P > 0.0005$ m, the assumption of gap closing is correct.
Substituting u_P into Equation (E19), we obtain $N_{st} = 3423.2$ N, which implies that the steel is in compression, as expected. We can now find the axial stress in steel, as before.

COMMENT

1. In the force method as well as in the displacement method the number of unknowns was 1 as the degree of redundancy and the number of degrees of freedom were 1. This is not always the case. In the next example the number of degrees of freedom is less than the degree of redundancy, and hence the displacement method will be easier to implement.

EXAMPLE 4.11

Three steel bars A, B, and C ($E = 200$ GPa) have lengths $L_A = 4$ m, $L_B = 3$ m, and $L_C = 2$ m, as shown in Figure 4.31. All bars have the same cross-sectional area of 500 mm^2. Determine (a) the elongation in bar B; (b) the normal stress in bar C.

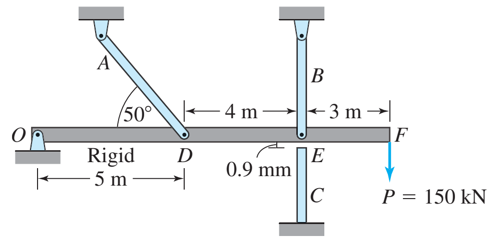


Figure 4.31 Geometry in Example 4.11.

DISPLACEMENT METHOD: PLAN

Assume that the gap is closed. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 We assume that the force P is sufficient to close the gap.

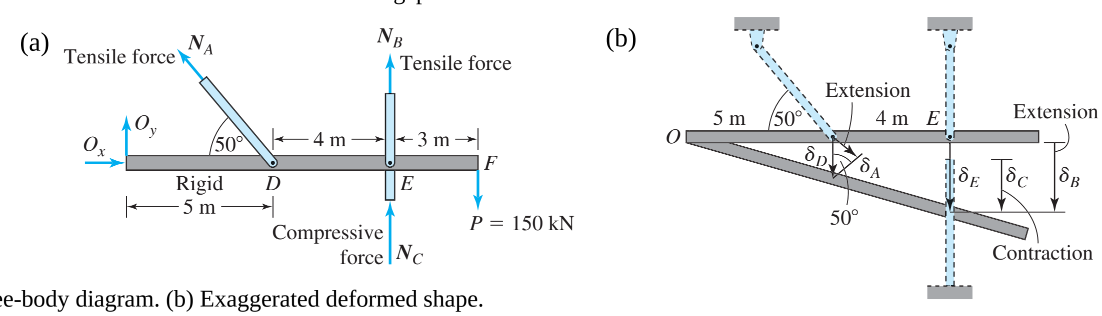


Figure 4.32 (a) Free-body diagram. (b) Exaggerated deformed shape.

Step 2 We draw the free-body diagram of the rigid bar in Figure 4.32*a* with bars A and B in tension and bar C in compression. By equilibrium of moment at point O we obtain the equilibrium equation shown in Equation (E1).

$$N_A \sin 50(5) + N_B(9) + N_C(9) - P(12) = 0 \qquad \text{or} \qquad 3.83N_A + 9N_B + 9N_C = 1800(10^3) \tag{E1}$$

Step 3 Figure 4.32*b* shows an exaggerated deformed shape with bars A and B as extension and bar C as contraction to be consistent with the forces drawn in Figure 4.32*a*. Noting that the gap is 0.0009 m, we can write the compatibility equations relating the deformations of bars B and C in terms of the displacement of pin E,

$$\delta_B = \delta_E \tag{E2}$$

$$\delta_C = \delta_E - 0.0009 \text{ m} \tag{E3}$$

Using similar triangles in Figure 4.32*b* we relate the displacements of point D and E,

$$\frac{\delta_D}{5\text{ m}} = \frac{\delta_E}{9\text{ m}} \tag{E4}$$

Using small-strain approximation we can relate the deformation of bar *A* to the displacement of point *D*,

$$\delta_D = \frac{\delta_A}{\sin 50} \tag{E5}$$

Substituting Equation (E5) into Equation (E4), we obtain

$$\frac{(\delta_A/\sin 50)}{5\text{ m}} = \frac{\delta_E}{9\text{ m}} \qquad \text{or} \qquad \delta_A = 0.4256\delta_E \tag{E6}$$

Step 4 The axial rigidity of all bars is $EA = [200\ (10^9)\ \text{N/m}^2]\ [500\ (10^{-6})\ \text{m}^2] = 100 \times 10^6\ \text{N}$. Using Equation (4.21) we can write,

$$N_A = \frac{100(10^6)}{4}\delta_A\ \text{N} = 25(10^6)\delta_A\ \text{N} \tag{E7}$$

$$N_B = \frac{100(10^6)}{3}\delta_B = 33.33(10^6)\delta_B\ \text{N} \tag{E8}$$

$$N_C = \frac{100(10^6)}{2}\delta_C = 50.00(10^6)\delta_C\ \text{N} \tag{E9}$$

Step 5 Substituting Equations (E6), (E2), and (E3) into Equations (E7), (E8), and (E9), we obtain

$$N_A = 25(10)^6(0.4256\delta_E) = 10.64(10^6)\delta_E\ \text{N} \tag{E10}$$

$$N_B = 33.33(10^6)\delta_E = 33.33(10^6)\delta_E\ \text{N} \tag{E11}$$

$$N_C = 50.00(10^6)(\delta_E - 0.0009) = [50.00(10^6)\delta_E - 45(10^3)]\ \text{N} \tag{E12}$$

Substituting Equations (E10), (E11), and (E12) into Equation (E1) we obtain the displacement of pin *E*,

$$3.83(10.64)(10^6)\delta_E + 9(33.33)(10^6)\delta_E + 9[50.00(10^6)\delta_E - 45(10^3)] = 1800(10^3) \qquad \text{or} \qquad \delta_E = 2.788(10^{-3})\ \text{m}$$

ANS. $\delta_E = 2.8$ mm

Step 6 The assumption of gap closure is correct as $\delta_E = 2.8$ mm whereas the gap is only 0.9 mm.
From Equation (12) we obtain the internal axial force in bar *C*, from which we obtain the axial stress in bar *C*,

$$N_C = 50.00(10^6)[2.788(10^{-3})] - 45(10^3) = 94.4(10^3)$$

$$\sigma_C = \frac{N_C}{A_C} = \frac{94.4(10^3)\ \text{N}}{500(10^{-6})\ \text{m}^2} = 188.8 \times 10^6\ N/\text{m}^2 \tag{E13}$$

ANS. $\sigma_C = 189$ MPa (C)

COMMENTS

1. Equation (E4) is a relationship of points on the rigid bar. Equations (E2), (E3), and (E5) relate the motion of points on the rigid bar to the deformation of the rods. This two-step process helps break the complexity into simpler steps.
2. The degree of freedom for this system is 1. In place of δ_E as an unknown, we could have used the displacement of any point on the rigid bar or the rotation angle of the bar, as all of these quantities are related.

FORCE METHOD: PLAN

We assume that the force *P* is sufficient to close the gap. If this assumption is correct, then bar *C* will be in compression. We follow the procedure outlined in Section 4.3.4 to solve the problem.

SOLUTION

Step 1 Assume that the gap closes.
Step 2 Figure 4.32*a* shows the free-body diagram of the rigid bar. By equilibrium we obtain Equation (E1), rewritten here for convenience.

$$3.83N_A + 9N_B + 9N_C = 1800(10^3)\ \text{N} \tag{E14}$$

Equation (14) has three unknowns, hence the degree of redundancy is 2. We will need two compatibility equations.
Step 3 We draw the deformed shape, as shown in Figure 4.32*b*, and obtain relationships between points on the rigid bar and the deformation of the bars. Then by eliminating δ_E from Equations (E3) and (E6) and using Equation (E2) we obtain the compatibility equations,

$$\delta_C = \delta_B - 0.0009\ \text{m} \tag{E15}$$

$$\delta_A = 0.4256\delta_B \tag{E16}$$

Step 4 We write the deformations of the bars in terms of the internal forces for each member,

$$\delta_A = \frac{N_A L_A}{E_A A_A} = \frac{N_A(4\text{ m})}{100(10^6)\text{ N}} = 0.04\boldsymbol{N}_A(10^{-6})\text{ m} \tag{E17}$$

$$\delta_B = \frac{N_B L_B}{E_B A_B} = \frac{N_B(3\text{ m})}{100(10^6)\text{ N}} = 0.03\boldsymbol{N}_B(10^{-6})\text{ m} \tag{E18}$$

$$\delta_C = \frac{N_C L_C}{E_C A_C} = \frac{N_C(2\text{ m})}{100(10^6)\text{ N}} = 0.02\boldsymbol{N}_C(10^{-6})\text{ m} \tag{E19}$$

Step 5 Substituting Equations (E17), (E18), and (E19) into Equations (E15) and (E16), we obtain

$$0.02\ \boldsymbol{N}_C = 0.03\ \boldsymbol{N}_B - 900\text{ N} \qquad \text{or} \qquad \boldsymbol{N}_C = 1.5\boldsymbol{N}_B - 45(10^3)\text{ N} \tag{E20}$$

$$0.04\ \boldsymbol{N}_A = 0.4256(0.03\ \boldsymbol{N}_B) \qquad \text{or} \qquad \boldsymbol{N}_A = 0.3192\ \boldsymbol{N}_B \tag{E21}$$

Solving Equations (E14), (E20), and (E21), we obtain the internal forces:

$$\boldsymbol{N}_A = 29.67(10^3)\text{ N} \qquad \boldsymbol{N}_B = 92.95(10^3)\text{ N} \qquad \boldsymbol{N}_C = 94.43(10^3)\text{ N} \tag{E22}$$

Step 6 The positive value of $\boldsymbol{N}_C$ confirms it is compressive and the assumption of the gap being closed is valid.

Substituting $\boldsymbol{N}_B$ from Equation (E22) into Equation (E18), we find the deformation of bar *B*, which is the same as the displacement of pin *E*,

$$\delta_E = \delta_B = 0.03[92.95(10^3)](10^{-6}) = 2.788(10^3)\text{ m} \tag{E23}$$

ANS. $\delta_E = 2.8\text{ mm}$

The calculation for the normal stress in bar *C* is as before.

COMMENT

1. Equations (E14), (E20), and (E21) represent three equations in three unknowns. Had we used the reaction forces O_x and O_y in Figure 4.32 as the unknowns, we could have generated two equations in two unknowns, consistent with the fact that this system has a degree of redundancy of 2. But as we saw, the displacement method required only one unknown. Thus our choice of method of solution should be dictated by the number of degrees of freedom and the number of degrees of static redundancy. For fewer degrees of freedom we should use the displacement method; for fewer degrees of static redundancy we should use the force method.

Consolidate your knowledge

1. With the book closed, write a procedure for solving a statically indeterminate problem by the force method.
2. With the book closed, write a procedure for solving a statically indeterminate problem by the displacement method.

PROBLEM SET 4.3

4.45 A rigid bar is hinged at *C* as shown in Figure P4.45. The modulus of elasticity of bar *A* is *E* = 30,000 ksi, the cross-sectional area is *A* = 1.25 in.^{2}, and the length is 24 in. Determine the applied force *F* if point *B* moves upward by 0.002 in.

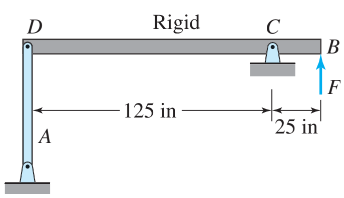


Figure P4.45

4.46 A rigid bar is hinged at *C* as shown in Figure P4.46. The modulus of elasticity of bar *A* is *E* = 30,000 ksi, the cross-sectional area is *A* = 1.25 in.^{2}, and the length is 24 in. Determine the applied force *F* if point *B* moves upward by 0.002 in.

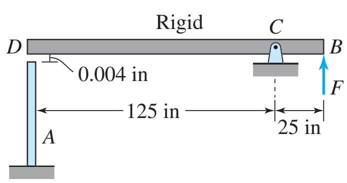


Figure P4.46

4.47 A rigid bar is hinged at C as shown in Figure P4.47. The modulus of elasticity of bar A is $E = 100$ GPa, the cross-sectional area is $A = 15$ mm^2, and the length is 1.2 m. Determine the applied force F if point B moves to the left by 0.75 mm.

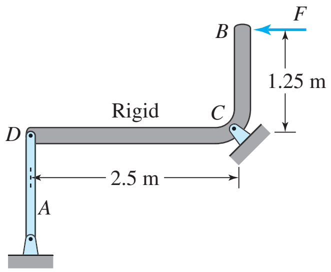


Figure P4.47

4.48 A rigid bar is hinged at C as shown in Figure P4.48. The modulus of elasticity of bar A is $E = 100$ GPa, the cross-sectional area is $A = 15$ mm^2, and the length is 1.2 m. Determine the applied force F if point B moves to the left by 0.75 mm.

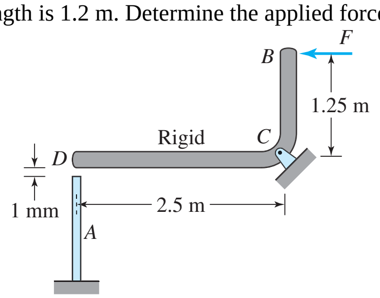


Figure P4.48

4.49 The roller at P in Figure P4.49 slides in the slot due to the force $F = 20$ kN. Member AP has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Determine the displacement of the roller.

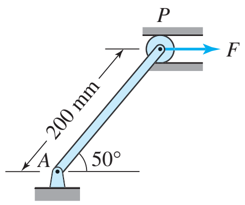


Figure P4.49

4.50 The roller at P in Figure P4.50 slides in the slot due to the force $F = 20$ kN. Member AP has a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. Determine the displacement of the roller.

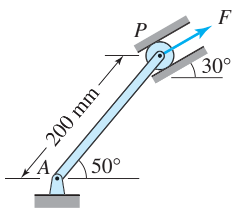
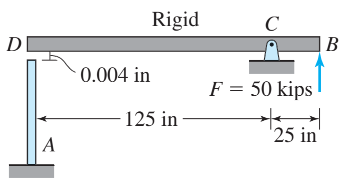


Figure P4.50

4.51 A rigid bar is hinged at C as shown in Figure P4.51. The modulus of elasticity of bar A is $E = 30{,}000$ ksi, the cross-sectional area is $A = 1.25$ in.^{2}, and the length is 24 in. Determine the axial stress in bar A and the displacement of point D on the rigid bar.



FigureP4.51

4.52 A rigid bar is hinged at C as shown in Figure P4.52. The modulus of elasticity of bar A is $E = 30{,}000$ ksi, the cross-sectional area is $A = 1.25$ in.^{2}, and the length is 24 in. Determine the axial stress in bar A and the displacement of point D on the rigid bar.

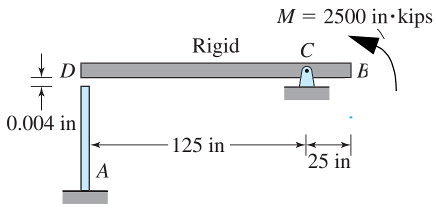
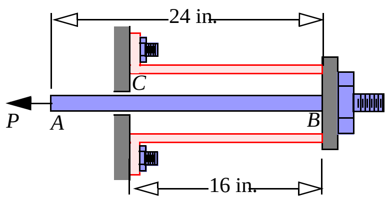


Figure P4.52

4.53 A steel (E = 30,000 ksi, ν = 0.28) rod passes through a copper (E = 15,000 ksi, ν = 0.35) tube as shown in Figure P4.53. The steel rod has a diameter of 1/2 in., and the tube has an inside diameter of 3/4 in. and a thickness of 1/8 in. If the applied load is P = 2.5 kips, determine (a) the movement of point A (b) the change in diameter of the steel rod.



FigureP4.53

4.54 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/2 in. as shown in Figure P4.54. The bar is horizontal before the force is applied. Determine the angle of rotation of the bar from the horizontal when a force P= 5 kips is applied.

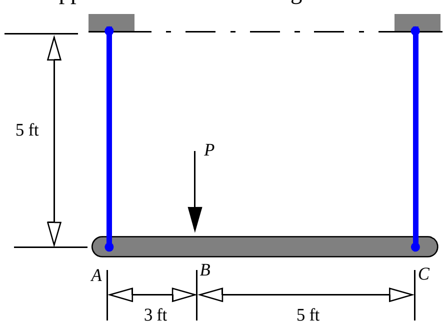


Figure P4.54

4.55 Two rigid beams are supported by four axial steel (E = 210 GPa) rods of diameter 10 mm, as shown in Figure P4.55. Determine the angle of rotation of the bars from the horizontal no load position when a force of P = 5 kN is applied.

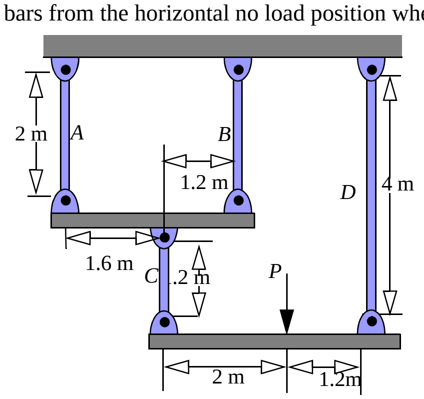


Figure P4.55

4.56 Two rigid beams are supported by four axial steel rods (E = 210 GPa, σ_{yield} = 210 MPa) of diameter 20 mm, as shown in Figure P4.55. For a factor of safety of 1.5, determine the maximum value of force F that can be applied without causing any rod to yield.

4.57 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/2 in., as shown in Figure P4.57. Determine the extensions of cables *CE* and *BD* when a force *P*= 5 kips is applied.

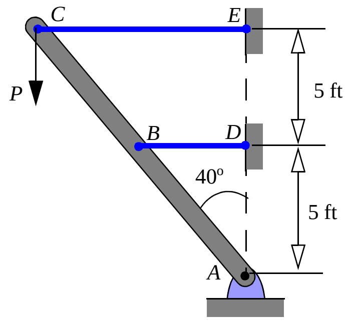


Figure P4.57

4.58 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) as shown in Figure P4.57. The yield stress of aluminum is 40 ksi. If the applied force *P* = 10 kips, determine the minimum diameter of cables *CE* and *BD* to the nearest 1/16 in.

4.59 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/2 in. as shown in Figure P4.57. The yield stress of aluminum is 40 ksi. Determine the maximum force *P* to the nearest pound that can be applied.

4.60 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P4.60. Both bars have a cross-sectional area $A = 100\text{ mm}^2$ and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm. Determine the displacement of the roller and the axial stress in bar *AP*.

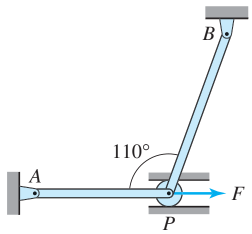


Figure P4.60

4.61 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P4.61. Both bars have a cross-sectional area $A = 100\text{ mm}^2$ and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm. Determine the displacement of the roller and the axial stress in bar *AP*.

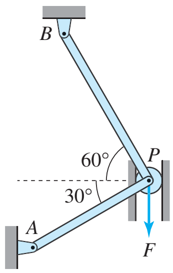


Figure P4.61

4.62 A force $F = 20$ kN is applied to the roller that slides inside a slot as shown in Figure P4.62. Both bars have a cross-sectional area $A = 100\text{ mm}^2$ and a modulus of elasticity $E = 200$ GPa. Bars *AP* and *BP* have lengths $L_{AP} = 200$ mm and $L_{BP} = 250$ mm. Determine the displacement of the roller and the axial stress in bar *AP*.

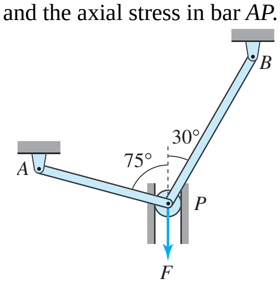


Figure P4.62

4.63 An aluminum (E = 70 GPa, σ_{yield} = 280 MPa, ν = 0.28) wire of diameter 0.5 mm is to hang two flower pots of equal mass as shown in Figure P4.63. (a) Determine the maximum mass of the pots to the nearest gram that can be hung if yielding is to be avoided in all wires. (b) For the maximum mass what is the percentage change in the diameter of the wire *B*C.

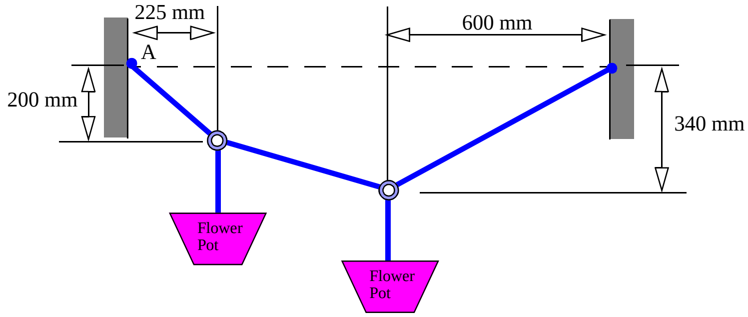
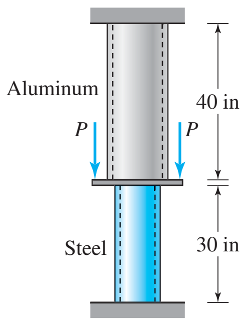


Figure P4.63

4.64 An aluminum (E = 70 GPa, σ_{yield} = 280 MPa, ν = 0.28) wire is to hang two flower pots of equal mass of 5 kg as shown in Figure P4.63. Determine the minimum diameter of the wires to the nearest 1/10 of a millimeter if yielding is to be avoided in all wires.

4.65 An aluminum hollow cylinder (E_{al} = 10,000 ksi, ν_{al} = 0.25) and a steel hollow cylinder (E_{st} = 30,000 ksi, ν_{st} = 0.28) are securely fastened to a rigid plate, as shown in Figure P4.65. Both cylinders are made from $\frac{1}{8}$-in. thickness sheet metal. The outer diameters of the aluminum and steel cylinders are 4 in. and 3 in., respectively. For an applied load of *P* = 20 kips determine (a) the displacement of the rigid plate; (b) the change in diameter of each cylinder.



FigureP4.65

4.66 An aluminum hollow cylinder (E_{al} = 10,000 ksi, ν_{al} = 0.25) and a steel hollow cylinder (E_{st} = 30,000 ksi, ν_{st} = 0.28) are securely fastened to a rigid plate, as shown in Figure P4.65. Both cylinders are made from $\frac{1}{8}$-in. thickness sheet metal. The outer diameters of the aluminum and steel cylinders are 4 in. and 3 in., respectively. The allowable stresses in aluminum and steel are 10 ksi and 25 ksi, respectively. Determine the maximum force *P* that can be applied to the assembly.

4.67 A gap of 0.004 inch exists between the rigid bar and bar *A* before the force *F* is applied as shown in Figure P4.67. The rigid bar is hinged at point *C*. The lengths of bars *A* and *B* are 30 and 50 inches respectively. Both bars have an area of cross-section *A* = 1 in.^{2} and modulus of elasticity *E* = 30,000 ksi. Determine the axial stresses in bars *A* and *B* if *P* = 100 kips.

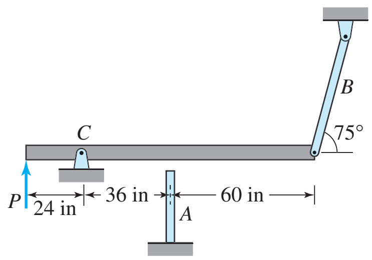


Figure P4.67

4.68 A gap of 0.004 inch exists between the rigid bar and bar *A* before the force *F* is applied as shown in Figure P4.67. The rigid bar is hinged at point *C*. The lengths of bars *A* and *B* are 30 in. and 50 in. respectively. Both bars have an area of cross-section *A* = 1 in.^{2} and modulus of elasticity *E* = 30,000 ksi. If the allowable normal stress in the bars is 20 ksi in tension or compression, determine the maximum force *P* that can be applied.

4.69 In Figure P4.69 a gap exists between the rigid bar and rod *A* before force *F* is applied. The rigid bar is hinged at point *C*. The lengths of bars *A* and *B* are 1 m and 1.5 m, and the diameters are 50 mm and 30 mm, respectively. The bars are made of steel with a modulus of elasticity $E = 200$ GPa and Poisson's ratio $\nu = 0.28$. If $F = 75$ kN determine (a) the deformation of the two bars; (b) the change in the diameters of the two bars.

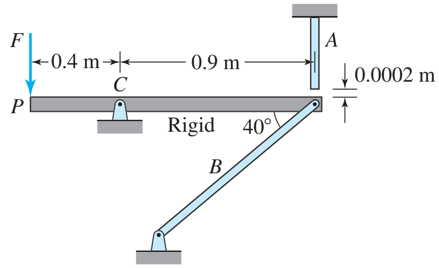


Figure P4.69

4.70 In Figure P4.69 a gap exists between the rigid bar and rod *A* before force *F* is applied. The rigid bar is hinged at point *C*. The lengths of bars *A* and *B* are 1 m and 1.5 m, and the diameters are 50 mm and 30 mm, respectively. The bars are made of steel with a modulus of elasticity $E = 200$ GPa and Poisson's ratio $\nu = 0.28$. If the allowable axial stresses in bars *A* and *B* are 110 MPa and 125 MPa, respectively, determine the maximum force *F* that can be applied.

4.71 A rectangular aluminum bar ($E = 10{,}000$ ksi), a steel bar ($E = 30{,}000$ ksi), and a brass bar ($E = 15{,}000$ ksi) are assembled as shown in Figure P4.71. All bars have the same thickness of 0.5 in. A gap of 0.02 in. exists before the load *P* is applied to the rigid plate. Assume that the rigid plate does not rotate. If $P = 15$ kips determine (a) the axial stress in steel; (b) the displacement of the rigid plate with respect to the right wall.

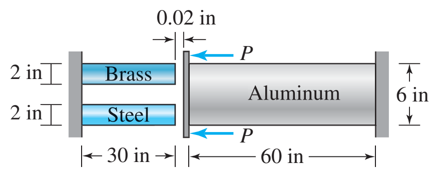


Figure P4.71

4.72 A rectangular aluminum bar ($E = 10{,}000$ ksi), a steel bar ($E = 30{,}000$ ksi), and a brass bar ($E = 15{,}000$ ksi) are assembled as shown in Figure P4.71. All bars have the same thickness of 0.5 in. A gap of 0.02 in. exists before the load *P* is applied to the rigid plate. Assume that the rigid plate does not rotate. If the allowable axial stresses in brass, steel, and aluminum are 8 ksi, 15 ksi, and 10 ksi, respectively, determine the maximum load *P*.

4.73 In Figure P4.73 bars *A* and *B* have cross-sectional areas of 400 mm^2 and a modulus of elasticity $E = 200$ GPa. A gap exists between bar *A* and the rigid bar before the force *F* is applied. If the applied force $F = 10$ kN determine: (a) the axial stress in bar *B;* (b) the deformation of bar *A*.

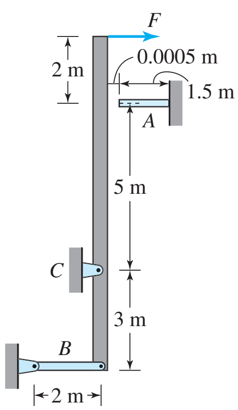


Figure P4.73

4.74 In Figure P4.73 bars *A* and *B* have cross-sectional areas of 400 mm^2 and a modulus of elasticity $E = 200$ GPa. A gap exists between bar *A* and the rigid bar before the force *F* is applied.Determine the maximum force *F* that can be applied if the allowable stress in member *B* is 120 MPa (C) and the allowable deformation of bar *A* is 0.25 mm.

4.75 A rectangular steel bar ($E = 30{,}000$ ksi, $\nu = 0.25$) of 0.5 in. thickness has a gap of 0.01 in. between the section at *D* and a rigid wall before the forces are applied as shown in Figure P4.75. Assuming that the applied forces are sufficient to close the gap, determine (a) the movement of rigid plate at *C* with respect to the left wall; (b) the change in the depth *d* of segment *CD*.

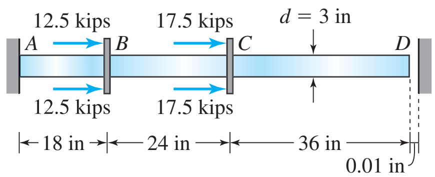


Figure P4.75

4.76 Three plastic members of equal cross sections are shown in Figure P4.76. Member *B* is smaller than members A by 0.5 mm. A distributed force is applied to the rigid plate, which moves downward without rotating. The moduli of elasticity for members *A* and *B* are 1.5 GPa and 2.0 GPa, respectively. Determine the axial stress in each member if the distributed force $W = 20$ MPa.

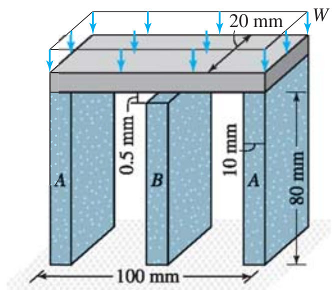


Figure P4.76

4.77 Three plastic members of equal cross sections are shown in Figure P4.76. Member *B* is smaller than members A by 0.5 mm. A distributed force is applied to the rigid plate, which moves downward without rotating. The moduli of elasticity for members *A* and *B* are 1.5 GPa and 2.0 GPa, respectively. Determine the maximum intensity of the distributed force that can be applied to the rigid plate if the allowable stresses in members *A* and *B* are 50 MPa and 30 MPa.

4.78 Figure P4.78 shows an aluminum rod (E = 70 GPa, ν = 0.25) inside a steel tube (E = 210 GPa, ν = 0.28). The aluminium rod is slightly longer than the steel tube and has a diameter of 40 mm. The steel tube has an inside diameter of 50 mm and is 10 mm thick. If the applied load $P =$ 200 kN, determine (a) the axial stresses in aluminium rod and steel tube; (b) the change in diameter of aluminium.

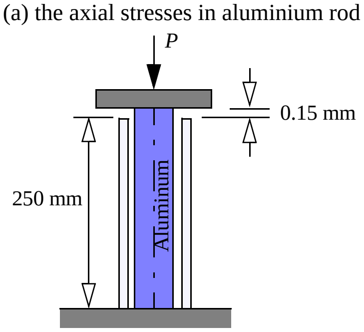


Figure P4.78

4.79 Figure P4.78 shows an aluminium rod (E = 70 GPa, σ_{yield} = 280 MPa) inside a steel tube (E = 210 GPa, σ_{yield} = 210 MPa). The aluminium rod is slightly longer than the steel tube and has a diameter of 40 mm. The steel tube has an inside diameter of 50 mm and is 10 mm thick. What is the maximum force *P* that can be applied without yielding either material.

4.80 A rigid bar *ABCD* hinged at one end and is supported by two aluminum cables (E = 10,000 ksi) with a diameter of 1/4 in. as shown in Figure P4.80. The bar is horizontal before the force is applied. Determine the angle of rotation of the bar from the horizontal when a force *P*= 10kips is applied.

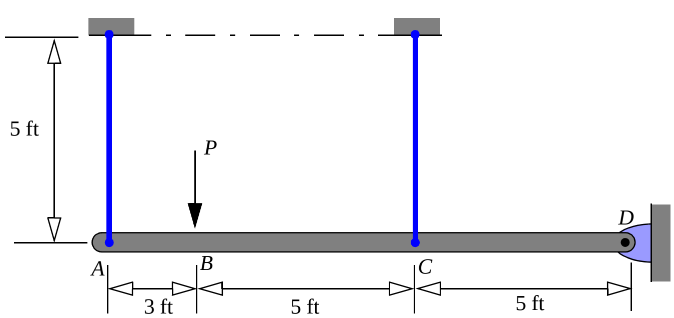


Figure P4.80

4.81 A suspended walkway is modelled as a rigid bar and supported by steel rods (E =30,000 ksi) as shown in Figure P4.81. The rods have a diameter of 2 in., and the nut has a contact area with the bottom of the walkway is 4 in.^{2} The weight of the walk per unit length is w = 725lb/ft. Determine (a) the axial stress in the steel rods; (b) the average bearing stress between the nuts at A and D and the walkway.

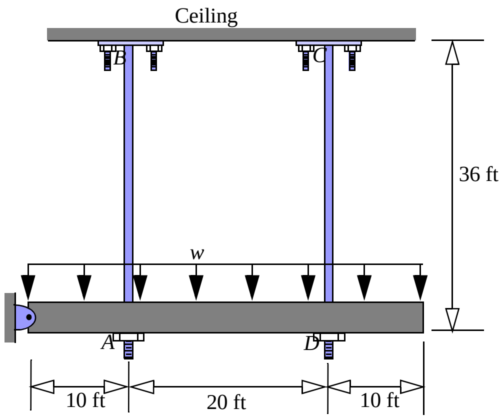


Figure P4.81

4.82 An aluminum circular bar (E_{al} = 70 GPa) and a steel tapered circular bar (E_{st} = 200 GPa) are securely attached to a rigid plate on which axial forces are applied, as shown in Figure P4.82. Determine (a) the displacement of the rigid plate; (b) the maximum axial stress in steel

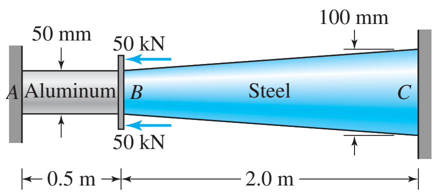


Figure P4.82

Design problems

4.83 A rigid bar hinged at point *O* has a force *P* applied to it, as shown in Figure P4.83. Bars *A* and *B* are made of steel (*E* = 30,000 ksi). The cross-sectional areas of bars *A* and *B* are $A_A = 1$ in.^{2} and $A_B = 2$ in.^{2}. If the allowable deflection at point *C* is 0.01 in. and the allowable stress in the bars is 25 ksi, determine the maximum force *P* that can be applied

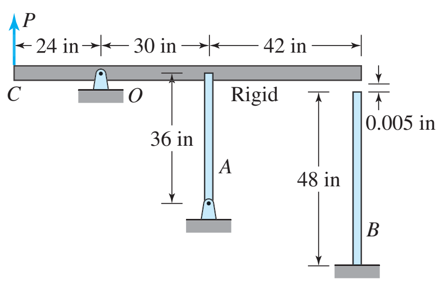


Figure P4.83

4.84 The structure at the base of a crane is modeled by the pin-connected structure shown in Figure P4.84. The allowable axial stresses in members *AC* and *BC* are 15 ksi, and the modulus of elasticity is 30,000 ksi. To ensure adequate stiffness at the base, the displacement of pin *C* in the vertical direction is to be limited to 0.1 in. Determine the minimum cross-sectional areas for members *AC* and *BC*.

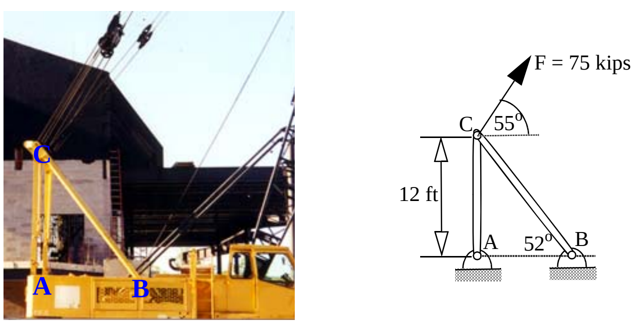


Figure P4.84

4.85 The landing wheel of a plane is modeled as shown in Figure P4.85. The pin at *C* is in double shear and has an allowable shear stress of 12 ksi. The allowable axial stress for link *BC* is 30 ksi. Determine the diameter of pin *C* and the effective cross-sectional area of link *BC*. (*Note:*

Attachments at *A* and *B* are approximated by pins to simplify analysis. There are two links represented by *BC*, one on either side of the hydraulic cylinder, which we are modeling as a single link with an effective cross-sectional area that is to be determined so that the free-body diagram is two-dimensional.).

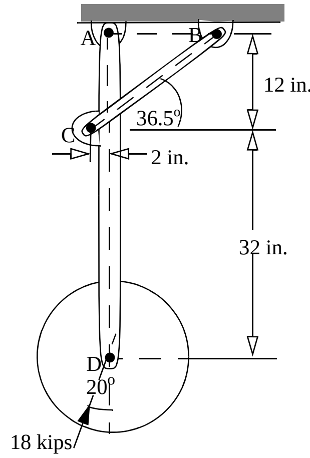


Figure P4.85

Stress concentration

4.86 The allowable shear stress in the stepped axial rod shown in Figure P4.86 is 20 ksi. If $F = 10$ kips, determine the smallest fillet radius that can be used at section *B*. Use the stress concentration graphs given in Section C.4.2.

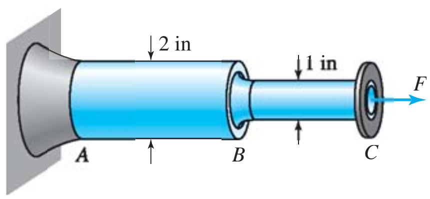


Figure P4.86

4.87 The fillet radius in the stepped circular rod shown in Figure P4.87 is 6 mm. Determine the maximum axial force *F* that can act on the rigid wheel if the allowable axial stress is 120 MPa and the modulus of elasticity is 70 GPa. Use the stress concentration graphs given in Section C.4.2.

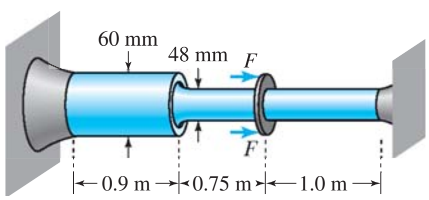


Figure P4.87

Fatigue

4.88 The fillet radius is 0.2 mm in the stepped steel circular rod shown in Figure P4.86. What should be the peak value of the cyclic load *F* to ensure a service life of one-half million cycles? Use the *S–N* curve shown in Figure 3.36

4.89 The aluminum axial rod in Figure P4.87 is subjected to a cyclic load *F*. Determine the peak value of *F* to ensure a service life of one million cycles. Use the *S–N* curves shown in Figure 3.36 and modulus of elasticity $E = 70$ GPa.

MoM in Action: Kansas City Walkway Disaster

On July 17, 1981, nearly 2000 people had gathered to watch a dance competition in the atrium of the Hyatt Regency Hotel in Kansas City, Missouri. At 7:05 P.M. a loud, sharp sound was heard throughout the building. Within minutes, the second- and fourth floor walkways crashed to the ground, killing 114 people and injuring over 200 others. The worst structural failure in the history of United States had taken place. It is a tragic story of multiple design failures – and of failure in professional ethics as well.

Three suspended walkways spanned the hotel atrium, a large open area of approximately 117 ft by 145 ft and 50 ft high. The fourth- floor walkway was directly above the second-floor walkway, while the third-floor walkway (Figure 4.33*a*) was offset 15 ft from the plane of the other two. Each walkway was 120 ft long and 8.6 ft wide, with four 30-ft intervals between support steel rods of diameter 1.25 in. A square box beam was constructed by welding two channel beams, and a hole was drilled through for the supporting axial rods, as shown in Figure 4.33*b*. In the original design (Figure 4.33b), a single continuous steel axial rod passed through the second- and fourth-floor walkways and was attached to the ceiling truss. This design required that the axial rod between the walkways be threaded so that the nuts underneath the box beams could be installed. As designed, the fourth-floor connection had to support only loads from the fourth-floor walkway, while the ceiling truss would support the total load of the second- and fourth-floor walkways together.

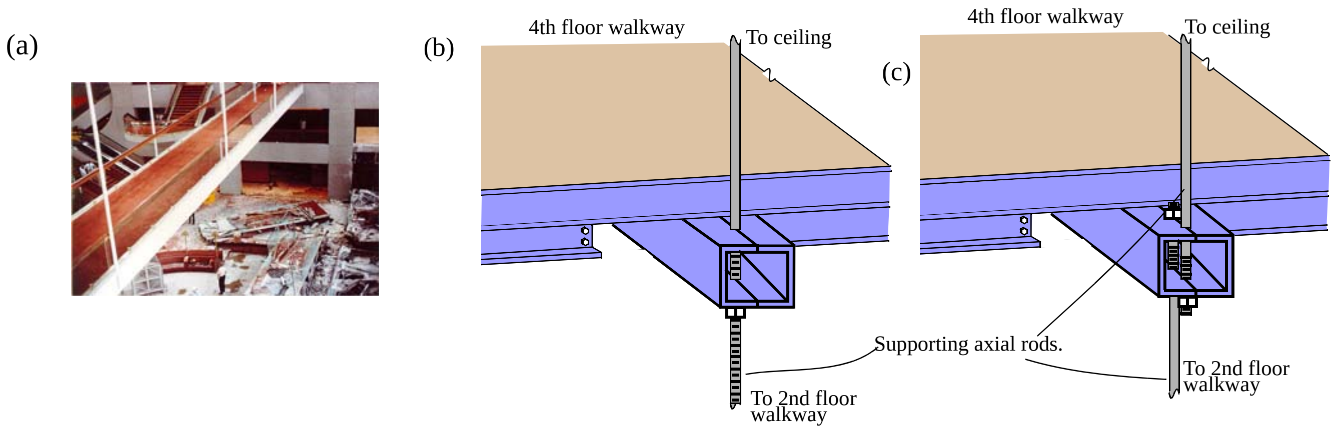


Figure 4.33 Kansas City Hyatt Regency walkways. (a) 3rd floor walkway (Courtesy Dr. Lee Lowery Jr.)(b) Original design connection (c) Fabricated design connection.

The first failure in design was that the box-beam connection could support only 60% of the load specified by the Kansas City building code. Haven's Steel Co. did not want to thread the length of the axial rod between second and fourth floor, so the design was changed to that shown in Figure 4.33*c* in which only the ends of the axial rod were threaded. The change was approved over the phone, with no recalculation of the new design. This transferred the load of the second floor to the box beams of the fourth floor, which thus supported the loads of both walkways at once. This revised design could support only 30% of the load specified by code. Further compounding the design failure, the axial rod passed through the weld, the weakest structural point in the box beam. Close inspection of the connections would have shown the overstressing of the box beam, as was in fact observed in the third-floor walkway (which did not collapse). This inspection was not done.

On that fatal night people stood on the walkways watching the dance competition, with still more spectators on the ground floor. The loud noise preceding the crash was the nut punching though the box beam of the fourth-floor walkway. First one walkway crashed into the other beneath it, and then the two fell to the ground upon the people below.

Engineers Duncan and Gillium were charged with gross negligence, misconduct, and unprofessional conduct, and both their license and their firm's license to practice were revoked in the states of Missouri and Kansas. The tragedy has become a model for the study of engineering design errors and ethics.

4.4* INITIAL STRESS OR STRAIN

Members in a statically indeterminate structure may have an initial stress or strain before the loads are applied. These initial stresses or strains may be intentional or unintentional and can be caused by several factors. A good design must account for these factors by calculating the acceptable levels of prestress.

Nuts on a bolt are usually finger-tightened to hold an assembly in place. At this stage the assembly is usually stress free. The nuts are then given additional turns to pretension the bolts. When a nut is tightened by one full rotation, the distance it moves is called the **pitch**. Alternatively, pitch is the distance between two adjoining peaks on the threads. One reason for pretensioning is to prevent the nuts from becoming loose and falling off. Another reason is to introduce an initial stress that will be opposite in sign to the stress that will be generated by the loads. For example, a cable in a bridge may be pretensioned by tightening the nut and bolt systems to counter the slackening in the cable that may be caused due to wind or seasonal temperature changes.

If during assembly a member is shorter than required, then it will be forced to stretch, thus putting the entire structure into a prestress. Tolerances for the manufacture of members must be prescribed to ensure that the structure is not excessively pre-stressed.

In prestressed concrete, metal bars are initially stretched by applying tensile forces, and then concrete is poured over these bars. After the concrete has set, the applied tensile forces are removed. The initial prestress in the bars is redistributed, putting the concrete in compression. Concrete has good compressive strength but poor tensile strength. After prestressing, the concrete can be used in situations where it may be subjected to tensile stresses.

EXAMPLE 4.12

Bars A and B in the assembly shown in Figure 4.34 are made of steel with a modulus of elasticity $E = 200$ GPa, a cross-sectional area $A = 100\ \text{mm}^2$, and a length $L = 2.5$ m. Bar A is pulled by 3 mm to fill the gap before the force F is applied. (a) Determine the initial axial stress in both bars. (b) If the applied force $F = 10$ kN, determine the total axial stress in both bars.

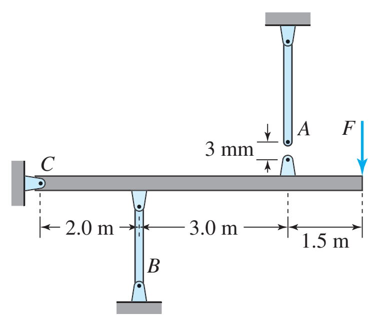


Figure 4.34 Two-bar mechanism in Example 4.12.

PLAN

(a) We can use the force method to solve the problem. After the gap has been closed, the two bars will be in tension. The degree of static redundancy for this problem is 1. We can write one compatibility equation and one equilibrium equation of the moment about C and solve the problem. (b) We can consider calculating the internal forces with just F, assuming the gap has closed and the system is stress free before F is applied. Bar B will be in compression and bar A will be in tension due to the force F. The internal forces in the bars can be found as in part (a). The initial stresses in part (a) can be superposed on the stresses due to solely F, to obtain the total axial stresses.

SOLUTION

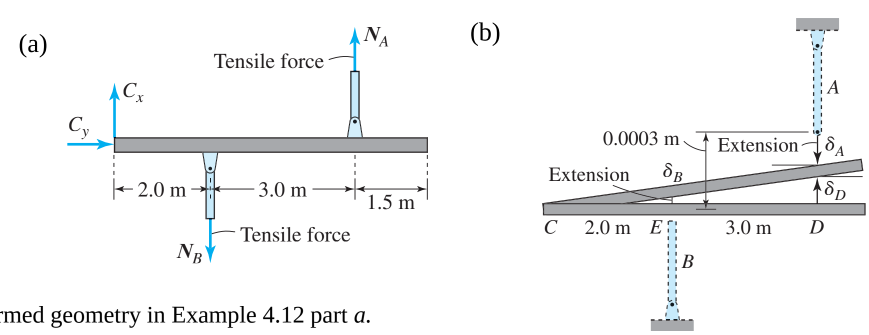


Figure 4.35 (a) Free-body diagram (b) Deformed geometry in Example 4.12 part *a*.

(a) We draw the free-body diagram of the rigid bars with both bars in tension as shown in Figure 4.35a. By moment equilibrium about point C we obtain Equation (E1).

$$N_A(5\ \text{m}) = N_B(2\ \text{m})\ 1234 \tag{E1}$$

Figure 4.35*b* shows the approximate deformed shape. The movement of point *E* is equal to the deformation of bar *B*. The movements of points *E* and *D* on the rigid bar can be related by similar triangles to obtain:

$$\frac{\delta_D}{5\text{ m}} = \frac{\delta_B}{2\text{ m}} \tag{E2}$$

The sum of extension of bar *A* and the movement of point *D* are then equal to the gap:

$$\delta_D + \delta_A = 0.003\text{ m} \tag{E3}$$

From Equations (E2) and (E3) we obtain

$$2.5\delta_B + \delta_A = 0.003\text{ m} \tag{E4}$$

The deformation of bars *A* and *B* can be written as

$$\delta_A = \frac{N_A L_A}{E_A A_A} = \frac{N_A(2.5\text{ m})}{[200(10^9)\text{ N/m}^2][100(10^{-6})\text{ m}^2]} = 0.125N_A(10^{-6})\text{ m} \tag{E5}$$

$$\delta_B = \frac{N_B L_B}{E_B A_B} = \frac{N_B(2.5\text{ m})}{[200(10^9)\text{ N/m}^2][100(10^{-6})\text{ m}^2]} = 0.125N_B(10^{-6})\text{ m} \tag{E6}$$

Substituting Equations (E5) and (E6) into (E4) we obtain

$$2.5(0.125N_B(10^{-6})\text{ m}) + 0.125N_A(10^{-6})\text{ m} = 0.003\text{ m} \qquad \text{or} \qquad 2.5N_B + N_A = 24{,}000 \tag{E7}$$

Solving Equations (E1) and (E7), we obtain the internal forces,

$$N_A = 3310.3\text{ N} \qquad N_B = 8275.9\text{ N} \tag{E8}$$

The stresses in *A* and *B* can now be found:

$$\sigma_A = \frac{N_A}{A_A} = 33.1(10^6)\text{ N/m}^2 \qquad \sigma_B = \frac{N_B}{A_B} = 82.7(10^6)\text{ N/m}^2 \tag{E9}$$

ANS. $\sigma_A = 33.1\text{ MPa (T)} \quad \sigma_B = 82.7\text{ MPa (T)}$

(b) In the calculations that follow, the purpose of the overbars is to distinguish the variables from those in part *a*. We draw the free-body diagram of the rigid bars in Figure 4.36a, with bars *A* in tension and bar *B* in compression, and by moment equilibrium about point *C* obtain

$$F(6.5\text{ m}) - \bar{N}_A(5\text{ m}) - \bar{N}_B(2\text{ m}) = 0 \qquad \text{or} \qquad 5\bar{N}_A + 2\bar{N}_B = 65{,}000\text{ N} \tag{E10}$$

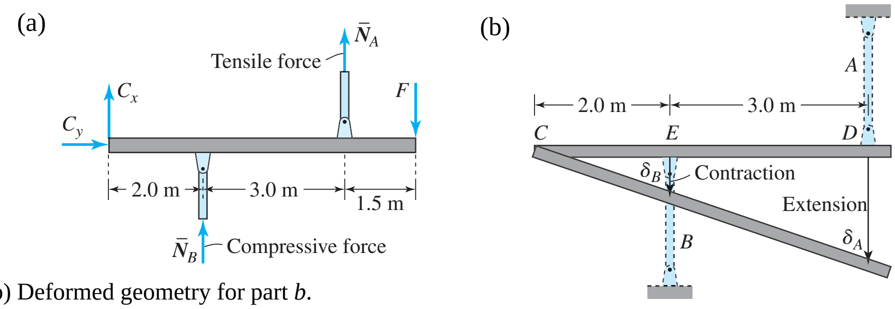


Figure 4.36 (a) Free-body diagram; (b) Deformed geometry for part *b*.

We draw the approximate deformed shape as shown in Figure 4.36b. For this part of the problem the movements of points *D* and *E* are equal to the deformation of the bar. By similar triangles we obtain

$$\frac{\bar{\delta}_A}{5\text{ m}} = \frac{\bar{\delta}_B}{2\text{ m}} \tag{E11}$$

The relation between deformation and internal forces is as before, as shown in Equations (E5) and (E6). Substituting Equations (E5) and (E6) into Equation (E11), we obtain

$$0.125\bar{N}_B(10^{-6}) = 0.4(0.125\bar{N}_A)(10^{-6}) \qquad \text{or} \qquad \bar{N}_B = 0.4\bar{N}_A \tag{E12}$$

Solving Equations (E10) and (E12), we obtain

$$\bar{N}_A = 11.20(10^3)\text{ N} \qquad \bar{N}_B = 4.48(10^3)\text{ N} \tag{E13}$$

The stresses in *A* and *B* are then

$$\bar{\sigma}_A = \frac{\bar{N}_A}{A_A} = 112\text{ MPa (T)} \qquad \bar{\sigma}_B = \frac{\bar{N}_B}{A_B} = 44.8\text{ MPa (C)} \tag{E14}$$

The total axial stress can now be obtained by superposing the stresses in Equations (E9) and (E14).

ANS. $(\sigma_A)_{\text{total}} = 145.1\text{ MPa (T)} \qquad (\sigma_B)_{\text{total}} = 37.9\text{ MPa (T)}$

COMMENTS

1. We solved the problem twice, to incorporate the initial stress (strain) due to misfit and then to account for the external load. Since the problem is linear, it should not matter how we reach the final equilibrium position. In the next section we will see that it is possible to solve the problem only once, but it would require an understanding of how initial strain is accounted for in the theory.
2. Consider a slightly different problem. In Figure 4.37, after the nut is finger-tight, it is given an additional quarter turn before the force F is applied. The pitch of the threads is 12 mm. We are required to find the initial axial stress in both bars and the total axial stress. The nut moves by pitch times the number of turns—that is, 3 mm. If we initially ignore the force F and bar B, then the movement of the nut forces the rigid bar to move by the same amount as the gap in Figure 4.34. The mechanisms of introducing the initial strains are different for the problems in Figures 4.34 and 4.37, but the results of the two problems will be identical at equilibrium. The strain due to the tightening of a nut may be hard to visualize, but the analogous problem of strain due to misfit can be visualized and used as an alternative visualization aid.

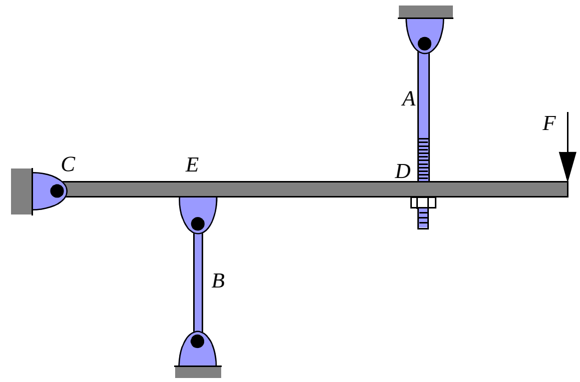


Figure 4.37 Problem similar to Example 4.12.

4.5* TEMPERATURE EFFECTS

Length changes due to temperature variations introduce stresses caused by the constraining effects of other members in a statically indeterminate structure. There are a number of similarities for the purpose of analysis between initial strain and thermal strain. Thus we shall rederive our theory to incorporate initial strain. We once more assume that plane sections remain plane and parallel and we have small strain; that is, Assumptions 1 and 2 are valid. Hence the total strain at any cross section is uniform and only a function of x, as in Equation (4.4). We further assume that the material is isotropic and linearly elastic—that is, Assumptions 3 and 4 are valid. We drop Assumption 5 to account for initial strain ε_0 at a point and write the stress–strain relationship as

$$\varepsilon_{xx} = \frac{du}{dx} = \frac{\sigma_{xx}}{E} + \varepsilon_0 \tag{4.22}$$

Substituting Equation (4.22) into Equation (4.1), assuming that the material is homogeneous and the initial strain ε_0 is uniform across the cross section, we have

$$\boldsymbol{N} = \int_A \left(E\frac{du}{dx} - E\varepsilon_0\right) dA = \frac{du}{dx}\int_A E\ dA - \int_A E\varepsilon_0\ dA = \frac{du}{dx}EA - EA\varepsilon_0 \quad \text{or} \tag{4.23}$$

$$\frac{du}{dx} = \frac{\boldsymbol{N}}{EA} + \varepsilon_0 \tag{4.24}$$

Substituting Equation (4.24) into Equation (4.22), we obtain a familiar relationship:

$$\sigma_{xx} = \frac{\boldsymbol{N}}{A} \tag{4.25}$$

If Assumptions 7 through 9 are valid, and if ε_0 does not change with x, then all quantities on the right-hand side of Equation (4.25) are constant between x_1 and x_2, and by integration we obtain

$$u_2 - u_1 = \frac{\boldsymbol{N}(x_2 - x_1)}{EA} + \varepsilon_0(x_2 - x_1) \tag{4.26}$$

or alternatively,

$$\boxed{\delta = \frac{NL}{EA} + \varepsilon_0 L} \tag{4.27}$$

Equations (4.25) and (4.27) imply that the initial strain affects the deformation but does not affect the stresses. This seemingly paradoxical result has different explanations for the thermal strains and for strains due to misfits or to pretensioning of the bolts.

First we consider the strain ε_0 due to temperature changes. If a body is homogeneous and unconstrained, then no stresses are generated due to temperature changes, as observed in Section 3.9. This observation is equally true for statically determinate structures. The determinate structure simply expands or adjusts to account for the temperature changes. But in an indeterminate structure, the deformation of various members must satisfy the compatibility equations. The compatibility constraints cause the internal forces to be generated, which in turn affects the stresses.

In thermal analysis $\varepsilon_0 = \alpha \Delta T$. An increase in temperature corresponds to extension, whereas a decrease in temperature corresponds to contraction. Equation (4.27) assumes that N is positive in tension, and hence extensions due to ε_0 are positive and contractions are negative. However, if on the free-body diagram N is shown as a compressive force, then δ is shown as contraction in the deformed shape. Consistency requires that contraction due to ε_0 be treated as positive and extension as negative in Equation (4.27). The sign of $\varepsilon_0 L$ due to temperature changes must be consistent with the force N shown on the free-body diagram.

We now consider the issue of initial strains caused by factors discussed in Section 4.4. If we start our analysis with the undeformed geometry even when there is an initial strain or stress, then the implication is that we have imposed a strain that is opposite in sign to the actual initial strain before imposing external loads. To elaborate this issue of sign, we put $\delta = 0$ in Equation (4.27) to correspond to the undeformed state. Also note that N and ε_0 must have opposite signs for the two terms on the right-hand side to combine, yielding a result of zero. But strain and internal forces must have the same sign. For example, if a member is short and has to be pulled to overcome a gap due misfit, then at the undeformed state the bar has been extended and is in tension before external loads are applied. The problem can be corrected only if we think of ε_0 as negative to the actual initial strain. Thus prestrains (stresses) can be analyzed by using ε_0 as negative to the actual initial strain in Equation (4.27).

If we have external forces in addition to the initial strain, then we can solve the problem in two ways. We can find the stresses and the deformation due to initial strain and due to external forces individually, as we did in Section 4.4, and superpose the solution. The advantage of such an approach is that we have a good intuitive feel for the solution process. The disadvantage is that we have to solve the problem twice. Alternatively we could use Equation (4.27) and solve the problem once, but we need to be careful with our signs, and the approach is less intuitive and more mathematical.

EXAMPLE 4.13

Bars A and B in the mechanism shown in Figure 4.38 are made of steel with a modulus of elasticity E = 200 GPa, a coefficient of thermal expansion $\alpha = 12\ \mu/^\circ\text{C}$, a cross-sectional area $A = 100\ \text{mm}^2$, and a length L = 2.5 m. If the applied force F = 10 kN and the temperature of bar A is decreased by 100°C, find the total axial stress in both bars.

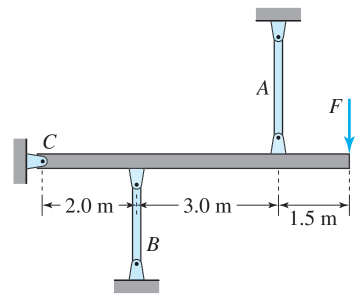


Figure 4.38 Two-bar mechanism in Example 4.13.

PLAN

We can use the force method to solve this problem. The problem has 1 degree of redundancy. We can write one compatibility equation and, using (4.27), get one equation relating the internal forces. By taking the moment about point C in the free-body diagram of the rigid bar, we can obtain the remaining equation and solve the problem.

SOLUTION

The axial rigidity and the thermal strain are

$$EA = [200(10^9)\ \text{N/m}^2][100(10^{-6})\ \text{m}^2] = 20(10^6)\ \text{N} \tag{E1}$$

$$\varepsilon_0 = \alpha\ \Delta T = 12(10^{-6})(-100) = -1200(10^{-6}) \tag{E2}$$

We draw the free-body diagram of the rigid bar with bar A in tension and bar B in compression as shown in Figure 4.39*a*. By moment equilibrium about point C we obtain

$$F(6.5\ \text{m}) - N_A(5\ \text{m}) - N_B(2\ \text{m}) = 0 \qquad \text{or} \qquad 5N_A + 2N_B = 65(10^3)\ \text{N} \tag{E3}$$

We draw the approximate deformed shape in Figure 4.39*b*. Noting that the movements of points D and E are equal to the deformation of the bars we obtain from similar triangles

$$\frac{\delta_A}{5\ \text{m}} = \frac{\delta_B}{2\ \text{m}} \tag{E4}$$

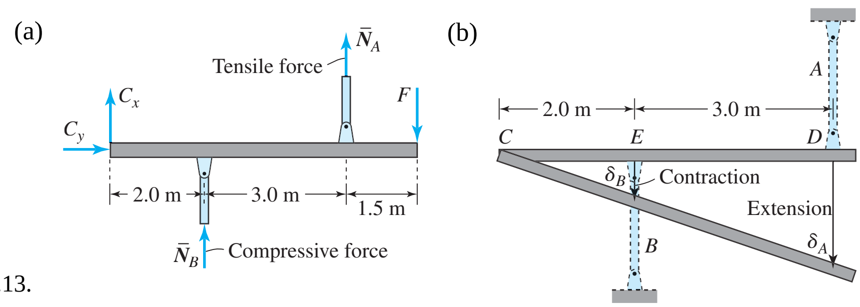


Figure 4.39 Free-body diagram in Example 4.13.

The deformations of bars A and B can be written as

$$\delta_A = \frac{N_A L_A}{E_A A_A} + \varepsilon_0 L_A = \frac{N_A(2.5\ \text{m})}{20(10^6)} - 1200(2.5\ \text{m})(10^{-6}) = (0.125N_A - 3000)10^{-6}\ \text{m} \tag{E5}$$

$$\delta_B = \frac{N_B L_B}{E_B A_B} = \frac{N_B(2.5\ \text{m})}{20(10^6)} = 0.125N_B(10^{-6})\ \text{m} \tag{E6}$$

Substituting Equations (E5) and (E6) into Equation (E4), we obtain

$$0.125N_B(10^{-6})\ \text{m} = 0.4(0.125N_A - 3000)10^{-6}\ \text{m} \qquad \text{or} \qquad N_B = 0.4N_A - 9600 \tag{E7}$$

Solving Equations (E3) and (E7), we obtain

$$N_A = 14.51(10^3)\ \text{N} \qquad N_B = -3.79(10^3)\ \text{N} \tag{E8}$$

Noting that we assumed that bar B is in compression, the sign of N_B in Equation (E8) implies that it is in tension. The stresses in A and B can now be found by dividing the internal forces by the cross-sectional areas.

ANS. $\sigma_A = 145.1\ \text{MPa (T)}$ $\sigma_B = 37.9\ \text{MPa (T)}$

COMMENTS

1. In Figure 4.34 the prestrain in member A is $0.0003/2.5 = 1200 \times 10^{-6}$ extension. This means that $\varepsilon_0 = -1200 \times 10^{-6}$. Substituting this value we obtain Equation (E5). Nor will any other equation in this example change for problems represented by Figures 4.34 and 4.37. Thus it is not surprising that the results of this example are identical to those of Example 4.12. But unlike Example 4.12, we solved the problem only once.
2. It would be hard to guess intuitively that bar B will be in tension, because the initial strain is greater than the strain caused by the external force F. But this observation is obvious in the two solutions obtained in Example 4.12.
3. To calculate the initial strain using the method in this example, it is recommended that the problem be formulated initially in terms of the force F. Then to calculate initial strain, substitute $F = 0$. This recommendation avoids some of the confusion that will be caused by a change of the sign of ε_0 in the initial strain calculations.

PROBLEM SET 4.4

Initial strains

4.90 During assembly of a structure, a misfit between bar *A* and the attachment of the rigid bar was found, as shown in Figure P4.90. If bar *A* is pulled and attached, determine the initial stress introduced due to the misfit. The modulus of elasticity of the circular bars *A* and *B* is $E =$ 10,000 ksi and the diameter is 1 in.

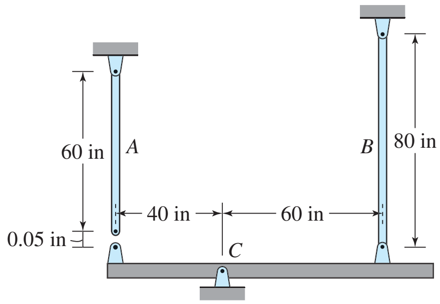


Figure P4.90

4.91 Bar *A* was manufactured 2 mm shorter than bar *B* due to an error. The attachment of these bars to the rigid bar would cause a misfit of 2 mm. Calculate the initial stress for each assembly, shown in Figure P4.91. Which of the two assembly configurations would you recommend? Use a modulus of elasticity $E = 70$ GPa and a diameter of 25 mm for the circular bars.

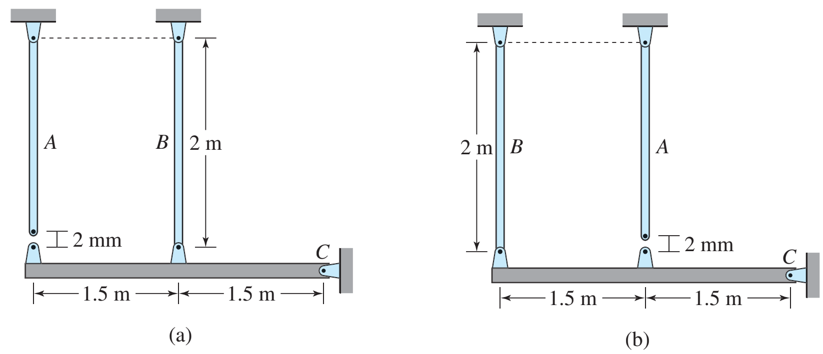


Figure P4.91

4.92 A steel bolt is passed through an aluminum sleeve as shown in Figure P4.92. After assembling the unit by finger-tightening (no deformation) the nut is given a $\frac{1}{4}$ turn. If the pitch of the threads is 3.0 mm, determine the initial axial stress developed in the sleeve and the bolt. The moduli of elasticity for steel and aluminum are $E_{st} = 200$ GPa and $E_{al} = 70$ GPa and the cross-sectional areas are $A_{st} = 500\ \text{mm}^2$ and $A_{al} = 1100\ \text{mm}^2$.

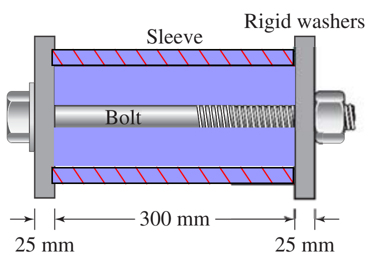


Figure P4.92

4.93 The rigid bar shown in Figure P4.93 is horizontal when the unit is put together by finger-tightening the nut. The pitch of the threads is 0.125 in. The properties of the bars are listed in Table 4.93. Develop a table in steps of quarter turns of the nut that can be used for prescribing the pretension in bar *B*. The maximum number of quarter turns is limited by the yield stress.

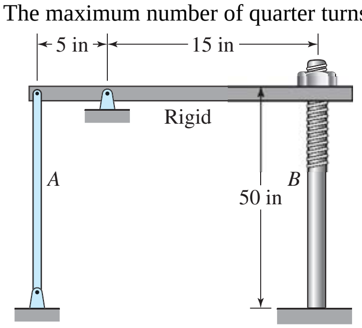


Figure P4.93

TABLE P4.93 **Material properties**

	Bar *A*	**Bar *B***
Modulus of elasticity	10,000 ksi	30,000 ksi
Yield stress	24 ksi	30 ksi
Cross-sectional area	0.5 in^2	0.75 in^2

Temperature effects

4.94 The temperature for the bar in Figure P4.94 increases as a function of *x*: $\Delta T = T_L x^2 / L^2$. Determine the axial stress and the movement of a point at $x = L/2$ in terms of the length *L*, the modulus of elasticity *E*, the cross-sectional area *A*, the coefficient of thermal expansion α, and the increase in temperature at the end T_L.

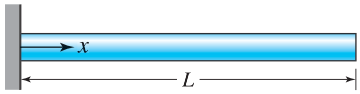


Figure P4.94

4.95 The temperature for the bar in Figure P4.95 increases as a function of *x*: $\Delta T = T_L x^2 / L^2$. Determine the axial stress and the movement of a point at $x = L/2$ in terms of the length *L*, the modulus of elasticity *E*, the cross-sectional area *A*, the coefficient of thermal expansion α, and the increase in temperature at the end T_L.

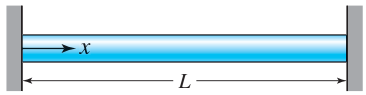


Figure P4.95

4.96 The tapered bar shown in Figure P4.96 has a cross-sectional area that varies with *x* as $A = K(L - 0.5x)^2$. If the temperature of the bar increases as $\Delta T = T_L x^2 / L^2$, determine the axial stress at midpoint in terms of the length *L*, the modulus of elasticity *E*, the cross-sectional area *A*, the parameter *K*, the coefficient of thermal expansion α, and the increase in temperature at the end T_L.

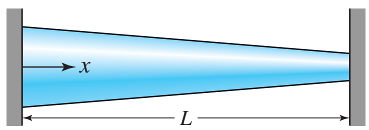


Figure P4.96

4.97 Three metallic rods are attached to a rigid plate, as shown in Figure P4.97. The temperature of the rods is lowered by 100°F after the forces are applied. Assuming the rigid plate does not rotate, determine the movement of the rigid plate. The material properties are listed in Table 4.98.

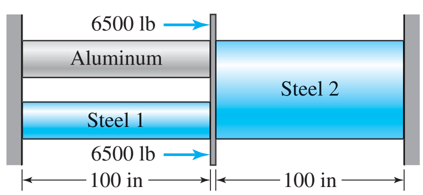


Figure P4.97

4.98 Solve Problem 4.92 assuming that in addition to turning the nut, the temperature of the assembled unit is raised by 40°C. The coefficients of thermal expansion for steel and aluminum are $\alpha_{st} = 12\ \mu/°C$ and $\alpha_{al} = 22.5\ \mu/°C$.

TABLE P4.98 **Material properties**

	Area (in.^{2})	***E* (ksi)**	**α (10^{-6}/°F)**
Aluminum	4	10,000	12.5
Steel 1	4	30,000	6.6
Steel 2	12	30,000	6.6

4.99 Determine the axial stress in bar *A* of Problem 4.93 assuming that the nut is turned 1 full turn and the temperature of bar *A* is decreased by 80°F. The coefficient of thermal expansion for bar *A* is $\alpha_{st} = 22.5\ \mu/°F$.

4.6* STRESS APPROXIMATION

Many applications are based on strength design. As was demonstrated in Examples 1.4 and 1.5, we can obtain stress formulas starting with a stress approximation across the cross section and use these in strength design. But how do we deduce a stress behavior across the cross section? In this section we consider the clues that we can use to deduce approximate stress behavior.

In Section 4.7 we will show how to apply these ideas to thin-walled pressure vessels. Section 5.4 on the torsion of thin-walled tubes is another application of the same ideas.

Think of each stress component as a mathematical function to be approximated. The simplest approximation of a function (the stress component) is to assume it to be a constant, as was done in Figure 1.16*a* and *b*. The next level of complexity is to assume a stress component as a linear function, as was done in Figure 1.16*c* and *d*. If we continued this line of thinking, we would next assume a quadratic or higher-order polynomial. The choice of a polynomial for approximating a stress component is dictated by several factors, some of which are discussed in this section.

4.6.1 Free Surface

A segment of a body that has no forces acting on the surface is shown in Figure 4.40. If we consider a point on the surface and draw a stress cube, then the surface with the outward normal in the *z* direction will have no stresses, and we have a situation of plane stress at that point. Because the points on which no forces are acting can be identified by inspection, these points provide us with a clue to making assumptions regarding stress behavior, as will be demonstrated next.

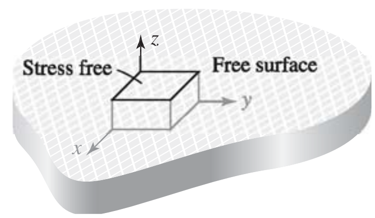


Figure 4.40 Free surface and plane stress.

The drill shown in Figure 4.41 has point *A* located just outside the material that is being drilled. Point A is on a free surface, hence all stress components on this surface, including the shear stress, must go to zero. Point *B* is at the tip of the drill, the point at which the material is being sheared off, that is, at point *B* the shear stress must be equal to the shear strength of the material. Now we have two points of observation. The simplest curve that can be fitted through two points is a straight line. A linear approximation of shear stress, as shown in Figure 4.41, is a better approximation than the uniform behavior we assumed in Example 1.6. It can be confirmed that with linear shear stress behavior, the minimum torque will be 188.5 in. · kips, which is half of what we obtained in Example 1.6. Only experiment can confirm whether the stress approximation in Figure 4.41 is correct. If it is not, then the experimental results would suggest other equations to consider.

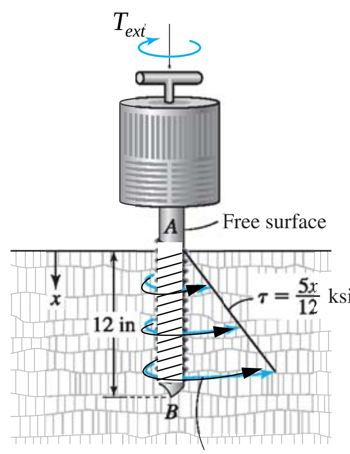


Figure 4.41 Using free surface to guide stress approximation.

4.6.2 Thin Bodies

The smaller the region of approximation, the better is the accuracy of the analytical model. If the dimensions of a cross section are small compared to the length of the body, then assuming a constant or a linear stress distribution across the cross section will introduce small errors in the calculation of internal forces and moments, such as in pins discussed in Section 1.1.2. We now consider another small region of approximation, termed thin bodies. A body is called **thin** if its thickness is an order of magnitude (factor of 10) smaller than the other dimensions.

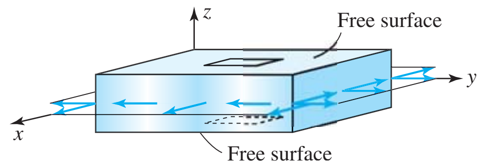


Figure 4.42 Plane stress assumption in thin plates.

Figure 4.42 shows a segment of a plate with loads in the *x* and *y* directions. The top and bottom surfaces of the plate are free surfaces, that is, plane stress exists on both surfaces. This does not imply that a point in the middle of the two surfaces is also in a state of plane stress, but if the plate is thin compared to its other dimensions then to simplify analysis, it is reasonable to assume that the entire plate is in plane stress. The other stress components are usually assumed uniform or linear in the thickness direction in thin bodies.

The assumption of plane stress is made in thin bodies even when there are forces acting on one of the surfaces in the thickness direction. The assumption is justified if the maximum stresses in the *xy* plane turn out to be an order of magnitude greater than the applied distributed load. But the validity of the assumption can be checked only after the stress formula has been developed. Some examples of thin bodies are the skin of an aircraft, the floors and ceilings of buildings, and thin-walled cylindrical or spherical pressure vessels.

4.6.3 Axisymmetric Bodies

A body whose geometry, material properties, and loading are symmetric with respect to an axis is called an **axisymmetric body**. The stress components which are produced cannot depend upon the angular location in an axisymmetric body. In other words, the stress components must also be symmetric with respect to the axis. By using this argument of axisymmetry in thin bodies, we can get good stress approximation, as will be demonstrated by a simple example below and further elaborated in Section 4.7.

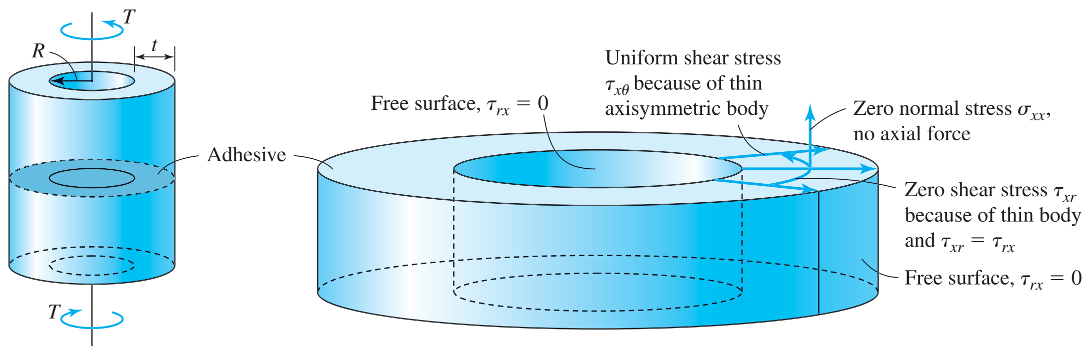


Figure 4.43 Deducing stress behavior in adhesively bonded thin cylinders.

Consider all the stress components acting in the adhesive layer between two thin cylinders subjected to a torque, as shown in Figure 4.43. The shear stress in the radial direction τ_{xr} is assumed to be zero because the symmetric counterpart of this shear stress, τ_{rx}, has to be zero on the inside and outside free surfaces of this thin body. Because the problem is axisymmetric, the normal stress σ_{xx} and the tangential shear stress $\tau_{x\theta}$ cannot depend on the angular coordinate. But a uniform axial stress σ_{xx} would produce an internal axial force. Because no external axial force exists, we approximate the axial stress as zero. Because of thinness, the tangential shear stress $\tau_{x\theta}$ is assumed to be constant in the radial direction. In Example 1.6 we developed the stress formula relating $\tau_{x\theta}$ to the applied torque. In Section 5.4, in a similar manner, we shall deduce the behavior of the shear stress distribution in thin-walled cylindrical bodies of arbitrary cross sections.

4.6.4 Limitations

All analytical models depend on assumptions and are approximations. They are mathematical representations of nature and have errors in their predictions. Whether the approximation is acceptable depends on the accuracy needed and the experimental results. If all we are seeking is an order-of-magnitude value for stresses, then assuming a uniform stress behavior in most cases will give us an adequate answer. But constructing sophisticated models based on stress approximation alone is difficult, if not impossible. Further, an assumed stress distribution may correspond to a material deformation that is physically impossible. For example, the approximation might require holes or corners to form inside the material. Another difficulty is validating the assumption. We need to approximate six independent stress components, which are difficult to visualize and, being internal, cannot be measured directly. These difficulties can be overcome by approximating not the stress but the displacement that can be observed experimentally as discussed in Section 3.2.

We conclude this section with the following observations:

1. A point is in plane stress on a free surface.
2. Some of the stress components must tend to zero as the point approaches the free surface.
3. A state of plane stress may be assumed for thin bodies.
4. Stress components may be approximated as uniform or linear in the thickness direction for thin bodies.
5. A body that has geometry, material properties, and loads that are symmetric about an axis must have stresses that are also symmetric about the axis.

4.7* THIN-WALLED PRESSURE VESSELS

Cylindrical and spherical pressure vessels are used for storage, as shown in Figure 4.44, and for the transportation of fluids and gases. The inherent symmetry and the assumption of thinness make it possible to deduce the behavior of stresses to a first approximation. The argument of symmetry implies that stresses cannot depend on the angular location. By limiting ourselves to thin walls, we can assume uniform radial stresses in the thickness direction. The net effect is that all shear stresses in cylindrical or spherical coordinates are zero, the radial normal stress can be neglected, and the two remaining normal stresses in the radial and circumferential directions are constant. The two unknown stress components can be related to pressure by static equilibrium.

(a)

(b)

Figure 4.44 Gas storage tanks.

The "thin-wall" limitation implies that the ratio of the mean radius R to the wall thickness t is greater than 10. The higher the ratio of R/t, the better is the prediction of our analysis.

4.7.1 Cylindrical Vessels

Figure 4.45 shows a thin cylinder subjected to a pressure of p. The stress element on the right in Figure 4.45 shows the stress components in the cylindrical coordinate system (r, θ, x) on four surfaces. The outer surface of the cylinder is stress free. Hence the shear stresses $\tau_{r\theta}$ and τ_{rx} and the normal stress σ_{rr} are all zero on the outer surface (at A). On the inner surface (at B) there is only a radial force due to pressure p, but there are no tangential forces. Hence on the inner surface the shear stresses $\tau_{r\theta}$ and τ_{rx} are zero. Since the wall is thin, we can assume that the shear stresses $\tau_{r\theta}$ and τ_{rx} are zero across the thickness. The radial normal stress varies from a zero value on the outer surface to a value of the pressure on the inner surface. At the end of our derivation we will justify that the radial stress σ_{rr} can be neglected as it is an order of magnitude less than the other two normal stresses σ_{xx} and $\sigma_{\theta\theta}$. A nonzero value of $\tau_{\theta x}$ will either result in a torque or movement of points the θ direction. Since there is no applied torque, and the movement of a point cannot depend on the angular location because of symmetry, we conclude that the shear stress $\tau_{\theta x}$ is zero.

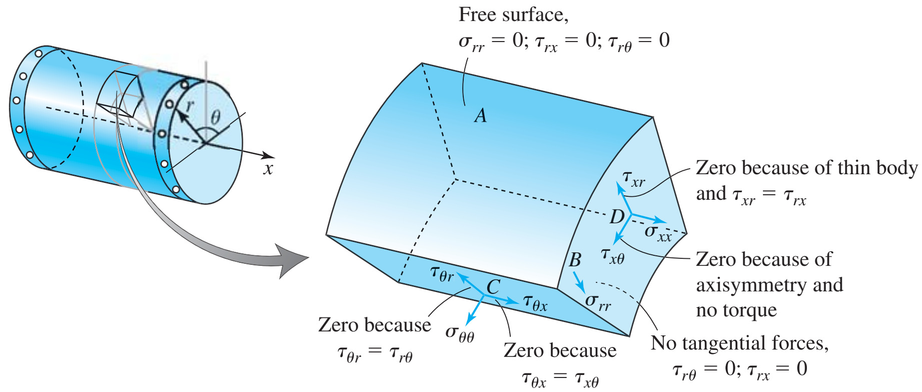


Figure 4.45 Stress element in cylindri-

Thus all shear stresses are zero, while the radial normal stress is neglected.The axial stress σ_{xx} and the hoop stress $\sigma_{\theta\theta}$ are assumed uniform across the thickness and across the circumference, as these cannot depend upon angular location. Figure 4.67*a* shows this state of stress. We could start with a differential element and find the internal forces by integrating σ_{xx} and $\sigma_{\theta\theta}$ over appropriate areas. But as these two stresses are uniform across the entire circumference, we can reach the same conclusions by considering two free-body diagrams shown in Figure 4.46*b* and *c*.

By equilibrium of forces on the free-body diagram in Figure 4.46*b*, we obtain $2\sigma_{\theta\theta}(t\,dx) = p(2R)\,dx$, or

$$\boxed{\sigma_{\theta\theta} = \frac{pR}{t}} \tag{4.28}$$

By equilibrium of forces on the free-body diagram in Figure 4.46*c* we obtain $\sigma_{xx}(2\pi R)(t) = p(\pi R^2)$, or

$$\boxed{\sigma_{xx} = \frac{pR}{2t}} \tag{4.29}$$

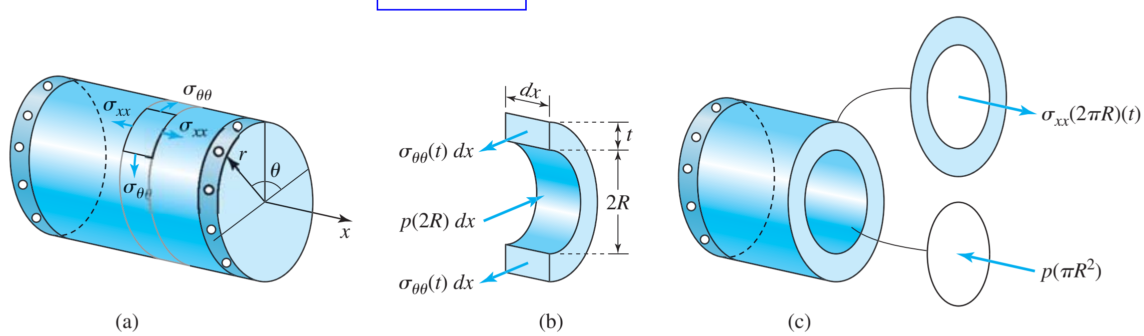


Figure 4.46 Stress analysis in thin cylindrical pressure vessels.

With $R/t > 10$ the stresses σ_{xx} and $\sigma_{\theta\theta}$ are greater than the maximum value of radial stress σ_{rr} (=p) by factors of at least 5 and 10, respectively. This justifies our assumption of neglecting the radial stress in our analysis.

The axial stress σ_{xx} and the hoop stress $\sigma_{\theta\theta}$ are always tensile under internal pressure. The formulas may be used for small applied external pressure but with the following caution. External pressure causes compressive normal stresses that can cause the cylinder to fail due to buckling. The buckling phenomenon is discussed in Chapter 11.

Although the normal stresses are assumed not to vary in the circumferential or thickness direction, our analysis does not preclude variations in the axial direction (*x* direction). But the variations in the *x* direction must be gradual. If the variations are very rapid, then our assumption that stresses are uniform across the thickness will not be valid, as can be shown by a more rigorous three-dimensional elasticity analysis.

4.7.2 Spherical Vessels

We use the spherical coordinate system (r, θ, ϕ) for our analysis, as shown in Figure 4.47*a*. Proceeding in a manner similar to the analysis of cylindrical vessels, we deduce the following:

1. All shear stresses are zero:

$$\tau_{r\phi} = \tau_{\phi r} = 0 \qquad \tau_{r\theta} = \tau_{\theta r} = 0 \qquad \tau_{\theta\phi} = \tau_{\phi\theta} = 0 \tag{4.30}$$

2. Normal radial stress σ_{rr} varies from a zero value on the outside to the value of the pressure on the inside. We will once more neglect the radial stress in our analysis and justify it posterior.
3. The normal stresses $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$ are equal and are constant over the entire vessel. We set $\sigma_{\theta\theta} = \sigma_{\phi\phi} = \sigma$.

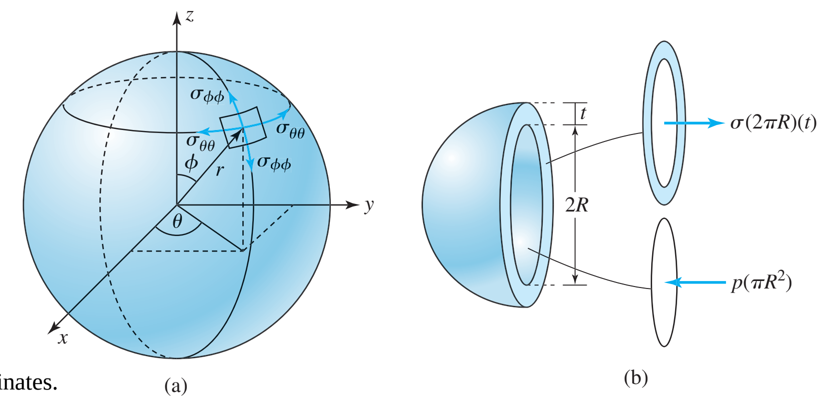


Figure 4.47 Stress analysis in thin spherical coordinates.

As all imaginary cuts through the center are the same, we consider the free-body diagram shown in Figure 4.47*b*. By equilibrium of forces we obtain $\sigma(2\pi R)(t) = p\pi R^2$, or

$$\boxed{\sigma = \frac{pR}{2t}} \tag{4.31}$$

With $R/t > 10$ the normal stress σ is greater than the maximum value of radial stress σ_{rr} $(=p)$ by a factor of at least 5. This justifies our assumption of neglecting the radial stress in our analysis. At each and every point the normal stress in any circumferential direction is the same for thin spherical pressure vessels.

EXAMPLE 4.14

The lid is bolted to the tank in Figure 4.48 along the flanges using 1-in.-diameter bolts. The tank is made from sheet metal that is $\frac{1}{2}$ in. thick and can sustain a maximum hoop stress of 24 ksi in tension. The normal stress in the bolts is to be limited to 60 ksi in tension. A manufacturer can make tanks of diameters from 2 ft to 8 ft in steps of 1 ft. Develop a table that the manufacturer can use to advise customers of the size of tank and the number of bolts per lid needed to hold a desired gas pressure.

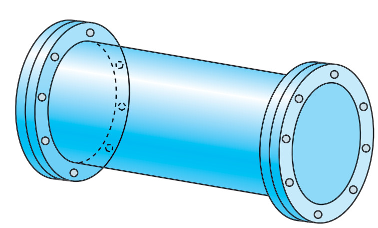


Figure 4.48 Cylindrical tank in Example 4.14.

PLAN

Using Equation (4.28) we can establish a relationship between the pressure p and the radius R (or diameter D) of the tank through the limiting value on hoop stress. We can relate the number of bolts needed by noting that the force due to pressure on the lid is carried equally by the bolts.

SOLUTION

The area of the bolts can be found as shown in (E1).

$$A_{bolt} = \pi(1 \text{ in.})^2/4 = (\pi/4) \text{ in.}^2 \quad \text{(E1)}$$

From Equation (4.28) we obtain (E2).

$$\sigma_{\theta\theta} = \frac{pR}{1/2} \le 24{,}000 \text{ psi} \quad \text{or} \quad p \le \frac{24,000}{D} \text{ psi} \quad \text{(E2)}$$

Figure 4.49 shows the free-body diagram of the lid. By equilibrium of forces we obtain (E3).

$$nN_{bolt} = N_{lid} \quad \text{or} \quad n\sigma_{bolt}\left(\frac{\pi}{4}\right) = p(\pi R^2) \quad \text{or} \quad \sigma_{bolt} \le \frac{4pR^2}{n} \quad \text{or} \quad \sigma_{bolt} = \frac{pD^2}{n} \le 60{,}000 \quad \text{(E3)}$$

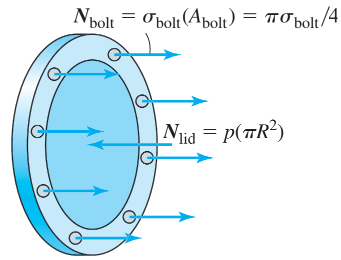


Figure 4.49 Relating forces in bolts and lid in Example 4.14.

Substituting (E3) into (E2) we obtain (E4).

$$\frac{24,000D}{n} \le 60{,}000 \quad \text{or} \quad n \ge 0.4D \quad \text{(E4)}$$

We consider the values of D from 24 in to 96 in. in steps of 12 in and calculate the values of p and n from Equations (E2) and (E4). We report the values of p by rounding downward to the nearest integer that is a factor of 5, and the values of n are reported by rounding upward to the nearest integer, as given in Table 4.2.

TABLE 4.2 **Results of Example 4.14**

Tank Diameter D (ft)	Maximum Pressure p (psi)	Minimum Number of Bolts n
2	1000	10
3	665	15
4	500	20
5	400	24
6	330	30
7	280	34
8	250	39

COMMENT

1. We rounded downwards for p and upwards for n to satisfy the inequalities of Equations (E2) and (E4). Intuitively we know that smaller pressure and more bolts will result is a safer pressure tank.

PROBLEM SET 4.5

Thin-walled pressure vessels

4.100 Fifty rivets of 10-mm diameter are used for attaching caps at each end on a 1000-mm mean diameter cylinder, as shown in Figure P4.100. The wall of the cylinder is 10 mm thick and the gas pressure is 200 kPa. Determine the hoop stress and the axial stress in the cylinder and the shear stress in each rivet.

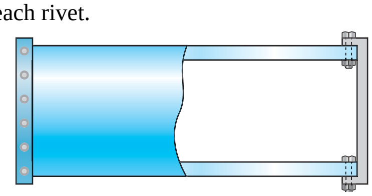


Figure P4.100

4.101 A pressure tank 15 ft long and with a mean diameter of 40 in is to be fabricated from a $\frac{1}{2}$-in.-thick sheet. A 15-ft-long, 8-in.-wide, $\frac{1}{2}$-in.-thick plate is bonded onto the tank to seal the gap, as shown in Figure P4.101. What is the shear stress in the adhesive when the pressure in the tank is 75 psi? Assume uniform shear stress over the entire inner surface of the attaching plate.

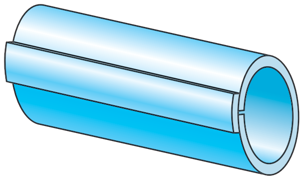


Figure P4.101

Design problems

4.102 A 5-ft mean diameter spherical tank has a wall thickness of $\frac{3}{4}$ in. If the maximum normal stress is not to exceed 10 ksi, determine the maximum permissible pressure.

4.103 In a spherical tank having a 500-mm mean radius and a thickness of 40 mm, a hole of 50-mm diameter is drilled and then plugged using adhesive of 1.2-MPa shear strength to form a safety pressure release mechanism (Figure P4.103). Determine the maximum allowable pressure and the corresponding hoop stress in the tank material.

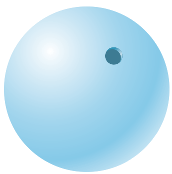


Figure P4.103

4.104 A 20-in. mean diameter pressure cooker is to be designed for a 15-psi pressure (Figure P4.104). The allowable normal stress in the cylindrical pressure cooker is to be limited to 3 ksi. Determine the minimum wall thickness of the pressure cooker. A $\frac{1}{2}$-lb weight on top of the nozzle is used to control the pressure in the cooker. Determine the diameter *d* of the nozzle.

Figure P4.104

4.105 The cylindrical gas tank shown in Figure P4.105 is made from 8-mm-thick sheet metal and must be designed to sustain a maximum normal stress of 100 MPa. Develop a table of maximum permissible gas pressures and the corresponding mean diameters of the tank in steps of 100 mm between diameter values of 400 mm and 900 mm.

Figure P4.105

4.106 A pressure tank 15 ft long and a mean diameter of 40 in. is to be fabricated from a $\frac{1}{2}$-in.-thick sheet. A 15-ft-long, 8-in.-wide, $\frac{1}{2}$-in.-thick plate is to be used for sealing the gap by using two rows of 90 rivets each. If the shear strength of the rivets is 36 ksi and the normal stress in the tank is to be limited to 20 ksi, determine the maximum pressure and the minimum diameter of the rivets that can be used.

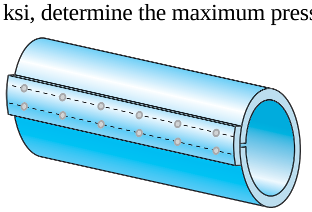


Figure P4.106

4.107 A pressure tank 5 m long and a mean diameter of 1 m is to be fabricated from a 10 mm thick sheet as shown in Figure P4.106. A 5 m-long, 200 mm wide, 10-mm-thick plate is to be used for sealing the gap by using two rows of 100 rivets each. The shear strength of the rivets is 300 MPa and the yield strength of the tank material is 200 MPa. Determine the maximum pressure and the minimum diameter of the rivets to the nearest millimeter that can be used for a factor of safety of 2.

4.8* CONCEPT CONNECTOR

The finite-element method (FEM) is a popular numerical technique for the stress and deformation analysis of planes, ships, automobiles, buildings, bridges, machines, and medical implants, as well as for earthquakes predictions. It is used in both static and dynamic analysis and both linear and nonlinear analysis as well. A whole industry is devoted to developing FEM software, and many commercial packages are already available, including software modules in computer-aided design (CAD), computer-aided manufacturing (CAM), and computer-aided engineering (CAE). This section briefly describes the main ideas behind one version of FEM.

4.8.1 The Finite Element Method

In the *stiffness method* FEM is based on the displacement method, while in the *flexibility method* it is based on the force method. Most commercial FEM software is based on the displacement method.

In the displacement method, the unknowns are the displacements of points called *nodes*, and a set of linear equations represents the force equilibrium at the nodes. For example, the unknowns could be the displacements of pins in a truss, and the linear equations could be the equilibrium equations at each joint written in terms of the displacements. In FEM, however, the equilibrium equations are derived by requiring that the nodal displacements minimize the potential energy of the structure. First equations are created for small, finite elements whose assembly represents the body, and these lead to equations for the entire body. It is assumed that the displacement in an element can be described by a polynomial. Figure 4.50 shows the linear and quadratic displacements in a one-dimensional rod.

Linear element:

$$u(x) = a_0 + a_1 x$$

$$u(x) = u_1\left(\frac{x - x_2}{x_1 - x_2}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right)$$

$$u(x) = u_1\phi_1(x) + u_2\phi_2(x)$$

Quadratic element:

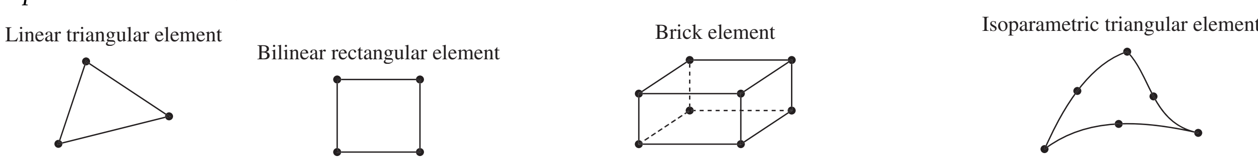
$$u(x) = a_0 + a_1 x + a_2 x^2$$

$$u(x) = u_1\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right) + u_2\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right) + u_3\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)$$

$$u(x) = u_1\phi_1(x) + u_2\phi_2(x) + u_3\phi_3(x)$$

Figure 4.50

The constants a_i in the polynomials can be found in terms of the nodal displacement values u_i and nodal coordinates x_i as shown in Figure 4.50. The polynomial functions ϕ_i that multiply the nodal displacements are called *interpolation functions*, because we can now interpolate the displacement values from the nodal values. Sometimes the same polynomial functions are also used for representing the shapes of the elements. Then the interpolation functions are also referred to as *shape functions.* When the same polynomials represent the displacement and the shape of an element, then the element is called an *isoparametric* element.



$u(x, y) = a_0 + a_1x + a_2y$

$u(x, y) = a_0 + a_1x + a_2y + a_3xy$

$u(x, y, z) = a_0 + a_1x + a_2y + a_3z + a_4xy + a_5yz + a_6xz + a_7xyz$

$u(x, y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2$

Figure 4.51 Examples of elements in finite-element method.

Figure 4.51 shows some popular elements in two and three dimensions. Strains from the displacements can be found by using Equations (2.9a) through (2.9i). The strains are substituted into potential energy, which is then minimized to generate the algebraic equations.

A FEM program consists of three major modules:

1. In first module, called the *pre-processor,* the user: creates the geometry; creates a *mesh* which the discretized geometry of elements; applies loads; and applies the boundary conditions. Figure 4.52 shows a finite-element mesh for a bracket constructed using three-dimensional tetrahedron elements. The bottom of the bracket is welded to another member. The load that is transferred through the bolt must be measured or estimated before a solution can be found. The bottom of the bracket is then modeled as points with zero displacements.
2. In the second module called *solver* the algebraic equations are created and solved. Once the nodal displacements solved are know then stresses are obtained.
3. In the third module called the *post-processor* the results of displacements and stresses are displayed in a variety of forms that are specified by the user.

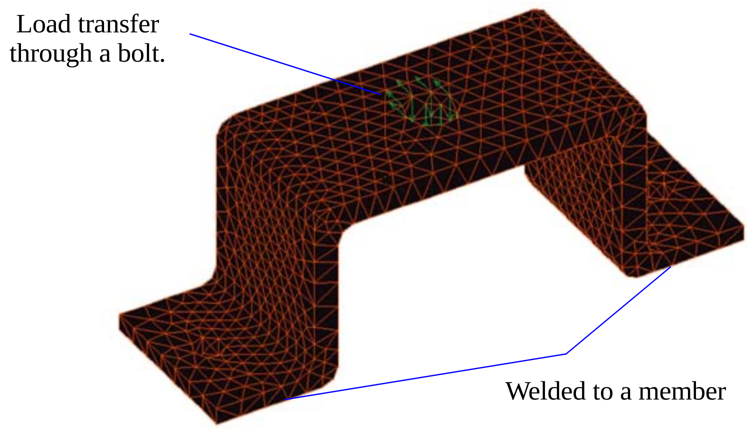


Figure 4.52 Finite-element mesh of bracket. (Courtesy Professor C. R. Vilmann.)

4.9 CHAPTER CONNECTOR

In this chapter we established formulas for deformations and stresses in axial members. We saw that the calculation of stresses and relative deformations requires the calculation of the internal axial force at a section. For statically determinate axial members, the internal axial force can be calculated either (1) by making an imaginary cut and drawing an appropriate free-body diagram or (2) by drawing an axial force diagram.

In statically indeterminate structures there are more unknowns than there are equilibrium equations. Compatibility equations have to be generated from approximate deformed shapes to solve a statically indeterminate problem. In the displacement method the equilibrium and compatibility equations are written either in terms of the deformation of axial members or in terms of the displacements of points on the structure, and the set of equations is solved. In the force method the equilibrium and compatibility equations are written either in terms of internal forces in the axial members or in terms of the reactions at the support of the structure, and the set of equations is again solved.

In Chapter 8, on stress transformation, we shall consider problems in which we first find the axial stress using the stress formula in this chapter and then find stresses on inclined planes, including planes with maximum shear stress. In Chapter 9, on strain transformation, we shall find the axial strain and the strains in the transverse direction due to Poisson's effect. We will then consider strains in different coordinate systems, including coordinate systems in which shear strain is a maximum. In Section 10.1 we shall consider the combined loading problems of axial, torsion, and bending and the design of simple structures that may be determinate or indeterminate.

POINTS AND FORMULAS TO REMEMBER

- Theory is limited to (i) slender members, (ii) regions away from regions of stress concentration, (iii) members in which the variation in cross-sectional areas and external loads is gradual, (iv) members on which axial load is applied such that there is no bending.
- $$N = \int_A \sigma_{xx}\, dA \quad (4.1) \qquad u = u(x) \quad (4.3) \qquad \text{Small strain: } \varepsilon_{xx} = \frac{du(x)}{dx} \quad (4.4)$$
- where u is the axial displacement, which is positive in the positive x direction, ε_{xx} is the axial strain, σ_{xx} is the axial stress, and N is the internal axial force over cross section A.
- Axial strain ε_{xx} is uniform across the cross section.
- Equations (4.1), (4.3), and (4.4) do not change with material model.
- Formulas below are valid for material that is linear, elastic, isotropic, with no inelastic strains:
- Homogeneous cross-section:
- $$\frac{du}{dx} = \frac{N}{EA} \quad (4.7) \qquad \sigma_{xx} = \frac{N}{A} \quad (4.8) \qquad u_2 - u_1 = \frac{N(x_2 - x_1)}{EA} \quad (4.10)$$
- where EA is the axial rigidity of the cross section.
- If N, E, or A change with x, then find deformation by integration of Equation (4.7).
- If N, E, and A do not change between x_1 and x_2, then use Equation (4.10) to find deformation.
- For homogeneous cross sections all external loads must be applied at the centroid of the cross section, and centroids of all cross sections must lie on a straight line.

•Structural analysis: $$\delta = \frac{NL}{EA} \quad (4.21)$$

- where δ is the deformation in the original direction of the axial bar.
- If N is a tensile force, then δ is elongation. If N is a compressive force, then δ is contraction.
- Degree of static redundancy is the number of unknown reactions minus the number of equilibrium equations.
- If degree of static redundancy is not zero, then we have a statically indeterminate structure.
- Compatibility equations are a geometric relationship between the deformation of bars derived from the deformed shapes of the structure.
- The number of compatibility equations in the analysis of statically indeterminate structures is always equal to the degree of redundancy.
- The direction of forces drawn on the free-body diagram must be consistent with the deformation shown in the deformed shape of the structure.
- The variables necessary to describe the deformed geometry are called degrees of freedom.
- In the displacement method, the displacements of points are treated as unknowns. The number of unknowns is equal to the degrees of freedom.
- In the force method, reaction forces are the unknowns. The number of unknowns is equal to the degrees of redundancy.

CHAPTER FIVE
TORSION OF SHAFTS

Learning objectives

1. Understand the theory, its limitations, and its applications in design and analysis of torsion of circular shafts.
2. Visualize the direction of torsional shear stress and the surface on which it acts.

When you ride a bicycle, you transfer power from your legs to the pedals, and through shaft and chain to the rear wheel. In a car, power is transferred from the engine to the wheel requiring many shafts that form the drive train such as shown in Figure 5.1*a*. A shaft also transfers torque to the rotor blades of a helicopter, as shown in Figure 5.1*b*. Lawn mowers, blenders, circular saws, drills—in fact, just about any equipment in which there is circular motion has shafts.

Any structural member that transmits torque from one plane to another is called a **shaft**. This chapter develops the simplest theory for torsion in circular shafts, following the logic shown in Figure 3.15, but subject to the limitations described in Section 3.13. We then apply the formulas to the design and analysis of statically determinate and indeterminate shafts.

(a) (b)

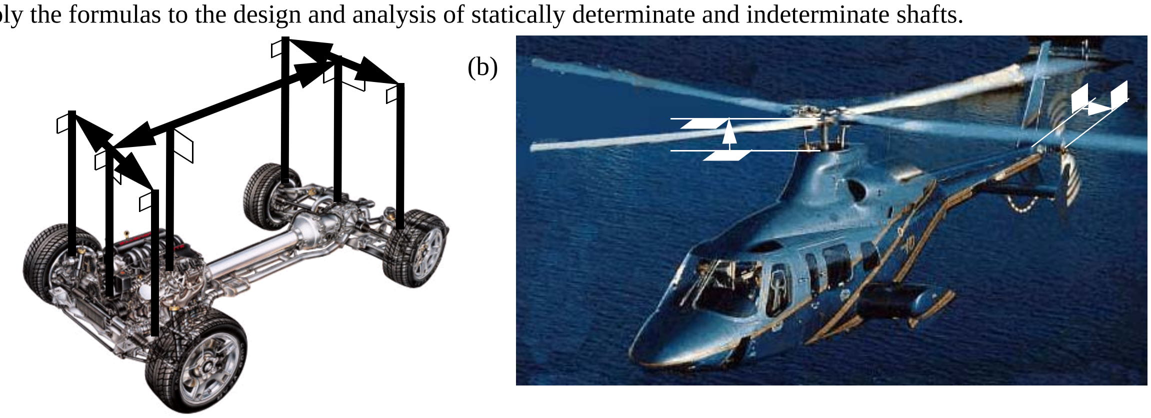


Figure 5.1 Transfer of torques between planes.

5.1 PRELUDE TO THEORY

As a prelude to theory, we consider several numerical examples solved using the logic discussed in Section 3.2. Their solution will highlight conclusions and observations that will be formalized in the development of the theory in Section 5.2.

- Example 5.1 shows the kinematics of shear strain in torsion. We apply the logic described in Figure 3.15, for the case of discrete bars attached to a rigid plate.
- Examples 5.2 and 5.3 extend the of calculation of shear strain to continuous circular shafts.
- Example 5.4 shows how the choice of a material model affects the calculation of internal torque. As we shall see the choice affects only the stress distribution, leaving all other equations unchanged. Thus the strain distribution, which is a kinematic relationship, is unaffected. So is static equivalency between shear stress and internal torque, and so are the equilibrium equations relating internal torques to external torques. Though we shall develop the simplest theory using Hooke's law, most of the equations here apply to more complex models as well.

EXAMPLE 5.1

The two thin bars of hard rubber shown in Figure 5.2 have shear modulus $G = 280$ MPa and cross-sectional area of 20 mm^2. The bars are attached to a rigid disc of 20-mm radius. The rigid disc is observed to rotate about its axis by an angle of 0.04 rad due to the applied torque T_{ext}. Determine the applied torque T_{ext}.

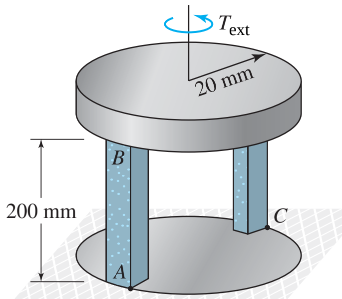


Figure 5.2 Geometry in Example 5.1.

PLAN

We can relate the rotation ($\Delta\phi = 0.04$) of the disc, the radius ($r = 0.02$ m) of the disc, and the length (0.2 m) of the bars to the shear strain in the bars as we did in Example 2.7. Using Hooke's law, we can find the shear stress in each bar. By assuming uniform shear stress in each bar, we can find the shear force. By drawing the free-body diagram of the rigid disc, we can find the applied torque T_{ext}.

SOLUTION

1. *Strain calculations:* Figure 5.3 shows an approximate deformed shape of the two bars. By symmetry the shear strain in bar C will be same as that in bar A. The shear strain in the bars can be calculated as in Example 2.7:

$$BB_1 = (0.02\text{ m})\ \Delta\phi = 0.0008\text{ m} \qquad \tan\gamma_A \approx \gamma_A = \frac{BB_1}{AB} = 0.004\text{ rad} \tag{E1}$$

$$\gamma_C = \gamma_A = 0.004\text{ rad} \tag{E2}$$

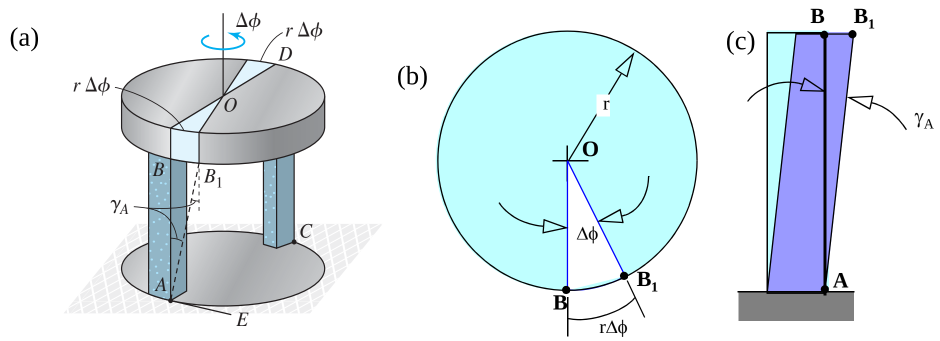


Figure 5.3 Exaggerated deformed geometry: (a) 3-D; (b) Top view; (c) Side view.

2. *Stress calculations:* From Hooke's law we can find the shear stresses as

$$\tau_A = G_A\gamma_A = [280(10^6)\text{ N/m}^2](0.004) = 1.12(10^6)\ \text{ N/m}^2 \tag{E3}$$

$$\tau_C = G_C\gamma_C = [280(10^6)\text{ N/m}^2](0.004) = 1.12(10^6)\ \text{ N/m}^2 \tag{E4}$$

3. *Internal forces:* We obtain the shear forces by multiplying the shear stresses by the cross-sectional area $A = 20 \times 10^{-6}\ \text{m}^2$:

$$\mathbf{V}_A = A_A\tau_A = [1.12(10^6)\ \text{ N/m}^2][20(10^{-6})\text{ m}^2] = 22.4\text{ N} \tag{E5}$$

$$\mathbf{V}_C = A_C\tau_C = [1.12(10^6)\ \text{ N/m}^2][20(10^{-6})\text{ m}^2] = 22.4\text{ N} \tag{E6}$$

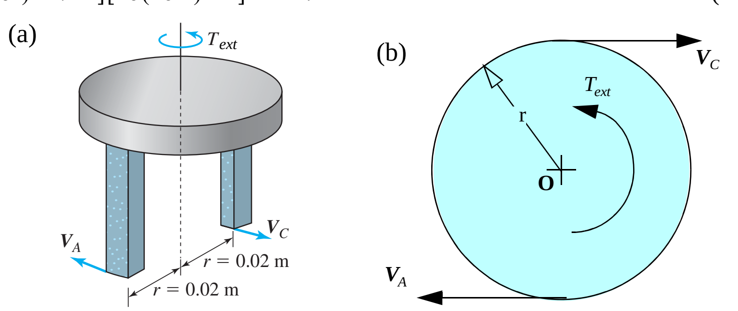


Figure 5.4 Free-body diagram: (a) 3-D; (b) Top view.

4. *External torque:* We draw the free-body diagram by making imaginary cuts through the bars, as shown in Figure 5.4. By equilibrium of moment about the axis of the disc through O, we obtain Equation (E7).

$$T_{ext} = rV_A + rV_C = (0.02\ \text{m})(22.4\ \text{N}) + (0.02\ \text{m})(22.4\ \text{N}) \tag{E7}$$

ANS. $T_{ext} = 0.896\ \text{N} \cdot \text{m}$

COMMENTS

1. In Figure 5.3 we approximated the arc BB_1 by a straight line, and we approximated the tangent function by its argument in Equation (E1). These approximations are valid only for small deformations and small strains. The net consequence of these approximations is that the shear strain along length AB_1 is uniform, as can be seen by the angle between any vertical line and line AB_1 at any point along the line.
2. The shear stress is assumed uniform across the cross section because of thin bars, but it is also uniform along the length because of the approximations described in comment 1.
3. The shear stress acts on a surface with outward normal in the direction of the length of the bar, which is also the axis of the disc. The shear force acts in the tangent direction to the circle of radius *r*. If we label the direction of the axis *x*, and the tangent direction θ, then the shear stress is represented by $\tau_{x\theta}$, as in Section 1.2
4. The sum in Equation (E7) can be rewritten as $\sum_{i=1}^{2} r\tau\ \Delta A_i$, where τ is the shear stress acting at the radius *r*, and ΔA_i is the cross-sectional area of the i^{th} bar. If we had *n* bars attached to the disc at the same radius, then the total torque would be given by $\sum_{i=1}^{n} r\tau\ \Delta A_i$. As we increase the number of bars *n* to infinity, the assembly approaches a continuos body. The cross-sectional area ΔA_i becomes the infinitesimal area *dA*, and the summation is replaced by an integral. We will formalize the observations in Section 5.1.1.
5. In this example we visualized a circular shaft as an assembly of bars. The next two examples further develop this idea.

EXAMPLE 5.2

A rigid disc of 20-mm diameter is attached to a circular shaft made of hard rubber, as shown in Figure 5.5. The left end of the shaft is fixed into a rigid wall. The rigid disc was rotated counterclockwise by 3.25°. Determine the average shear strain at point *A*.

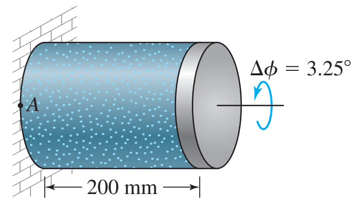


Figure 5.5 Geometry in Example 5.2.

PLAN

We can visualize the shaft as made up of infinitesimally thick bars of the type shown in Example 5.1. We relate the shear strain in the bar to the rotation of the disc, as we did in Example 5.1.

SOLUTION

We consider one line on the bar, as shown in Figure 5.6. Point *B* moves to point B_1. The right angle between *AB* and *AC* changes, and the change represents the shear strain γ. As in Example 5.1, we obtain the shear strain shown in Equation (E2):

$$\Delta\phi = \frac{3.25°\pi}{180°} = 0.05672\ \text{rad} \qquad BB_1 = r\ \Delta\phi = (10\ \text{mm})\ \Delta\phi = 0.5672\ \text{mm} \tag{E1}$$

$$\tan\gamma = \gamma = \frac{BB_1}{AB} = \frac{0.5672\ \text{mm}}{200\ \text{mm}} = 0.002836\ \text{rad} \tag{E2}$$

ANS. $\gamma = 2836\ \mu\text{rad}$

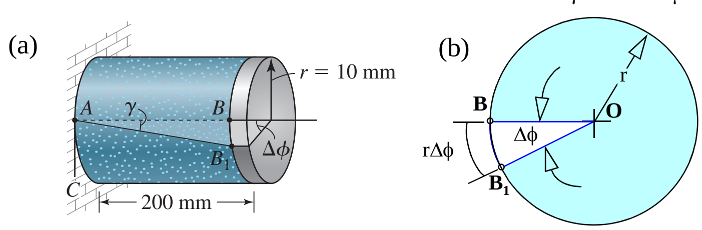


Figure 5.6 Deformed shape: (a) 3-D; (b) End view.

COMMENTS

1. As in Example 5.1, we assumed that the line *AB* remains straight. If the assumption were not valid, then the shear strain would vary in the axial direction.

2. The change of right angle that is being measured by the shear strain is the angle between a line in the axial direction and the tangent at any point. If we designate the axial direction x and the tangent direction θ (i.e., use polar coordinates), then the shear strain with subscripts will be $\gamma_{x\theta}$.
3. The value of the shear strain does not depend on the angular position as the problem is axisymmetric.
4. If we start with a rectangular grid overlaid on the shaft, as shown in Figure 5.7*a*, then each rectangle will deform by the same amount, as shown in Figure 5.7*b*. Based on the argument of axisymmetry, we will deduce this deformation for any circular shaft under torsion in the next section.

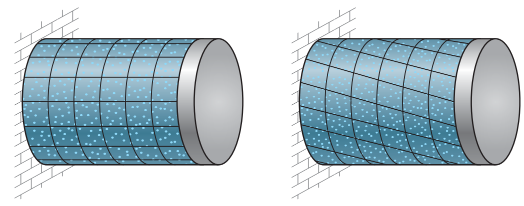


Figure 5.7 Deformation in torsion of (*a*) an un-deformed shaft. (*b*) a deformed shaft.

EXAMPLE 5.3

Three cylindrical shafts made from hard rubber are securely fastened to rigid discs, as shown in Figure 5.8. The radii of the shaft sections are $r_{AB} = 20$ mm, $r_{CD} = 15$ mm, and $r_{EF} = 10$ mm. If the rigid discs are twisted by the angles shown, determine the average shear strain in each section assuming the lines *AB*, *CD*, and *EF* remain straight.

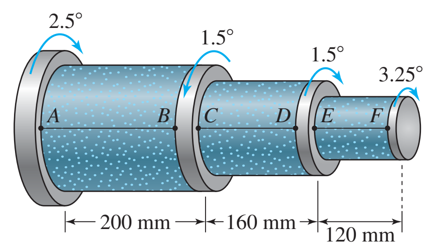


Figure 5.8 Shaft geometry in Example 5.3.

METHOD 1: PLAN

Each section of the shaft will undergo the deformation pattern shown in Figure 5.6, but now we need to account for the rotation of the disc at each end. We can analyze each section as we did in Example 5.2. In each section we can calculate the change of angle between the tangent and a line drawn in the axial direction at the point where we want to know the shear strain. We can then determine the sign of the shear strain using the definition of shear strain in Chapter 3.

SOLUTION

Label the left most disc as disc 1 and the rightmost disc, disc 4. The rotation of each disc in radians is as follows:

$$\phi_1 = \frac{2.5^\circ}{180^\circ}(3.142\text{ rad}) = 0.0436\text{ rad} \qquad \phi_2 = \frac{1.5^\circ}{180^\circ}(3.142\text{ rad}) = 0.0262\text{ rad}$$
$$\phi_3 = \frac{1.5^\circ}{180^\circ}(3.142\text{ rad}) = 0.0262\text{ rad} \qquad \phi_4 = \frac{3.25^\circ}{180^\circ}(3.142\text{ rad}) = 0.0567\text{ rad} \tag{E1}$$

Figure 5.9 shows approximate deformed shapes of the three segments,

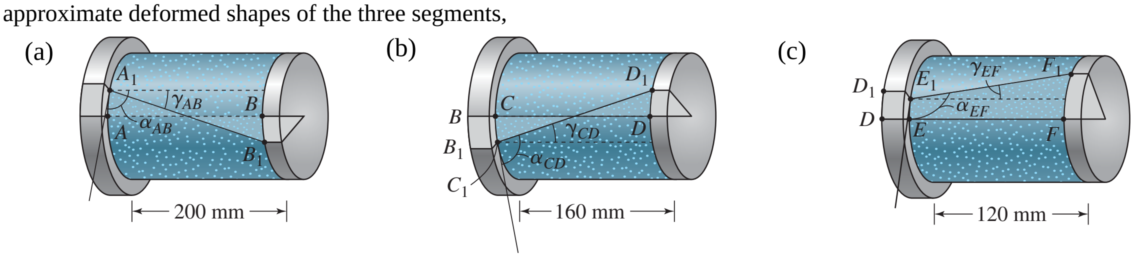


Figure 5.9 Approximate deformed shapes for Method 1 in Example 5.3 of segments (a) AB, (b) CD, and (c) EF.

Using Figure 5.9a we can find the shear strain in AB as

$$AA_1 = r_{AB}\phi_1 = (20\text{ mm})(0.0436) = 0.872\text{ mm} \qquad BB_1 = r_{AB}\phi_2 = (20\text{ mm})(0.0262) = 0.524\text{ mm} \tag{E2}$$

$$\text{:an}|\gamma_{AB}| \approx |\gamma_{AB}| = \frac{AA_1 + BB_1}{AB} = \frac{0.872\text{ mm} + 0.524\text{ mm}}{200\text{ mm}} \tag{E3}$$

The shear strain is positive as the angle γ_{AB} represents a decrease of angle from right angle.

ANS. $\gamma_{AB} = 6980\ \mu\text{rad}$

Using Figure 5.9b we can find the shear strain in CD as

$$CC_1 = r_{CD}\phi_2 = (15\ \text{mm})(0.0262) = 0.393\ \text{mm} \qquad DD_1 = r_{CD}\phi_3 = (15\ \text{mm})(0.0262) = 0.393\ \text{mm} \tag{E4}$$

$$\text{an}|\gamma_{CD}| \approx |\gamma_{CD}| = \frac{CC_1 + DD_1}{CD} = \frac{0.393\ \text{mm} + 0.393\ \text{mn}}{160\ \text{mm}} \tag{E5}$$

The shear strain is negative as the angle γ_{CD} represents an increase of angle from right angle.

ANS. $\gamma_{CD} = -4913\ \mu\text{rad}$

Using Figure 5.9c we can find the shear strain in EF as

$$EE_1 = r_{EF}\phi_3 = (10\ \text{mm})(0.0262) = 0.262\ \text{mm} \qquad FF_1 = r_{EF}\phi_4 = (10\ \text{mm})(0.0567) = 0.567\ \text{mm} \tag{E6}$$

$$\text{an}|\gamma_{EF}| \approx |\gamma_{EF}| = \frac{FF_1 - EE_1}{EF} = \frac{0.567\ \text{mm} - 0.262\ \text{mm}}{120\ \text{mm}} \tag{E7}$$

The shear strain is negative as the angle γ_{EF} represents an increase of angle from right angle.

ANS. $\gamma_{EF} = -2542\ \mu\text{rad}$

METHOD 2: PLAN

We assign a sign to the direction of rotation, calculate the relative deformation of the right disc with respect to the left disc, and analyze the entire shaft.

We draw an approximate deformed shape of the entire shaft, as shown in Figure 5.10. Let the counterclockwise rotation with respect to the x axis be positive and write each angle with the correct sign,

$$\phi_1 = -0.0436\ \text{rad} \qquad \phi_2 = 0.0262\ \text{rad} \qquad \phi_3 = -0.0262\ \text{rad} \qquad \phi_4 = -0.0567\ \text{rad} \tag{E8}$$

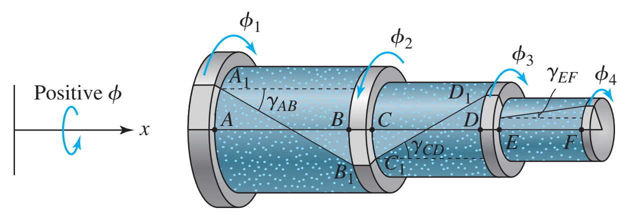


Figure 5.10 Shear strain calculation by Method 2 in Example 5.3.

We compute the relative rotation in each section and multiply the result by the corresponding section radius to obtain the relative movement of two points in a section. We then divide by the length of the section as we did in Example 5.2.

$$\Delta\phi_{AB} = \phi_2 - \phi_1 = 0.0698 \qquad \gamma_{AB} = \frac{r_{AB}\,\Delta\phi_{AB}}{AB} = \frac{(20\ \text{mm})\ (0.0698)}{(200\ \text{mm})} = 0.00698\ \text{rad} \tag{E9}$$

$$\Delta\phi_{CD} = \phi_3 - \phi_2 = -0.0524 \qquad \gamma_{CD} = \frac{r_{CD}\,\Delta\phi_{CD}}{CD} = \frac{(15\ \text{mm})\ (-0.0524)}{160\ \text{mm}} = -0.004913\ \text{rad} \tag{E10}$$

$$\Delta\phi_{EF} = \phi_4 - \phi_3 = -0.0305 \qquad \gamma_{EF} = \frac{r_{EF}\,\Delta\phi_{EF}}{EF} = \frac{(10\ \text{mm})\ (-0.0305)}{120\ \text{mm}} = -0.002542\ \text{rad} \tag{E11}$$

ANS. $\gamma_{AB} = 6980\ \mu\text{rad} \qquad \gamma_{CD} = -4913\ \mu\text{rad} \qquad \gamma_{EF} = -2542\ \mu\text{rad}$

COMMENTS

1. Method 1 is easier to visualize, but the repetitive calculations can be tedious. Method 2 is more mathematical and procedural, but the repetitive calculations are easier. By solving the problems by method 2 but spending time visualizing the deformation as in method 1, we can reap the benefits of both.
2. We note that the shear strain in each section is directly proportional to the radius and the relative rotation of the shaft and inversely proportional to its length.

5.1.1 Internal Torque

Example 5.1 showed that the shear stress $\tau_{x\theta}$ can be replaced by an equivalent torque using an integral over the cross-sectional area. In this section we formalize that observation.

Figure 5.11 shows the shear stress distribution $\tau_{x\theta}$ that is to be replaced by an equivalent internal torque $\boldsymbol{T}$. Let ρ represent the radial coordinate, that is, the radius of the circle at which the shear stress acts. The moment at the center due to the shear stress on the differential area is $\rho\tau_{x\theta}\,dA$. By integrating over the entire area we obtain the total internal torque at the cross section.

$$\boxed{T = \int_A \rho\, dV = \int_A \rho \tau_{x\theta}\, dA} \tag{5.1}$$

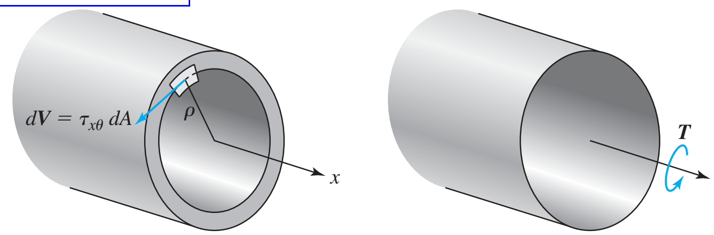


Figure 5.11 Statically equivalent internal torque.

Equation (5.1) is independent of the material model as it represents static equivalency between the shear stress on the entire cross section and the internal torque. If we were to consider a composite shaft cross section or nonlinear material behavior, then it would affect the value and distribution of $\tau_{x\theta}$ across the cross section. But Equation (5.1), relating $\tau_{x\theta}$ and T, would remain unchanged. Examples 5.4 will clarify the discussion in this paragraph.

EXAMPLE 5.4

A homogeneous cross section made of brass and a composite cross section of brass and steel are shown in Figure 5.12. The shear moduli of elasticity for brass and steel are $G_B = 40$ GPa and $G_S = 80$ GPa, respectively. The shear strain in polar coordinates at the cross section was found to be $\gamma_{x\theta} = 0.08\rho$, where ρ is in meters. (a)Write expressions for $\tau_{x\theta}$ as a function of ρ and plot the shear strain and shear stress distributions across both cross sections. (b) For each of the cross sections determine the statically equivalent internal torques.

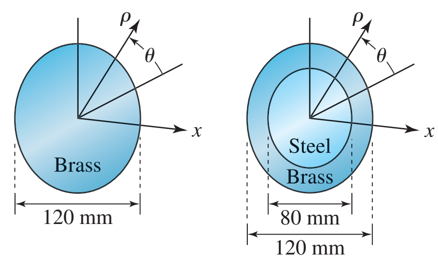


Figure 5.12 Homogeneous and composite cross sections in Example 5.4.

PLAN

(a) Using Hooke's law we can find the shear stress distribution as a function of ρ in each material. (b) Each of the shear stress distributions can be substituted into Equation (5.1) and the equivalent internal torque obtained by integration.

SOLUTION

(a) From Hooke's law we can write the stresses as

$$(\tau_{x\theta})_{\text{brass}} = [40(10^9)\ \text{N/m}^2](0.08\rho) = 3200\rho\ \text{MPa} \tag{E1}$$

$$(\tau_{x\theta})_{\text{steel}} = [80(10^9)\ \text{N/m}^2](0.08\rho) = 6400\rho\ \text{MPa} \tag{E2}$$

For the homogeneous cross section the stress distribution is as given in Equation (E1), but for the composite section it switches between Equation (E2) and Equation (E1), depending on the value of ρ. We can write the shear stress distribution for both cross sections as a function of ρ, as shown below.

Homogeneous cross section:

$$\tau_{x\theta} = 3200\rho\ \text{MPa} \qquad 0.00 \le \rho < 0.06 \tag{E3}$$

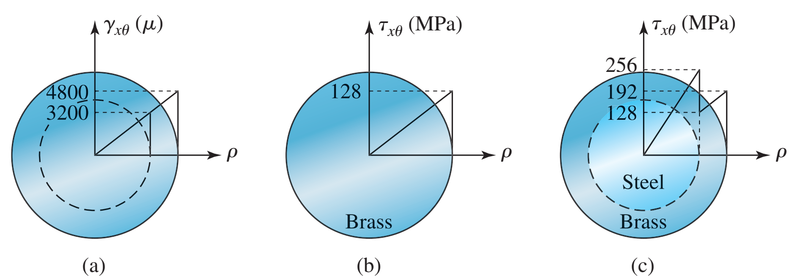


Figure 5.13 Shear strain and shear stress distributions in Example 5.4: (*a*) shear strain distribution; (*b*) shear stress distribution in homogeneous cross section; (c) shear stress distribution in composite cross section.

Composite cross section:

$$\tau_{x\theta} = \begin{cases} 6400\rho \text{ MPa} & 0.00 \le \rho < 0.04 \text{ m} \\ 3200\rho \text{ MPa} & 0.04 \text{ m} < \rho \le 0.06 \text{ m} \end{cases} \tag{E4}$$

The shear strain and the shear stress can now be plotted as a function of ρ, as shown in Figure 5.13(b). The differential area dA is the area of a ring of radius ρ and thickness $d\rho$, that is, $dA = 2\pi\rho\ d\rho$. Equation (5.1) can be written as

$$\boldsymbol{T} = \int_0^{0.06} \rho\tau_{x\theta}\,(2\pi\rho\ d\rho) \tag{E5}$$

Homogeneous cross section: Substituting Equation (E3) into Equation (E5) and integrating, we obtain the equivalent internal torque.

$$\boldsymbol{T} = \int_0^{0.06} \rho[3200\rho(10^6)\,](2\pi\rho\ d\rho) = [6400\pi(10^6)]\left(\frac{\rho^4}{4}\right)\Bigg|_0^{0.06} = 65.1(10^3)\text{ N}\cdot\text{m} \tag{E6}$$

ANS. $\boldsymbol{T} = 65.1 \text{ kN}\cdot\text{m}$

Composite cross section: Writing the integral in Equation (E5) as a sum of two integrals and substituting Equation (E3) we obtain the equivalent internal torque.

$$\boldsymbol{T} = \int_0^{0.06} \rho\tau_{x\theta}\,(2\pi\rho\ d\rho) = \underbrace{\int_0^{0.04} \rho\tau_{x\theta}\,(2\pi\rho\ d\rho)}_{\boldsymbol{T}_{\text{steel}}} + \underbrace{\int_{0.04}^{0.06} \rho\tau_{x\theta}\,(2\pi\rho\ d\rho)}_{\boldsymbol{T}_{\text{brass}}} \tag{E7}$$

$$T_{\text{steel}} = \int_0^{0.04} \rho[6400\rho(10^6)](2\pi\rho\ d\rho) = (12800\pi)(10^6)\left(\frac{\rho^4}{4}\right)\Bigg|_0^{0.04} = 25.7(10^3)\text{ N}\cdot\text{m} = 25.7\text{ kN}\cdot\text{m} \tag{E8}$$

$$T_{\text{brass}} = \int_{0.04}^{0.06} \rho[3200\rho(10^6)](2\pi\rho\ d\rho) = (6400\pi)(10^6)\left(\frac{\rho^4}{4}\right)\Bigg|_{0.04}^{0.06} = 52.3(10^3)\text{ N}\cdot\text{m} = 52.3\text{ kN}\cdot\text{m} \tag{E9}$$

$$T = T_{\text{steel}} + T_{\text{brass}} = 25.7\text{ kN}\cdot\text{m} + 52.3\text{ kN}\cdot\text{m} \tag{E10}$$

ANS. $T = 78\text{ kN}\cdot\text{m}$

COMMENTS

1. The example demonstrates that although the shear strain varies linearly across the cross section, the shear stress may not. In this example we considered material non homogeneity. In a similar manner we can consider other models, such as elastic–perfectly plastic, or material models that have nonlinear stress–strain curves.
2. The material models dictate the shear stress distribution across the cross section, but once the stress distribution is known, Equation (5.1) can be used to find the equivalent internal torque, emphasizing that Equation (5.1) does not depend on the material model.

PROBLEM SET 5.1

5.1 A pair of 48-in. long bars and a pair of 60-in. long bars are symmetrically attached to a rigid disc at a radius of 2 in. at one end and built into the wall at the other end, as shown in Figure P5.1. The shear strain at point A due to a twist of the rigid disc was found to be 3000 μrad. Determine the magnitude of shear strain at point D.

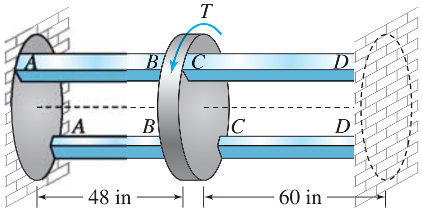


Figure P5.1

5.2 If the four bars in Problem 5.1 are made from a material that has a shear modulus of 12,000 ksi, determine the applied torque T on the rigid disc. The cross sectional areas of all bars are 0.25 in.^{2}.

5.3 If bars AB in Problem 5.1 are made of aluminum with a shear modulus G_{al} = 4000 ksi and bars CD are made of bronze with a shear modulus G_{br} = 6500 ksi, determine the applied torque T on the rigid disc. The cross-sectional areas of all bars are 0.25 in.^{2}.

5.4 Three pairs of bars are symmetrically attached to rigid discs at the radii shown in Figure P5.4. The discs were observed to rotate by angles $\phi_1 = 1.5°$, $\phi_2 = 3.0°$, and $\phi_3 = 2.5°$ in the direction of the applied torques T_1, T_2, and T_3, respectively. The shear modulus of the bars is 40 ksi and cross-sectional area is 0.04 in.^{2}. Determine the applied torques.

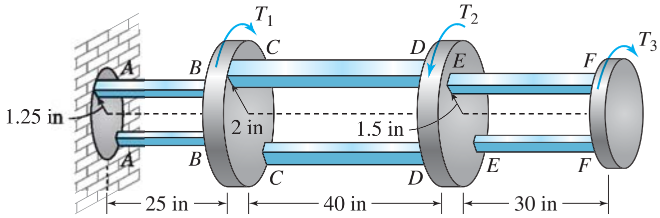


Figure P5.4

5.5 A circular shaft of radius r and length Δx has two rigid discs attached at each end, as shown in Figure P5.5. If the rigid discs are rotated as shown, determine the shear strain γ at point A in terms of r, Δx, and $\Delta\phi$, assuming that line AB remains straight, where $\Delta\phi = \phi_2 - \phi_1$.

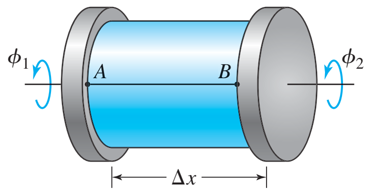


Figure P5.5

5.6 A hollow circular shaft made from hard rubber has an outer diameter of 4 in and an inner diameter of 1.5 in. The shaft is fixed to the wall on the left end and the rigid disc on the right hand is twisted, as shown in Figure P5.6. The shear strain at point A, which is on the outside surface, was found to be 4000 μrad. Determine the shear strain at point C, which is on the inside surface, and the angle of rotation. Assume that lines AB and CD remain straight during deformation.

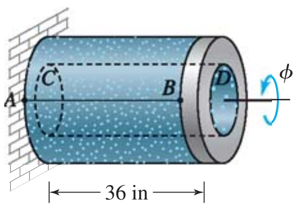


Figure P5.6

5.7 The magnitude of shear strains in the segments of the stepped shaft in Figure P5.7 was found to be $\gamma_{AB} = 3000$ μrad, $\gamma_{CD} = 2500$ μrad, and $\gamma_{EF} = 6000$ μrad. The radius of section AB is 150 mm, of section CD 70 mm, and of section EF 60 mm.Determine the angle by which each of the rigid discs was rotated.

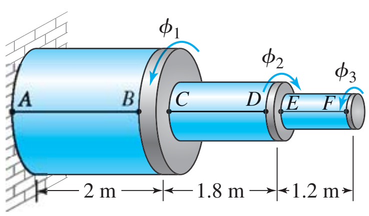


Figure P5.7

5.8 Figure P5.8 shows the cross section of a hollow aluminum (G= 26 GPa) shaft. The shear strain $\gamma_{x\theta}$ in polar coordinates at the section is $\gamma_{x\theta} = -0.06\rho$, where ρ is in meters. Determine the equivalent internal torque acting at the cross-section. Use d_i= 30 mm and d_o= 50 mm.

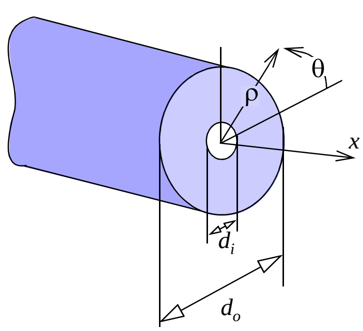


Figure P5.8

5.9 Figure P5.8 shows the cross section of a hollow aluminum (G = 26 GPa) shaft. The shear strain $\gamma_{x\theta}$ in polar coordinates at the section is $\gamma_{x\theta} = 0.05\rho$, where ρ is in meters. Determine the equivalent internal torque acting at the cross-section. Use d_i= 40 mm and d_o = 120 mm.

5.10 A hollow brass shaft ($G_B = 6500$ ksi) and a solid steel shaft ($G_S = 13{,}000$ ksi) are securely fastened to form a composite shaft, as shown in Figure P5.10.The shear strain in polar coordinates at the section is $\gamma_{x\theta} = 0.001\rho$, where ρ is in inches. Determine the equivalent internal torque acting at the cross section. Use $d_B = 4$ in. and $d_S = 2$ in.

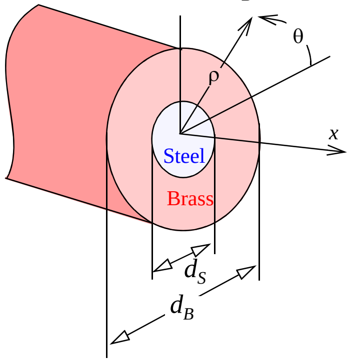


Figure P5.10

5.11 A hollow brass shaft ($G_B = 6500$ ksi) and a solid steel shaft ($G_S = 13{,}000$ ksi) are securely fastened to form a composite shaft, as shown in Figure P5.10.The shear strain in polar coordinates at the section is $\gamma_{x\theta} = -0.0005\rho$, where ρ is in inches. Determine the equivalent internal torque acting at the cross section. Use d_B = 6 in. and d_S = 4 in.

5.12 A hollow brass shaft ($G_B = 6500$ ksi) and a solid steel shaft ($G_S = 13{,}000$ ksi) are securely fastened to form a composite shaft, as shown in Figure P5.10.The shear strain in polar coordinates at the section is $\gamma_{x\theta} = 0.002\rho$, where ρ is in inches. Determine the equivalent internal torque acting at the cross section. Use d_B = 3 in. and d_S = 1 in.

5.13 A hollow titanium shaft ($G_{Ti} = 36$ *GPa*) and a hollow aluminum shaft ($G_{Al} = 26$ *GPa)* are securely fastened to form a composite shaft shown in Figure P5.13. The shear strain in polar coordinates at the section is $\gamma_{x\theta} = 0.04\rho$, where ρ is in meters. Determine the equivalent internal torque acting at the cross section. Use d_i = 50 mm, d_{Al} = 90 mm, and d_{Ti} = 100 mm.

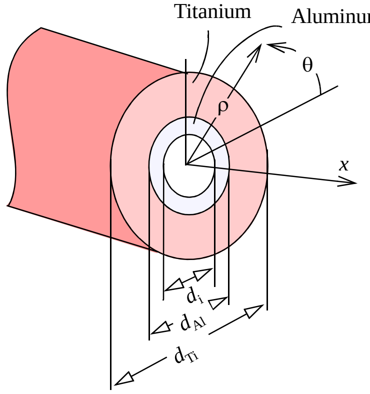


Figure P5.13

Stretch Yourself

5.14 A circular shaft made from elastic - perfectly plastic material has a torsional shear stress distribution across the cross section shown in Figure P5.14. Determine the equivalent internal torque.

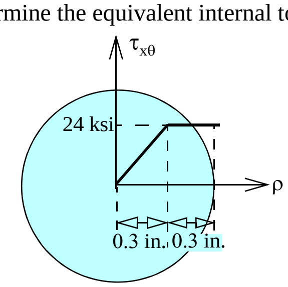


Figure P5.14

5.15 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches. The shaft is made from an elastic–perfectly plastic material, which has a yield stress $\tau_{yield} = 18$ ksi and a shear modulus $G = 12{,}000$ ksi. Determine the equivalent internal torque. (See Problem 3.144).

5.16 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches.The shaft is made form a bilinear material as shown in Figure 3.40. The material has a yield stress $\tau_{yield} = 18$ ksi and shear moduli $G_1 = 12{,}000$ ksi and $G_2 = 4800$ ksi. Determine the equivalent internal torque.(See Problem 3.145).

5.17 A solid circular shaft of 3-in. diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches.The shaft material has a stress–strain relationship given by $\tau = 243\gamma^{0.4}$ ksi. Determine the equivalent internal torque. (See Problem 3.146).

5.18 A solid circular shaft of 3-in diameter has a shear strain at a section in polar coordinates of $\gamma_{x\theta} = 2\rho\ (10^{-3})$, where ρ is the radial coordinate measured in inches. The shaft material has a stress–strain relationship given by $\tau = 12{,}000\gamma - 120{,}000\gamma^2$ ksi. Determine the equivalent internal torque. (See Problem 3.147).

5.2 THEORY OF TORSION OF CIRCULAR SHAFTS

In this section we develop formulas for deformation and stress in a circular shaft. We will follow the procedure in Section 5.1 but now with variables in place of numbers. The theory will be developed subject to the following limitations:

1. The length of the member is significantly greater than the greatest dimension in the cross section.
2. We are away from the regions of stress concentration.
3. The variation of external torque or change in cross-sectional areas is gradual except in regions of stress concentration.
4. External torques are not functions of time; that is, we have a static problem. (See Problems 5.55 and 5.56 for dynamic problems.)
5. The cross section is circular. This permits us to use arguments of axisymmetry in deducing deformation.

Figure 5.14 shows a circular shaft that is loaded by external torques T_1 and T_2 at each end and an external distributed torque $t(x)$, which has units of torque per unit length. The radius of the shaft $R(x)$ varies as a function of x. We expect that the internal torque $\boldsymbol{T}$ will be a function of x. ϕ_1 and ϕ_2 are the angles of rotation of the imaginary cross sections at x_1 and x_2, respectively.

The objectives of the theory are:

1. To obtain a formula for the relative rotation $\phi_2 - \phi_1$ in terms of the internal torque $\boldsymbol{T}$.
2. To obtain a formula for the shear stress $\tau_{x\theta}$ in terms of the internal torque $\boldsymbol{T}$.

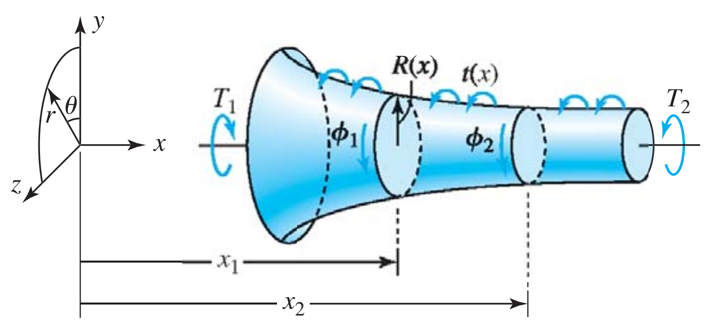


Figure 5.14 Circular shaft.

To account for the variations in $t(x)$ and $R(x)$ we will take $\Delta x = x_2 - x_1$ as an infinitesimal distance in which these quantities can be treated as constants. The deformation behavior across the cross section will be approximated. The logic shown in Figure 5.15 and discussed in Section 3.2 will be used to develop the simplest theory for the torsion of circular shafts members. Assumptions will be identified as we move from one step to the next. These assumptions are the points at which complexities can be added to the theory, as discussed in the examples and Stretch Yourself problems.

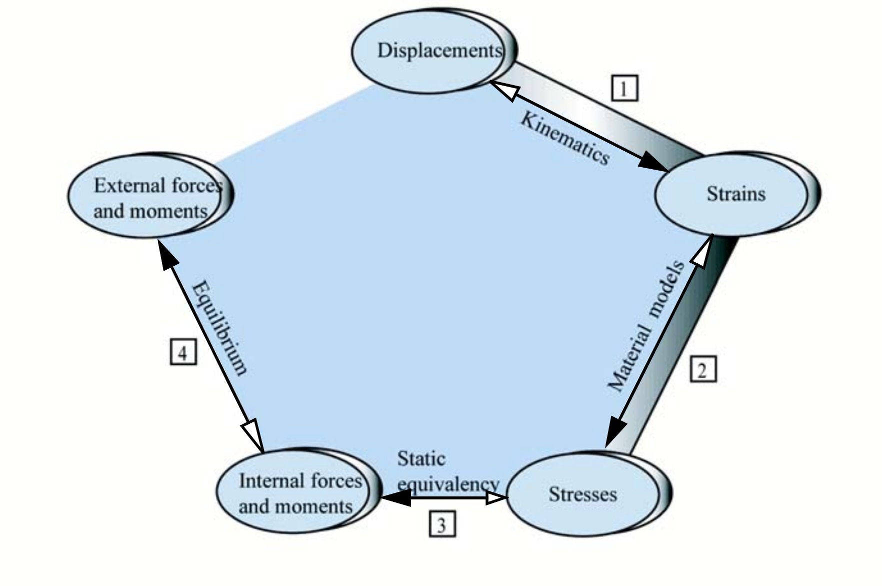


Figure 5.15 The logic of the mechanics of materials.

5.2.1 Kinematics

In Example 5.1 the shear strain in a bar was related to the rotation of the disc that was attached to it. In Example 5.2 we remarked that a shaft could be viewed as an assembly of bars. Three assumptions let us simulate the behavior of a cross section as a rotating rigid plate:

Assumption 1 Plane sections perpendicular to the axis remain plane during deformation.

Assumption 2 On a cross section, all radial lines rotate by equal angles during deformation.

Assumption 3 Radial lines remain straight during deformation.

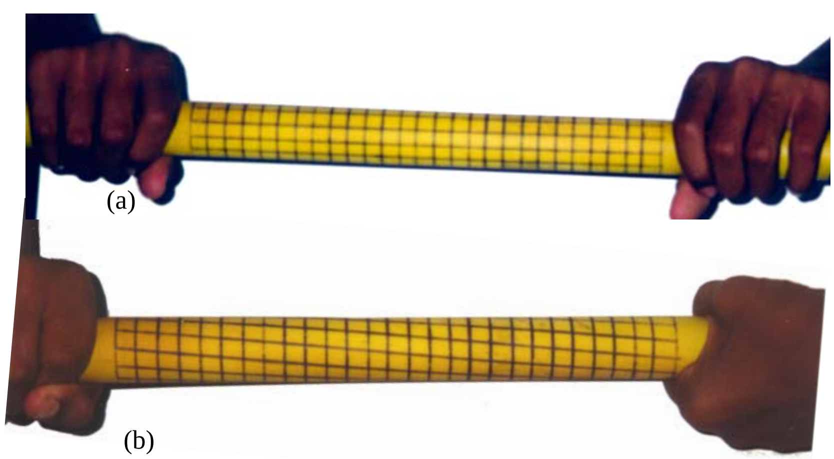


Figure 5.16 Torsional deformation: (*a*) original grid; (*b*) deformed grid. (Courtesy of Professor J. B. Ligon.)

Figure 5.16 shows a circular rubber shaft with a grid on the surface that is twisted by hand. The edges of the circles remain vertical lines during deformation. This observation confirms the validity of Assumption 1. Axial deformation due to torsional loads is called **warping**. Thus, circular shafts do not warp. Shafts with noncircular cross section warp, and this additional deformation leads to additional complexities. (See Problem 5.53).

The axisymmetry of the problem implies that deformation must be independent of the angular rotation. Thus, all radials lines must behave in exactly the same manner irrespective of their angular position, thus, Assumptions 2 and 3 are valid for circular shafts. Figure 5.17 shows that all radial lines rotate by the same **angle of twist** ϕ. We note that if all lines rotate by equal amounts on the cross section, then ϕ does not change across the cross section and hence can only be a function of x

$$\phi = \phi(x) \tag{5.2}$$

Sign Convention: ϕ is considered positive counterclockwise with respect to the x axis.

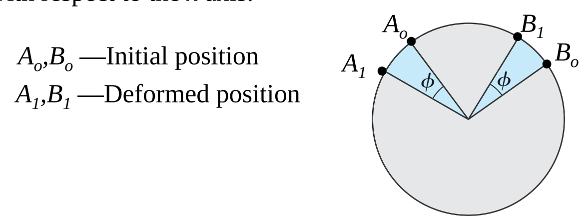


Figure 5.17 Equal rotation of all radial lines.

The shear strain of interest to us is the measure of the angle change between the axial direction and the tangent to the circle in Figure 5.16. If we use polar coordinates, then we are interested in the change in angle which is between the x and θ directions— in other words, $\gamma_{x\theta}$.

Assumptions 1 through 3 are analogous to viewing each cross section in the shaft as a rigid disc that rotates about its own axis. We can then calculate the shear strain as in Example 5.2, provided we have small deformation and strain.

Assumption 4 Strains are small.

We consider a shaft with radius ρ and length Δx in which the right section with respect to the left section is rotated by an angle $\Delta\phi$, as shown in Figure 5.18*a*. Using geometry we obtain the shear strain expression.

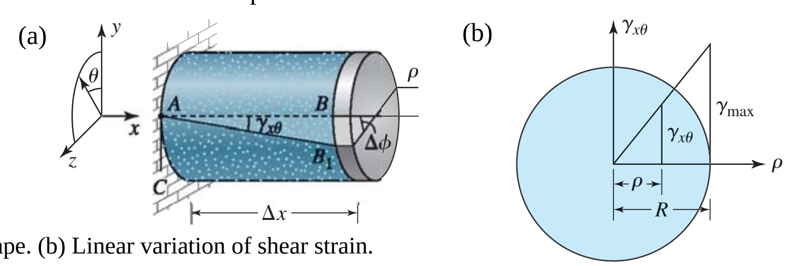


Figure 5.18 Shear strain in torsion. (a) Deformed shape. (b) Linear variation of shear strain.

$$\tan\gamma_{x\theta} \approx \gamma_{x\theta} = \lim_{AB \to 0}\left(\frac{BB_1}{AB}\right) = \lim_{\Delta x \to 0}\left(\frac{\rho\Delta\phi}{\Delta x}\right) \quad \text{or}$$

$$\boxed{\gamma_{x\theta} = \rho\frac{d\phi}{dx}} \tag{5.3}$$

where ρ is the radial coordinate of a point on the cross section. The subscripts x and θ emphasize that the change in angle is between the axial and tangent directions, as shown in Figure 5.18*a*. The quantity $d\phi/dx$ is called the **rate of twist**. It is a function of x only, because ϕ is a function of x only.

Equation (5.3) was derived from purely geometric considerations. If Assumptions 1 through 4 are valid, then Equation (5.3) is independent of the material. Equation (5.3) shows that the shear strain is a *linear function* of the radial coordinate ρ and reaches the maximum value γ_{max} at the outer surface ($\rho = \rho_{max} = R$), as shown in Figure 5.18*a*. Equation (5.4), an alternative form for shear strain, can be derived using similar triangles.

$$\gamma_{x\theta} = \frac{\gamma_{max}\rho}{R} \tag{5.4}$$

5.2.2 Material Model

Our motivation is to develop a simple theory for torsion of circular shafts. Thus we make assumptions regarding material behavior that will permit us to use the simplest material model given by Hooke's law.

Assumption 5 The material is linearly elastic.[1]

Assumption 6 The material is isotropic.

Substituting Equation (5.3) into Hooke's law, that is, $\tau = G\gamma$, we obtain

$$\tau_{x\theta} = G\rho\frac{d\phi}{dx} \tag{5.5}$$

Noting that θ is positive in the counterclockwise direction with respect to the x axis, we can represent the shear stress due to torsion on a stress element as shown in Figure 5.19. Also shown in Figure 5.19 are aluminum and wooden shafts that broke in torsion. The shear stress component that exceeds the shear strength in aluminum is $\tau_{x\theta}$. The shear strength of wood is weaker along the surface parallel to the grain, which for shafts is in the longitudinal direction. Thus $\tau_{\theta x}$ causes the failure in wooden shafts. The two failure surfaces highlight the importance of visualizing the torsional shear stress element.

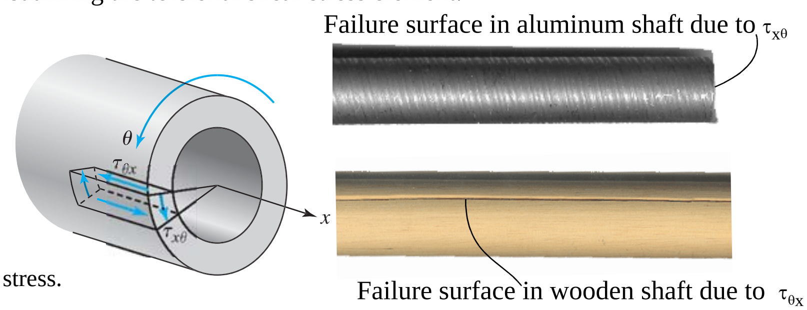


Figure 5.19 Stress element showing torsional shear stress.

5.2.3 Torsion Formulas

Substituting Equation (5.5) into Equation (5.1) and noting that $d\phi/dx$ is a function of x only, we obtain

$$\boldsymbol{T} = \int_A G\rho^2\frac{d\phi}{dx}\,dA = \frac{d\phi}{dx}\int_A G\rho^2\,dA \tag{5.6}$$

To simplify further, we would like to take G outside the integral, which implies that G cannot change across the cross section.

Assumption 7 The material is homogeneous across the cross section.[2]

From Equation (5.6) we obtain

$$\boldsymbol{T} = G\frac{d\phi}{dx}\int_A \rho^2\,dA = GJ\frac{d\phi}{dx} \tag{5.7}$$

where J is the **polar moment of inertia** for the cross section. As shown in Example 5.5, J for a circular cross section of radius R or diameter D is given by

$$\boxed{J = \int_A \rho^2\,dA = \frac{\pi}{2}R^4 = \frac{\pi}{32}D^4} \tag{5.8}$$

Equation (5.7) can be written as

$$\boxed{\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ}} \tag{5.9}$$

The higher the value of GJ, the smaller will be the deformation ϕ for a given value of the internal torque. Thus the rigidity of the shaft increases with the increase in GJ. A shaft may be made more rigid either by choosing a stiffer material (higher value of G) or by increasing the polar moment of inertia. The quantity GJ is called **torsional rigidity**.

Substituting Equation (5.9) into Equation (5.5), we obtain

$$\boxed{\tau_{x\theta} = \frac{\boldsymbol{T}\rho}{J}} \tag{5.10}$$

[1]See Problems 5.50 through 5.52 for nonlinear material behavior.
[2]In Problem. 5.49 this assumption is not valid.

The quantities $\boldsymbol{T}$ and J do not vary across the cross section. Thus the shear stress varies *linearly* across the cross section with ρ as shown in Figure 5.20. For a solid shaft, it is zero at the center where $\rho = 0$ and reaches a *maximum value on the outer surface* of the shaft where $\rho = R$,.

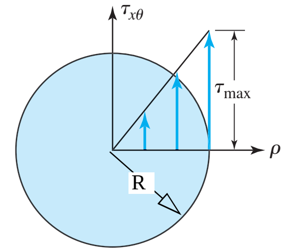


Figure 5.20 Linear variation of torsional shear stress.

Let the angle of rotation of the cross section at x_1 and x_2 be ϕ_1 and ϕ_2, respectively. By integrating Equation (5.9) we can obtain the relative rotation as:

$$\phi_2 - \phi_1 = \int_{\phi_1}^{\phi_2} d\phi = \int_{x_1}^{x_2} \frac{\boldsymbol{T}}{GJ}\, dx \tag{5.11}$$

To obtain a simple formula we would like to take the three quantities $\boldsymbol{T}$, G, and J outside the integral, which means that these quantities should not change with x. To achieve this simplicity we make the following assumptions:

Assumption 8 The material is homogeneous between x_1 and x_2. (G is constant)

Assumption 9 The shaft is not tapered between x_1 and x_2. (J is constant)

Assumption 10 The external (and hence also the internal) torque does not change with x between x_1 and x_2. ($\boldsymbol{T}$ is constant)

If Assumptions 8 through 10 are valid, then $\boldsymbol{T}$, G, and J are constant between x_1 and x_2, and from Equation (5.11) we obtain

$$\boxed{\phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{GJ}} \tag{5.12}$$

In Equation (5.12) points x_1 and x_2 must be chosen such that neither $\boldsymbol{T}$, G, nor J change between these points.

5.2.4 Sign Convention for Internal Torque

The shear stress was replaced by a statically equivalent internal torque using Equation (5.1). The shear stress $\tau_{x\theta}$ is positive on two surfaces. Hence the equivalent internal torque is positive on two surfaces, as shown in Figure 5.21. When we make the imaginary cut to draw the free-body diagram, then the internal torque must be drawn in the positive direction if we want the formulas to give the correct signs.

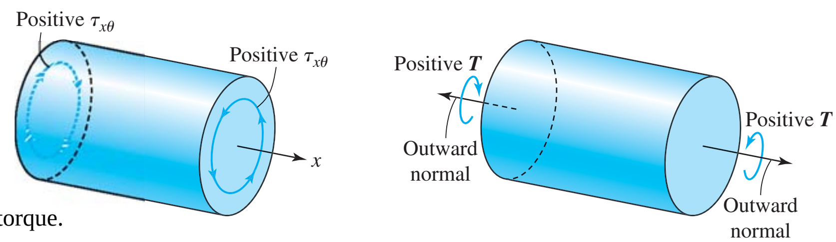


Figure 5.21 Sign convention for positive internal torque.

Sign Convention: Internal torque is considered positive counterclockwise with respect to the outward normal to the imaginary cut surface.

$\boldsymbol{T}$ may be found in either of two ways, as described next and elaborated further in Example 5.6.

1. $\boldsymbol{T}$ is always drawn counterclockwise with respect to the outward normal of the imaginary cut, as per our sign convention. The equilibrium equation is then used to get a positive or negative value for $\boldsymbol{T}$. The sign for relative rotation obtained from Equation (5.12) is positive counterclockwise with respect to the x axis. The direction of shear stress can be determined using the subscripts, as in Section 1.3.

2. ***T*** is drawn at the imaginary cut to equilibrate the external torques. Since inspection is used to determine the direction of ***T***, the direction of relative rotation in Equation (5.12) and the direction of shear stress $\tau_{x\theta}$ in Equation (5.10) must also be determined by inspection.

5.2.5 Direction of Torsional Stresses by Inspection.

The significant shear stress in the torsion of circular shafts is $\tau_{x\theta}$. All other stress components can be neglected provided the ratio of the length of the shaft to its diameter is on the order of 10 or more.

Figure 5.22*a* shows a segment of a shaft under torsion containing point *A*. We visualize point *A* on the left segment and consider the stress element on the left segment. The left segment rotates clockwise in relation to the right segment. This implies that point *A*, which is part of the left segment, is moving upward on the shaded surface. Hence the shear stress, like friction, on the shaded surface will be downward. We know that a pair of symmetric shear stress components points toward or away from the corner. From the symmetry, the shear stresses on the rest of the surfaces can be drawn as shown.

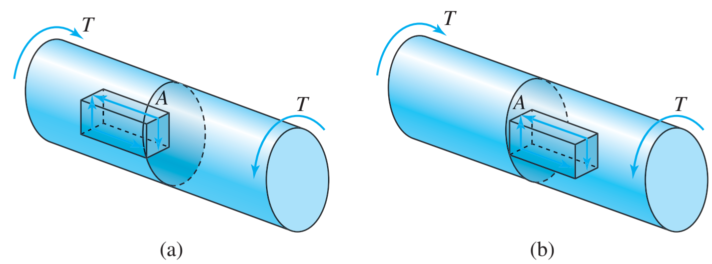


Figure 5.22 Direction of shear stress by inspection.

Suppose we had considered point *A* on the right segment of the shaft. In such a case we consider the stress element as part of the right segment, as shown in Figure 5.22*b*. The right segment rotates counterclockwise in relation to the left segment. This implies that point *A*, which is part of the right segment, is moving down on the shaded surface. Hence the shear stress, like friction, will be upward. Once more using the symmetry of shear stress components, the shear stress on the remaining surfaces can be drawn as shown.

In visualizing the stress surface, we need not draw the shaft segments in Figure 5.22. But care must be taken to identify the surface on which the shear stress is being considered. The shear stress on the adjoining imaginary surfaces have opposite direction. However, irrespective of the shaft segment on which we visualize the stress element, we obtain the same stress element, as shown in Figure 5.22. This is because the two stress elements shown represent the same point *A*.

An alternative way of visualizing torsional shear stress is to think of a coupling at an imaginary section and to visualize the shear stress directions on the bolt surfaces, as shown in Figure 5.23. Once the direction of the shear stress on the bolt surface is visualized, the remaining stress elements can be completed using the symmetry of shear stresses

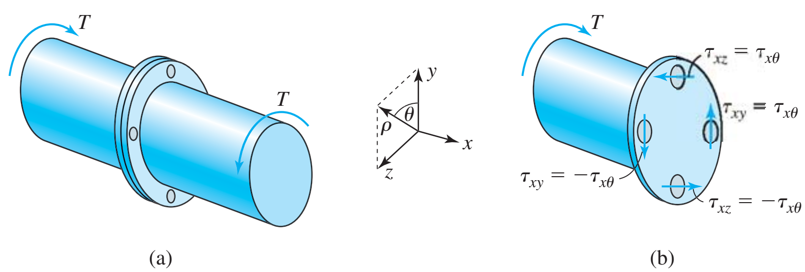


Figure 5.23 Torsional shear stresses.

After having obtained the torsional shear stress, either by using subscripts or by inspection, we can examine the shear stresses in Cartesian coordinates and obtain the stress components with correct signs, as shown in Figure 5.23*b*. This process of obtaining stress components in Cartesian coordinates will be important when we consider stress and strain transformation equations in Chapters 8 and 9, where we will relate stresses and strains in different coordinate systems.

The shear strain can be obtained by dividing the shear stress by *G*, the shear modulus of elasticity.

Consolidate your knowledge

1. Identify five examples of circular shafts from your daily life.
2. With the book closed, derive Equations 5.10 and 5.12, listing all the assumptions as you go along.

EXAMPLE 5.5

The two shafts shown in Figure 5.24 are of the same material and have the same amount of material cross-sectional areas A. Show that the hollow shaft has a larger polar moment of inertia than the solid shaft.

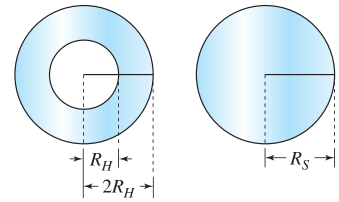


Figure 5.24 Hollow and solid shafts of Example 5.5.

PLAN

We can find the values of R_H and R_S in terms of the cross-sectional area A. We can then substitute these radii in the formulas for polar area moment to obtain the polar area moments in terms of A.

SOLUTION

We can calculate the radii R_H and R_S in terms of the cross sectional area A as

$$_H = \pi[(2R_H)^2 - R_H^2] = A \qquad \text{or} \qquad R_H^2 = \frac{A}{3\pi} \qquad \text{and} \qquad A_S = \pi R_S^2 = A \qquad \text{or} \qquad R_S^2 = \tag{E1}$$

The polar area moment of inertia for a hollow shaft with inside radius R_i and outside radius R_o can be obtained as

$$J = \int_A \rho^2\, dA = \int_{R_i}^{R_o} \rho^2 (2\pi\rho)\, d\rho = \frac{\pi}{2}\rho^4\Big|_{R_i}^{R_o} = \frac{\pi}{2}(R_o^4 - R_i^4) \tag{E2}$$

For the hollow shaft $R_o = 2R_H$ and $R_i = R_H$, whereas for the solid shaft $R_o = R_S$ and $R_i = 0$. Substituting these values into Equation (E2), we obtain the two polar area moments.

$$_H = \frac{\pi}{2}[(2R_H)^4 - R_H^4] = \frac{15}{2}\pi R_H^4 = \frac{15}{2}\pi\left(\frac{A}{3\pi}\right)^2 = \frac{5A^2}{6\pi} \qquad \text{and} \qquad J_S = \frac{\pi}{2}R_S^4 = \frac{\pi}{2}\left(\frac{A}{\pi}\right)^2 = \frac{A}{2} \tag{E3}$$

Dividing J_H by J_S we obtain

$$\frac{J_H}{J_S} = \frac{5}{3} = 1.67 \tag{E4}$$

ANS. As $J_H > J_S$ the polar moment for the hollow shaft is greater than that of the solid shaft for the same amount of material.

COMMENT

1. The hollow shaft has a polar moment of inertia of 1.67 times that of the solid shaft for the same amount of material. Alternatively, a hollow shaft will require less material (lighter in weight) to obtain the same polar moment of inertia. This reduction in weight is the primary reason why metal shafts are made hollow. Wooden shafts, however, are usually solid as the machining cost does not justify the small saving in weight.

EXAMPLE 5.6

A solid circular steel shaft (G_s = 12,000 ksi) of variable diameter is acted upon by torques as shown in Figure 5.25. The diameter of the shaft between wheels A and B and wheels C and D is 2 in., and the diameter of the shaft between wheels B and C is 4 in. Determine: (a) the rotation of wheel D with respect to wheel A; (b) the magnitude of maximum torsional shear stress in the shaft; (c) the shear stress at point E. Show it on a stress cube.

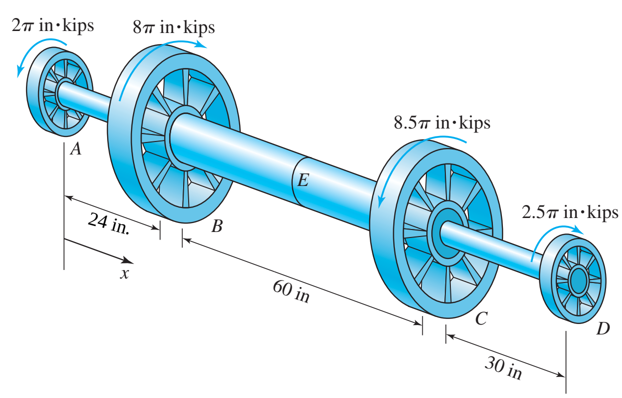


Figure 5.25 Geometry of shaft and loading in Example 5.6.

PLAN

By making imaginary cuts in sections *AB*, *BC*, and *CD* and drawing the free-body diagrams we can find the internal torques in each section. (a) We find the relative rotation in each section using Equation (5.12). Summing the relative rotations we can obtain $\phi_D - \phi_A$. (b) We find the maximum shear stress in each section using Equation (5.10), then by comparison find the maximum shear stress τ_{max} in the shaft. (c) In part (b) we found the shear stress in section *BC*. We obtain the direction of the shear stress either using the subscript or intuitively.

SOLUTION

The polar moment of inertias for each segment can be obtained as

$$J_{AB} = J_{CD} = \frac{\pi}{32}(2\text{ in.})^4 = \frac{\pi}{2}\text{ in.}^4 \qquad J_{BC} = \frac{\pi}{32}(4\text{ in.})^4 = 8\pi\text{ in.}^4 \tag{E1}$$

We make an imaginary cuts, draw internal torques as per our sign convention and obtain the free body diagrams as shown in Figure 5.25. We obtain the internal torques in each segment by equilibrium of moment about shaft axis:

$$\boldsymbol{T}_{AB} + 2\pi\text{ in.}\cdot\text{kips} = 0 \quad\text{or}\quad \boldsymbol{T}_{AB} = -2\pi\text{ in.}\cdot\text{kips} \tag{E2}$$

$$\boldsymbol{T}_{BC} + 2\pi\text{ in.}\cdot\text{kips} - 8\pi\text{ in.}\cdot\text{kips} = 0 \quad\text{or}\quad \boldsymbol{T}_{BC} = 6\pi\text{ in.}\cdot\text{kips} \tag{E3}$$

$$\boldsymbol{T}_{CD} + 2.5\pi\text{ in.}\cdot\text{kips} = 0 \quad\text{or}\quad \boldsymbol{T}_{CD} = -2.5\pi\text{ in.}\cdot\text{kips} \tag{E4}$$

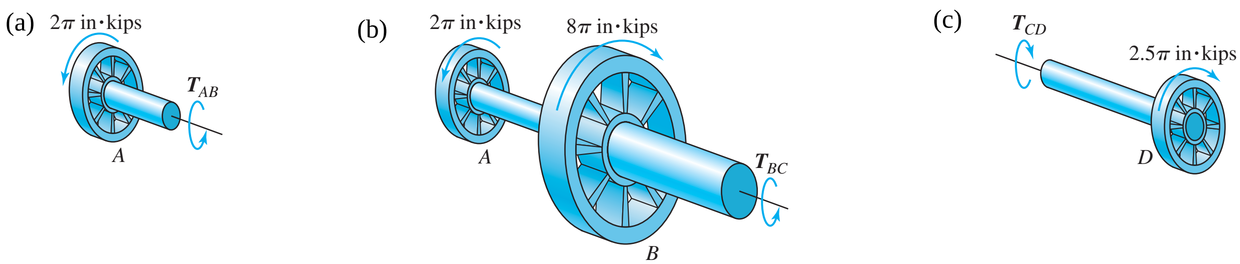


Figure 5.26 Free-body diagrams in Example 5.6 after an imaginary cut in segment (a) *AB*, (b) *BC*, and (c) *CD*.

(a) From Equation (5.12), we obtain the relative rotations of the end of segments as

$$\phi_B - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} = \frac{(-2\pi\text{ in.}\cdot\text{kips})(24\text{ in.})}{(12{,}000\text{ ksi})(\pi/2\text{ in.}^4)} = -8(10^{-3})\text{ rad} \tag{E5}$$

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{(6\pi\text{ in.}\cdot\text{kips})(60\text{ in.})}{(12{,}000\text{ ksi})(8\pi\text{ in.}^4)} = 3.75(10^{-3})\text{ rad} \tag{E6}$$

$$\phi_D - \phi_C = \frac{\boldsymbol{T}_{CD}(x_D - x_C)}{G_{CD}J_{CD}} = \frac{(-2.5\pi\text{ in.}\cdot\text{kips})(30\text{ in.})}{(12{,}000\text{ ksi})(\pi/2\text{ in.}^4)} = -12.5(10^{-3})\text{ rad} \tag{E7}$$

Adding Equations (E5), (E6), and (E7), we obtain the relative rotation of the section at *D* with respect to the section at *A*:

$$\phi_D - \phi_A = (\phi_B - \phi_A) + (\phi_C - \phi_B) + (\phi_D - \phi_C) = (-8 + 3.75 - 12.5)(10^{-3})\text{ rad} = -16.75(10^{-3})\text{ rad} \tag{E8}$$

ANS. $\phi_D - \phi_A = 0.01675\text{ rad cw}$

(b) The maximum torsional shear stress in section *AB* and CD will exist at $\rho = 1$ and in BC it will exist at $\rho = 2$. From Equation (5.10) we can obtain the maximum shear stress in each segment:

$$(\tau_{AB})_{max} = \frac{\boldsymbol{T}_{AB}(\rho_{AB})_{max}}{J_{AB}} = \frac{(-2\pi\text{ in.}\cdot\text{kips})(1\text{ in.})}{(\pi/2\text{ in.}^4)} = -4\text{ ksi} \tag{E9}$$

$$(\tau_{BC})_{max} = \frac{\boldsymbol{T}_{BC}(\rho_{BC})_{max}}{J_{BC}} = \frac{(6\pi\text{ in.}\cdot\text{kips})(2\text{ in.})}{(8\pi\text{ in.}^4)} = 1.5\text{ ksi} \tag{E10}$$

$$(\tau_{CD})_{max} = \frac{\boldsymbol{T}_{CD}(\rho_{CD})_{max}}{J_{CD}} = \frac{(-2.5\pi\text{ in.}\cdot\text{kips})(1\text{ in.})}{(\pi/2\text{ in.}^4)} = -5\text{ ksi} \tag{E11}$$

From Equations (E9), (E10), and (E11) we see that the magnitude of maximum torsional shear stress is in segment *CD*.

ANS. $|\tau_{max}| = 5\text{ ksi}$

(c) The direction of shear stress at point E can be determined as described below.

Shear stress direction using subscripts: In Figure 5.27*a* we note that $\tau_{x\theta}$ in segment *BC* is +1.5 ksi. The outward normal is in the positive x direction and the force has to be pointed in the positive θ direction (tangent direction), which at point *E* is downward.

Shear stress direction determined intuitively: Figure 5.27*b* shows a schematic of segment *BC*. Consider an imaginary section through *E* in segment *BC*. Segment *BE* tends to rotate clockwise with respect to segment *EC*. The shear stress will oppose the imaginary clockwise motion of segment *BE*; hence the direction will be counterclockwise, as shown.

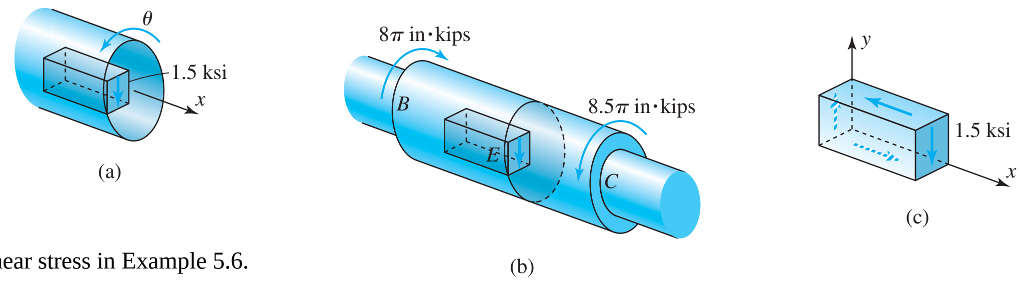


Figure 5.27 Direction of shear stress in Example 5.6.

We complete the rest of the stress cube using the fact that a pair of symmetric shear stresses points either toward the corner or away from the corner, as shown in Figure 5.27*c*.

COMMENTS

1. Suppose that we do not follow the sign convention for internal torque. Instead, we show the internal torque in a direction that counterbalances the external torque as shown in Figure 5.28. Then in the calculation of $\phi_D - \phi_A$ the addition and subtraction must be done manually to account for clockwise and counterclockwise rotation. Also, the shear stress direction must now be determined intuitively.

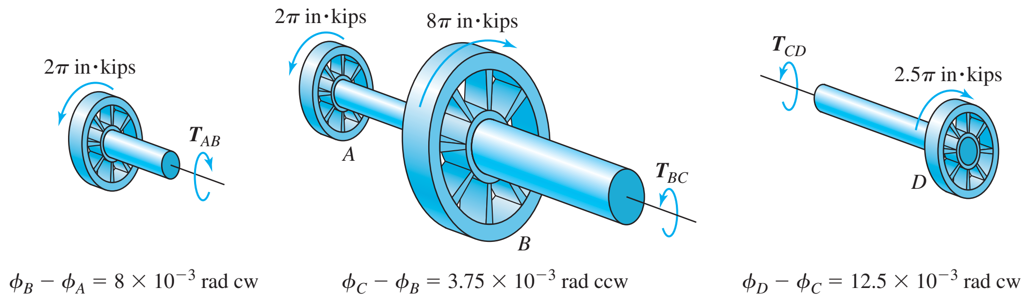


Figure 5.28 Intuitive analysis in Example 5.6.

2. An alternative perspective of the calculation of $\phi_D - \phi_A$ is as follows:

$$\phi_D - \phi_A = \int_{x_A}^{x_D} \frac{T}{GJ}\,dx = \int_{x_A}^{x_B} \frac{T_{AB}}{G_{AB}J_{AB}}\,dx + \int_{x_B}^{x_C} \frac{T_{BC}}{G_{BC}J_{BC}}\,dx + \int_{x_C}^{x_D} \frac{T_{CD}}{G_{CD}J_{CD}}\,dx$$

or, written compactly,

$$\Delta\phi = \sum_i \frac{T_i\,\Delta x_i}{G_i J_i} \tag{5.13}$$

3. Note that $T_{BC} - T_{AB} = 8\pi$ is the magnitude of the applied external torque at the section at *B*. Similarly $T_{CD} - T_{BC} = -8.5\pi$, which is the magnitude of the applied external torque at the section at *C*. In other words, the internal torques jump by the value of the external torque as one crosses the external torque from left to right. We will make use of this observation in the next section when plotting the torque diagram.

5.2.6 Torque Diagram

A **torque diagram** is a plot of the internal torque across the entire shaft. To construct torque diagrams we create a small torsion template to guide us in which direction the internal torque will jump. A **torsion template** is an infinitesimal segment of the shaft constructed by making imaginary cuts on either side of a supposed external torque.

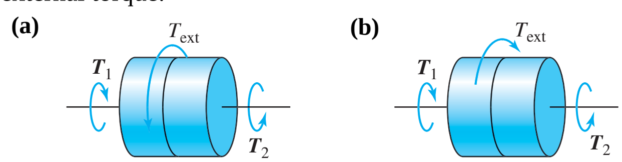


Figure 5.29 Torsion templates and equations.

Template Equations $T_2 = T_1 - T_{ext}$ $T_2 = T_1 + T_{ext}$

Figure 5.29 shows torsion templates. The external torque can be drawn either clockwise or counterclockwise. The ends of the torsion templates represent the imaginary cuts just to the left and just to the right of the applied external torque. The internal torques on these cuts are drawn according to the sign convention. An equilibrium equation is written, which we will call the **template equation**

If the external torque on the shaft is in the direction of the assumed torque shown on the template, then the value of $\boldsymbol{T}_2$ is calculated according to the template equation. If the external torque on the shaft is opposite to the direction shown, then $\boldsymbol{T}_2$ is calculated by changing the sign of T_{ext} in the template equation. Moving across the shaft using the template equation, we can then draw the torque diagram, as demonstrated in the next example.

EXAMPLE 5.7

Calculate the rotation of the section at *D* with respect to the section at *A* by drawing the torque diagram using the template shown in Figure 5.29.

PLAN

We can start the process by considering an imaginary extension on the left end. In the imaginary extension the internal torque is zero. Using the template in Figure 5.29*a* to guide us, we can draw the torque diagram.

SOLUTION

Let *LA* be an imaginary extension on the left side of the shaft, as shown in Figure 5.30. Clearly the internal torque in the imaginary section *LA* is zero, that is, $\boldsymbol{T}_1 = 0$. The torque at *A* is in the same direction as the torque T_{ext} shown on the template in Figure 5.29*a*. Using the template equation, we subtract the value of the applied torque to obtain a value of -2π in.·kips for the internal torque $\boldsymbol{T}_2$ just after wheel *A*. This is the starting value in the internal torque diagram.

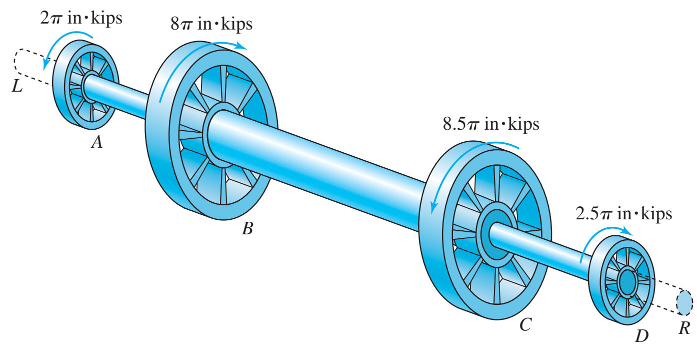


Figure 5.30 Imaginary extensions of the shaft in Example 5.7.

We approach wheel B with an internal torque value of -2π in.·kips, that is, $\boldsymbol{T}_1 = -2\pi$ in.·kips. The torque at B is in the opposite direction to the torque shown on the template in Figure 5.29*a* we add 8π in·kips to obtain a value of $+6\pi$ in.·kips for the internal torque just after wheel B.

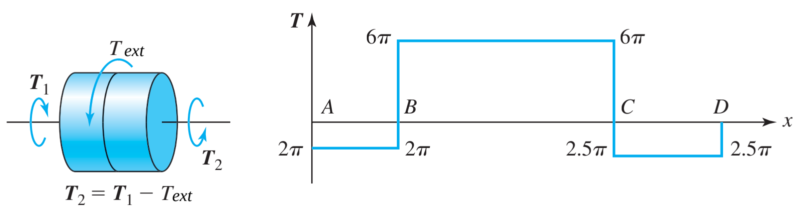


Figure 5.31 Torque diagram in Example 5.7.

We approach wheel *C* with a value of 6π in.·kips and note that the torque at *C* is in the same direction as that shown on the template in Figure 5.29*a*. Hence we subtract 8.5π in.·kips as per the template equation to obtain -2.5π in.·kips for the internal torque just after wheel *C*. The torque at *D* is in the same direction as that on the template, and on adding we obtain a zero value in the imaginary extended bar *DR* as expected, for the shaft is in equilibrium.

From Figure 5.31 the internal torque values in the segments are

$$\boldsymbol{T}_{AB} = -2\pi \text{ in.}\cdot\text{kips} \qquad \boldsymbol{T}_{BC} = 6\pi \text{ in.}\cdot\text{kips} \qquad \boldsymbol{T}_{CD} = -2.5\pi \text{ in.}\cdot\text{kips} \tag{E1}$$

To obtain the relative rotation of wheel *D* with respect to wheel *A*, we substitute the torque values in Equation (E1) into Equation (5.13):

$$\phi_D - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} + \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} + \frac{\boldsymbol{T}_{CD}(x_D - x_C)}{G_{CD}J_{CD}} \quad \text{or}$$

$$\phi_D - \phi_A = \frac{(-2\pi \text{ in.}\cdot\text{kips})(24 \text{ in.})}{(12{,}000 \text{ ksi})(\pi/2 \text{ in.}^4)} + \frac{(6\pi \text{ in.}\cdot\text{kips})(60 \text{ in.})}{(12{,}000 \text{ ksi})(8\pi \text{ in.}^4)} + \frac{(-2.5\pi \text{ in.}\cdot\text{kips})(30 \text{ in.})}{(12{,}000 \text{ ksi})(\pi/2 \text{ in.}^4)} = 16.75(10^{-3}) \text{ rad} \tag{E2}$$

ANS. $\phi_D - \phi_A = 0.01675$ rad cw

COMMENT

1. We could have created the torque diagram using the template shown in Figure 5.29*b* and the template equation. It may be verified that we obtain the same torque diagram. This shows that the direction of the applied torque T_{ext} on the template is immaterial.

EXAMPLE 5.8

A 1-m-long hollow shaft in Figure 5.32 is to transmit a torque of 400 N·m. The shaft can be made of either titanium alloy or aluminum. The shear modulus of rigidity G, the allowable shear stress τ_{allow}, and the density γ are given in Table 5.1. The outer diameter of the shaft must be 25 mm to fit existing attachments. The relative rotation of the two ends of the shaft is limited to 0.375 rad. Determine the inner radius to the nearest millimeter of the lightest shaft that can be used for transmitting the torque.

TABLE 5.1 **Material properties in Example 5.8**

Material	G (GPa)	τ_{allow} (MPa)	γ (Mg/m^3)
Titanium alloy	36	450	4.4
Aluminum	28	150	2.8

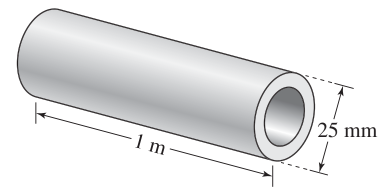


Figure 5.32 Shaft in Example 5.8.

PLAN

The change in inner radius affects only the polar moment J and no other quantity in Equations 5.10 and 5.12. For each material we can find the minimum polar moment J needed to satisfy the stiffness and strength requirements. Knowing the minimum J for each material we can find the maximum inner radius. We can then find the volume and hence the mass of each material and make our decision on the lighter shaft.

SOLUTION

We note that for both materials $\rho_{max} = 0.0125$ m and $x_2 - x_1 = 1$ m. From Equations 5.10 and 5.12 for titanium alloy we obtain limits on J_{Ti} shown below.

$$(\Delta\phi)_{Ti} = \frac{(400 \text{ N}\cdot\text{m})(1 \text{ m})}{[36(10^9) \text{ N/m}^2]J_{Ti}} \le 0.375 \text{ rad} \qquad \text{or} \qquad J_{Ti} \ge 29.63(10^{-9}) \text{ m}^4 \tag{E1}$$

$$(\tau_{max})_{Ti} = \frac{(400 \text{ N}\cdot\text{m})(0.0125 \text{ m})}{J_{Ti}} \le 450(10^6) \text{ N/m}^2 \qquad \text{or} \qquad J_{Ti} \ge 11.11(10^{-9}) \; m^4 \tag{E2}$$

Using similar calculations for the aluminum shaft we obtain the limits on J_{Al}:

$$(\Delta\phi)_{Al} = \frac{(400 \text{ N}\cdot\text{m})(1 \text{ m})}{[28(10^9) \text{ N/m}^2]\times J_{Al}} \le 0.375 \text{ rad} \qquad \text{or} \qquad J_{Al} \ge 38.10(10^{-9}) \text{ m}^4 \tag{E3}$$

$$(\tau_{max})_{Al} = \frac{(400 \text{ N}\cdot\text{m})(0.0125 \text{ m})}{J_{Al}} \le 150(10^6) \text{ N/m}^2 \qquad \text{or} \qquad J_{Al} \ge 33.33(10^{-9}) \text{ m}^4 \tag{E4}$$

Thus if $J_{Ti} \ge 29.63(10^{-9}) \text{ m}^4$, it will meet both conditions in Equations (E1) and (E2). Similarly if $J_{Al} \ge 38.10(10^{-9}) \text{ m}^4$, it will meet both conditions in Equations (E3) and (E4). The internal diameters D_{Ti} and D_{Al} can be found as follows:

$$J_{Ti} = \frac{\pi}{32}(0.025^4 - D_{Ti}^4) \ge 29.63(10^{-9}) \qquad D_{Ti} \le 17.3(10^{-3}) \text{ m} \tag{E5}$$

$$J_{Al} = \frac{\pi}{32}(0.025^4 - D_{Al}^4) \ge 38.10(10^{-9}) \qquad D_{Al} \le 7.1(10^{-3}) \text{ m} \tag{E6}$$

Rounding downward to the closest millimeter, we obtain

$$D_{Ti} = 17(10^{-3}) \text{ m} \qquad D_{Al} = 7(10^{-3}) \text{ m} \tag{E7}$$

We can find the mass of each material from the material density as

$$M_{Ti} = [4.4(10^6) \text{ g/m}^3]\left[\frac{\pi}{4}(0.025^2 - 0.017^2) \text{ m}^2\right](1 \text{ m}) = 1161 \; g \tag{E8}$$

$$M_{Al} = [2.8(10^6) \text{ g/m}^3]\left[\frac{\pi}{4}(0.025^2 - 0.007^2) \text{ m}^2\right](1 \text{ m}) = 1267 \; g \tag{E9}$$

From Equations (E8) and (E9) we see that the titanium alloy shaft is lighter.

ANS. A titanium alloy shaft should be used with an inside diameter of 17 mm.

COMMENTS

1. For both materials the stiffness limitation dictated the calculation of the internal diameter, as can be seen from Equations (E1) and (E3).
2. Even though the density of aluminum is lower than that titanium alloy, the mass of titanium is less. Because of the higher modulus of rigidity of titanium alloy we can meet the stiffness requirement using less material than for aluminum.

3. If in Equation (E5) we had $17.95(10^{-3})$ m on the right side, our answer for D_{Ti} would still be 17 mm because we have to round downward to ensure meeting the less-than sign requirement in Equation (E5).

EXAMPLE 5.9

The radius of a tapered circular shaft varies from $4r$ units to r units over a length of $40r$ units, as shown in Figure 5.33. The radius of the uniform shaft shown is r units. Determine (a) the angle of twist of wheel C with respect to the fixed end in terms of T, r, and G; (b) the maximum shear stress in the shaft.

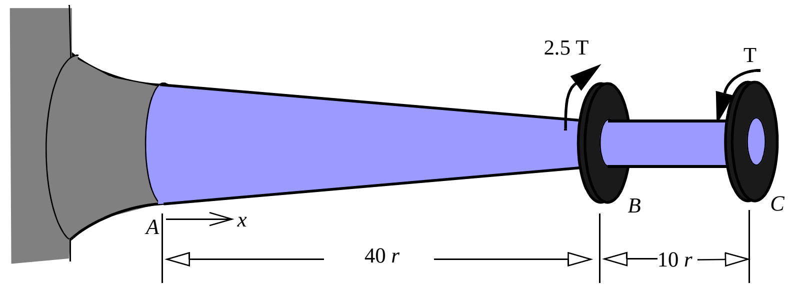


Figure 5.33 Shaft geometry in Example 5.9

PLAN

(a) We can find the relative rotation of wheel C with respect to wheel B using Equation (5.12). For section AB we obtain the polar moment J as a function of x and integrate Equation (5.9) to obtain the relative rotation of B with respect to A. We add the two relative rotations and obtain the relative rotation of C with respect to A. (b) As per Equation (5.10), the maximum shear stress will exist where the shaft radius is minimum (J is minimum) and T is maximum. Thus by inspection, the maximum shear stress will exist on a section just left of B.

SOLUTION

We note that R is a linear function of x and can be written as $R(x) = a + bx$. Noting that at $x = 0$ the radius $R = 4r$ we obtain $a = 4r$. Noting that $x = 40r$ the radius $R = r$ we obtain $b = -3r/(40r) = -0.075$. The radius R can be written as

$$R(x) = 4r - 0.075x \tag{E1}$$

Figure 5.34 shows the free body diagrams after imaginary cuts have been made and internal torques drawn as per our sign convention. By equilibrium of moment about the shaft axis we obtain the internal torques:

$$\boldsymbol{T}_{BC} = T \tag{E2}$$

$$\boldsymbol{T}_{AB} + 2.5T - T = 0 \qquad \text{or} \qquad \boldsymbol{T}_{AB} = -1.5T \tag{E3}$$

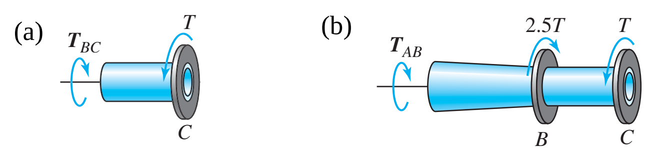


Figure 5.34 Free-body diagrams in Example 5.9 after imaginary cut in segment (a) BC (b) AB

The polar moment of inertias can be written as

$$J_{BC} = \frac{\pi}{2}r^4 \qquad J_{AB} = \frac{\pi}{2}R^4 = \frac{\pi}{2}(4r - 0.075x)^4 \tag{E4}$$

(a) We can find the relative rotation of the section at C with respect to the section at B using Equation (5.12):

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{T(10r)}{G(\pi/2)r^4} = \frac{6.366T}{Gr^3} \tag{E5}$$

Substituting Equations (E3) and (E4) into Equation (5.9) and integrating from point A to point B, we can find the relative rotation at the section at B with respect to the section at A:

$$\left(\frac{d\phi}{dx}\right)_{AB} = \frac{\boldsymbol{T}_{AB}}{G_{AB}J_{AB}} = \frac{-1.5T}{G(\pi/2)(4r - 0.075x)^4} \qquad \text{or} \qquad \int_{\phi_A}^{\phi_B} d\phi = \int_{x_A}^{x_B} -\frac{3T}{G\pi(4r - 0.075x)^4}\,dx \qquad \text{or}$$

$$\phi_B - \phi_A = -\frac{3T}{G\pi}\,\frac{1}{-3}\,\frac{1}{-0.075}\,\frac{1}{(4r - 0.075x)^3}\bigg|_0^{40r} = -\frac{T}{0.075G\pi}\left[\frac{1}{r^3} - \frac{1}{(4r)^3}\right] = -4.178\frac{T}{Gr^3} \tag{E6}$$

Adding Equations (E5) and (E6), we obtain

$$\phi_C - \phi_A = \frac{T}{Gr^3}(6.366 - 4.178) \tag{E7}$$

ANS. $\phi_C - \phi_A = 2.2\dfrac{T}{Gr^3}$ ccw

(b) Just left of the section at B we have $J_{AB} = \pi r^4/2$ and $\rho_{max} = r$. Substituting these values into Equation (5.10), we obtain the maximum torsional shear stress in the shaft as

$$\tau_{max} = \frac{-1.5Tr}{\pi r^4/2} = -\frac{0.955T}{r^3} \tag{E8}$$

ANS. $|\tau_{max}| = 0.955(T/r^3)$

Dimension check: The dimensional consistency[3] of the answer is checked as follows:

$$T \rightarrow O(FL) \qquad r \rightarrow O(L) \qquad G \rightarrow O\left(\frac{F}{L^2}\right) \qquad \phi \rightarrow O(\;) \qquad \frac{T}{Gr^3} \rightarrow O\left(\frac{FL}{\frac{F}{L^2}L^3}\right) \rightarrow O(\;) \rightarrow \text{checks}$$

$$\tau \rightarrow O\left(\frac{F}{L^2}\right) \qquad \frac{T}{r^3} \rightarrow O\left(\frac{FL}{L^3}\right) \rightarrow O\left(\frac{F}{L^2}\right) \rightarrow \text{checks}$$

COMMENT

1. The direction of the shear stress can be determined using subscripts or intuitively, as shown in Figure 5.35.

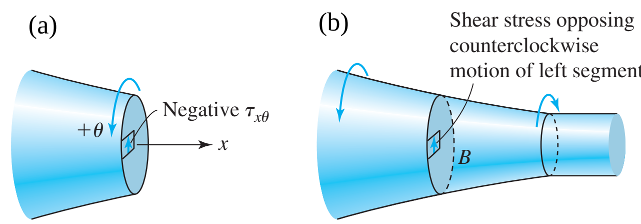


Figure 5.35 Direction of shear stress in Example 5.9: (a) by subscripts; (b) by inspection.

EXAMPLE 5.10

A uniformly distributed torque of q in.·lb/in. is applied to an entire shaft, as shown in Figure 5.36. In addition to the distributed torque a concentrated torque of $T = 3qL$ in.·lb is applied at section B. Let the shear modulus be G and the radius of the shaft r. In terms of q, L, G, and r, determine: (a) The rotation of the section at C. (b) The maximum shear stress in the shaft.

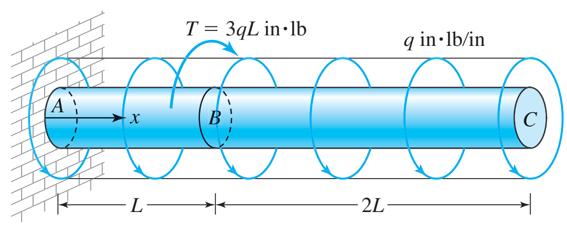


Figure 5.36 Shaft and loading in Example 5.10.

PLAN

(a) The internal torque in segments AB and BC as a function of x must be determined first. Then the relative rotation in each section is found by integrating Equation (5.9). (b) Since J and ρ_{max} are constant over the entire shaft, the maximum shear stress will exist on a section where the internal torque is maximum. By plotting the internal torque as a function of x we can determine its maximum value.

Figure 5.37 shows the free body diagrams after imaginary cuts are made at x distance from A and internal torques drawn as per our sign convention. We replace the distributed torque by an equivalent torque that is equal to the distributed torque intensity multiplied by the length of the cut shaft (the rectangular area). From equilibrium of moment about the shaft axis in Figure 5.37 we obtain the internal torques:

$$\boldsymbol{T}_{AB} + 3qL - q(3L - x) = 0 \quad \text{or} \quad \boldsymbol{T}_{AB} = -qx \tag{E1}$$

$$\boldsymbol{T}_{BC} - q(3L - x) = 0 \quad \text{or} \quad \boldsymbol{T}_{BC} = q(3L - x) \tag{E2}$$

[3] $O(\;)$ represents the dimension of the quantity on the left. F represents dimension for the force. L represents the dimension for length. Thus shear modulus, which has dimension of force (F) per unit area (L^2), is represented as $O(F/L^2)$.

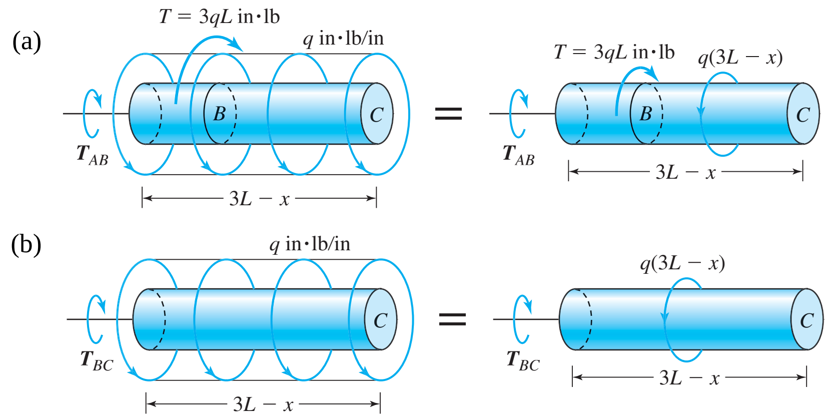


Figure 5.37 Free-body diagrams in Example 5.10 after imaginary cut in segment (a) *AB*, and (b) *BC*.

Integrating Equation (5.9) for each segment we obtain the relative rotations of segment ends as

$$\left(\frac{d\phi}{dx}\right)_{AB} = \frac{T_{AB}}{G_{AB}J_{AB}} = \frac{-qx}{G\pi r^4/2} \quad \text{or} \quad \int_{\phi_A}^{\phi_B} d\phi = -\int_{x_A=0}^{x_B=L} \frac{2qx}{G\pi r^4}\,dx \quad \text{or} \quad \phi_B - \phi_A = -\frac{qx^2}{G\pi r^4}\bigg|_0^L = -\frac{qL^2}{G\pi r^4} \tag{E3}$$

$$\left(\frac{d\phi}{dx}\right)_{BC} = \frac{T_{BC}}{G_{BC}J_{BC}} = \frac{q(3L-x)}{G\pi r^4/2} \quad \text{or} \quad \int_{\phi_B}^{\phi_C} d\phi = \int_{x_B=L}^{x_C=3L} \frac{2q(3L-x)}{G\pi r^4}\,dx \quad \text{or}$$

$$\phi_C - \phi_B = \frac{2q}{G\pi r^4}\left(3Lx - \frac{x^2}{2}\right)\bigg|_L^{3L} = \frac{2q}{G\pi r^4}\left[9L^2 - \frac{(3L)^2}{2} - 3L^2 + \frac{L^2}{2}\right] = \frac{4qL^2}{G\pi r^4} \tag{E4}$$

(a) Adding Equations (E3) and (E4), we obtain the rotation of the section at C with respect to the section at A:

$$\phi_C - \phi_A = -\frac{qL^2}{G\pi r^4} + \frac{4qL^2}{G\pi r^4} \tag{E5}$$

ANS. $\phi_C - \phi_A = = \left(\frac{3qL^2}{G\pi r^4}\right)\text{ccw}$

(b) Figure 5.38 shows the plot of the internal torque as a function of x using Equations (E1) and (E2). The maximum torque will occur on a section just to the right of B. From Equation (5.10) the maximum torsional shear stress is

$$\tau_{max} = \frac{T_{max}\rho_{max}}{J} = \frac{(2qL)(r)}{\pi r^4/2} \tag{E6}$$

ANS. $\tau_{max} = \frac{4qL}{\pi r^3}$

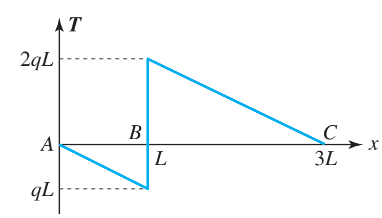


Figure 5.38 Torque diagram in Example 5.10.

Dimension check: The dimensional consistency (see footnote 12) of our answers is checked as follows:

$$q \rightarrow O\left(\frac{FL}{L}\right) \rightarrow O(F) \qquad r \rightarrow O(L) \qquad L \rightarrow O(L) \qquad G \rightarrow O\left(\frac{F}{L^2}\right)$$

$$\phi \rightarrow O(\,) \qquad \frac{qL^2}{Gr^4} \rightarrow O\left(\frac{FL^2}{(F/L^2)L^4}\right) \rightarrow O(\,) \rightarrow \text{checks} \qquad \tau \rightarrow O\left(\frac{F}{L^2}\right) \qquad \frac{qL}{r^3} \rightarrow O\left(\frac{FL}{L^3}\right) \rightarrow O\left(\frac{F}{L^2}\right) \rightarrow \text{checks}$$

COMMENT

1. A common mistake is to write the incorrect length of the shaft as a function of x in the free-body diagrams. It should be remembered that the location of the cut is defined by the variable x, which is measured from the common origin for all segments. Each cut produces two parts, and we are free to choose either part.

5.2.7* General Approach to Distributed Torque

Distributed torques are usually due to inertial forces or frictional forces acting on the surface of the shaft. The internal torque $\boldsymbol{T}$ becomes a function of x when a shaft is subjected to a distributed external torque, as seen in Example 5.10. If $t(x)$ is a simple function, then we can find $\boldsymbol{T}$ as a function of x by drawing a free-body diagram, as we did in Example 5.10. However, if the distributed torque $t(x)$ is a complex function (see Problems 5.39, 5.61, and 5.62), it may be easier to use the alternative solution method described in this section.

Consider an infinitesimal shaft element that is created by making two imaginary cuts at a distance dx from each other, as shown in Figure 5.39*a*.

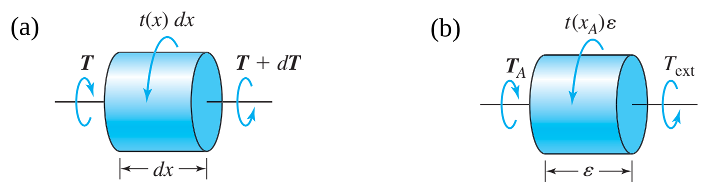


Figure 5.39 (a) Equilibrium of an infinitesimal shaft element. (b) Boundary condition on internal torque.

By equilibrium moments about the axis of the shaft, we obtain $(\boldsymbol{T}+d\boldsymbol{T})+t(x)\,dx-\boldsymbol{T} = 0$ or

$$\boxed{\frac{d\boldsymbol{T}}{dx}+t(x) = 0} \tag{5.14}$$

Equation (5.14) represents the equilibrium equation at any section x. It assumes that $t(x)$ is positive counterclockwise with respect to the x axis. The sign of $\boldsymbol{T}$ obtained from Equation (5.14) corresponds to the direction defined by the sign convention. If $t(x)$ is zero in a segment of a shaft, then the internal torque is constant in that segment.

Equation (5.14) can be integrated to obtain the internal torque $\boldsymbol{T}$. The integration constant can be found by knowing the value of the internal torque $\boldsymbol{T}$ at either end of the shaft. To obtain the value of $\boldsymbol{T}$ at the end of the shaft (say, point A), a free-body diagram is constructed after making an imaginary cut at an infinitesimal distance ε from the end, as shown in Figure 5.39*b*.We then write the equilibrium equation as

$$\lim_{\varepsilon \to 0}[T_{\text{ext}}-\boldsymbol{T}_A-t(x_A)\varepsilon] = 0 \qquad \text{or} \qquad \boldsymbol{T}_A = T_{\text{ext}} \tag{5.15}$$

Equation (5.15) shows that the distributed torque does not affect the boundary condition on the internal torque. The value of the internal torque $\boldsymbol{T}$ at the end of the shaft is equal to the concentrated external torque applied at the end. Equation (5.14) is a differential equation. Equation (5.15) is a boundary condition. A differential equation and all the conditions necessary to solve it is called the **boundary value problem.**

EXAMPLE 5.11

The external torque on a drill bit varies linearly to a maximum intensity of q in.·lb/in., as shown in Figure 5.40. If the drill bit diameter is d, its length L, and the modulus of rigidity G, determine the relative rotation of the end of the drill bit with respect to the chuck.

PLAN

The relative rotation of section B with respect to section A has to be found. We can substitute the given distributed torque in Equation (5.14) and integrate to find the internal torque as a function of x. We can find the integration constant by using the condition that at section B the internal torque will be zero. We can substitute the internal torque expression into Equation (5.9) and integrate from point A to point B to find the relative rotation of section B with respect to section A.

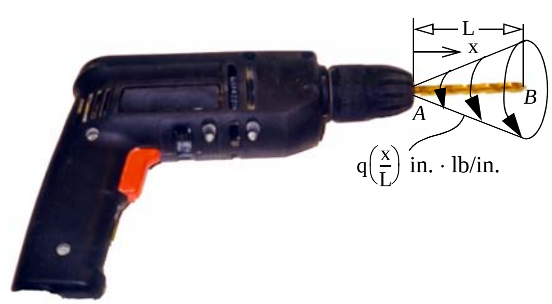


Figure 5.40 Distributed torque on a drill bit in Example 5.11.

SOLUTION

The distributed torque on the drill bit is counterclockwise with respect to the x axis. Thus we can substitute $t(x) = q(x/L)$ into Equation (5.14) to obtain the differential equation shown as Equation (E1). At point B, that is, at $x = L$, the internal torque should be zero as there is no concentrated applied torque at B.The boundary condition is shown as Equation (E2). The boundary value problem statement is

- Differential Equation

$$\frac{d\boldsymbol{T}}{dx} + q\frac{x}{L} = 0 \tag{E1}$$

- Boundary Condition

$$\boldsymbol{T}(x = L) = 0 \tag{E2}$$

Integrating Equation (E1) we obtain

$$\boldsymbol{T} = -q\frac{x^2}{2L} + c \tag{E3}$$

Substituting Equation (E2) into Equation (E3) we obtain the integration constant c as

$$-q\frac{L^2}{2L} + c = 0 \qquad \text{or} \qquad c = \frac{qL}{2} \tag{E4}$$

Substituting Equation (E4) into Equation (E3) we obtain internal torque as

$$T = \frac{q}{2L}(L^2 - x^2) \tag{E5}$$

Substituting Equation (E5) into Equation (5.9) and integrating we obtain the relative rotation of the section at B with respect to the section at A as

$$\frac{d\phi}{dx} = \frac{(q/2L)(L^2 - x^2)}{G\pi d^4/32} \qquad \text{or} \qquad \int_{\phi_A}^{\phi_B} d\phi = \frac{16q}{\pi G L d^4}\int_{x_A=0}^{x_B=L}(L^2 - x^2)\,dx \qquad \text{or} \qquad \phi_B - \phi_A = \frac{16q}{\pi G L d^4}\left(L^2 x - \frac{x^3}{3}\right)\Bigg|_0^L \tag{E6}$$

ANS. $\phi_B - \phi_A = \dfrac{32qL^2}{3\pi G d^4}$ ccw

Dimension check: The dimensional consistency (see footnote 12) of our answer is checked as follows:

$$t \rightarrow O\left(\frac{FL}{L}\right) \rightarrow O(F) \qquad d \rightarrow O(L) \qquad L \rightarrow O(L) \qquad G \rightarrow O\left(\frac{F}{L^2}\right) \qquad \phi \rightarrow O(\;) \qquad \frac{qL^2}{Gd^4} \rightarrow O\left(\frac{FL^2}{(F/L^2)L^4}\right) \rightarrow O(\;) \rightarrow \text{checks}$$

COMMENTS

1. No free-body diagram was needed to find the internal torque because Equation (5.14) is an equilibrium equation. It is therefore valid at each and every section of the shaft.
2. We could have obtained the internal torque by integrating Equation (5.14) from L to x as follows:

$$\int_{T_B=0}^{T} d\boldsymbol{T} = -\int_{x_B=L}^{x} t(x)\,dx = -\int_{L}^{x} q\left(\frac{x}{L}\right)dx = \frac{q}{2L}(L^2 - x^2)$$

3. The internal torque can also be found using a free-body diagram. We can make an imaginary cut at some location x and draw the free-body diagram of the right side. The distributed torque represented by $\int_x^L t(x)\,dx$ is the area of the trapezoid $BCDE$, and this observation can be used in drawing a statically equivalent diagram, as shown in Figure 5.41. Equilibrium then gives us the value of the internal torque as before. We can find the internal torque as shown.

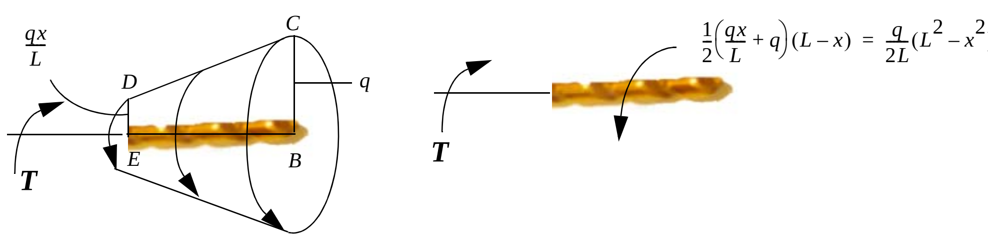


Figure 5.41 Internal torque by free-body diagram in Example 5.11.

4. The free-body diagram approach in Figure 5.41 is intuitive but more tedious and difficult than the use of Equation (5.14). As the function representing the distributed torque grows in complexity, the attractiveness of the mathematical approach of Equation (5.14) grows correspondingly.

MoM in Action: Drill, the Incredible Tool

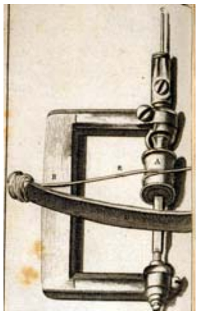
Drills have been in use for almost as long as humans have used tools. Early humans knew from experience that friction generated by torquing a wooden shaft could start a fire—a technique still taught in survivalist camps. Archeologists in Pakistan have found teeth perhaps 9000 years old showing the concentric marks of a flint stone drill. The Chinese used larger drills in the 3rd B.C.E. to extract water and oil from earth. The basic design—a chuck that delivers torque to the drill bit—has not changed, but their myriad uses to make holes from the very small to the very large continues to grow.

Early development of the drill was driven by the technology of delivering power to the drill bit. In 1728, French dentist Pierre Fauchard (Figure 5.42*a*) described how catgut twisted around a cylinder could power the rotary movement as a bow moved back and forth. However, hand drills like these operated at only about 15 rpm. George F. Harrington introduced the first motor-driven drill in 1864, powered by the spring action of a clock. George Green, an American dentist, introduced a pedal-operated pneumatic drill just four years later—and, in 1875, an electric drill. By 1914 dental drills could operate at 3000 rpm.

Other improvements took better understanding of the relationship between power, torsion, and shear stress in the drill bit (problems 5.45—5.47) and the material being drilled:

- The sharper the drill tip, the higher the shear stresses at the point, and the greater the amount of material that can be sheared. For most household jobs the angle of the drill tip is 118^{o}. For soft materials such as plastic, the angle is sharper, while for harder material such as steel the angle is shallower.
- For harder materials low speeds can prolong the life of drill bit. However, in dentistry higher speeds, of up to 500,000 rpm, reduce a patient’s pain.

(a)



(b)

Figure 5.42 (a) Pierre Fauchard drill. (b) Tunnel boring machine Matilda (Courtesy Erikt9).

- Slower speeds are also used to shear a large amount of material. Tunnel boring machines (TBM) shown in Figure 5.42*b* may operate at 1 to 10 rpm. The world’s largest TBM, with a diameter of 14.2 m, was used to drill the Elbe Tunnel in Hamburg, Germany. Eleven TBM’s drilled the three pipes of the English Channel, removing 10.5 million cubic yards of earth in seven years.
- Drill bits can be made of steel, tungsten carbide, polycrystalline diamonds, titanium nitrate, and diamond powder. The choice is dictated by the material to be drilled as well as the cost. Even household drills have different bits for wood, metal, or masonry.

Delivery and control of power to the drill bit are engineering challenges. So is removal of sheared material, not only to prevent the hole from plugging, but also because the material carries away heat, improving the strength and life of a drill bit. Yet the fundamental function of a drill remains: shearing through torsion.

PROBLEM SET 5.2

5.19 The torsional shear stress at point A on a solid circular homogenous cross-section was found to be τ_A = 120 MPa. Determine the maximum torsional shear stress on the cross-section.

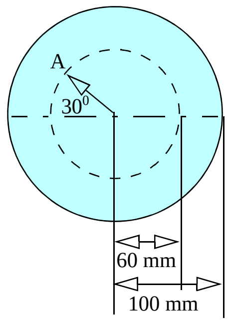


Figure P5.19

5.20 The torsional shear strain at point *A* on a homogenous circular section shown in Figure P5.20 was found to be 900 μ rads. Using a shear modulus of elasticity of 4000 ksi, determine the torsional shear stress at point *B*.

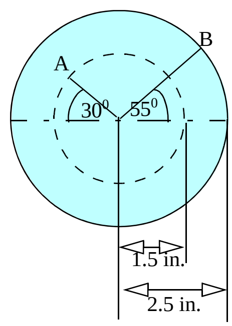
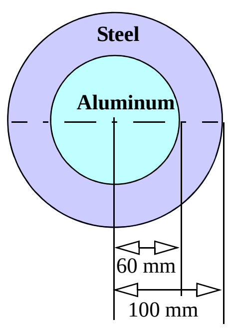


Figure P5.20

5.21 An aluminum shaft (G_{al}= 28 GPa) and a steel shaft (G_S=82 GPa) are securely fastened to form composite shaft with a cross section shown in Figure P5.21. If the maximum torsional shear strain in aluminum is 1500 μ rads, determine the maximum torsional shear strain in steel.



FigureP5.21

5.22 An aluminum shaft (G_{al}= 28 GPa) and a steel shaft (G_S=82 GPa) are securely fastened to form composite shaft with a cross section shown in Figure P5.21. If the maximum torsional shear stress in aluminum is 21 MPa, determine the maximum torsional shear stress in steel.

5.23 Determine the direction of torsional shear stress at points *A* and *B* in Figure P5.23 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

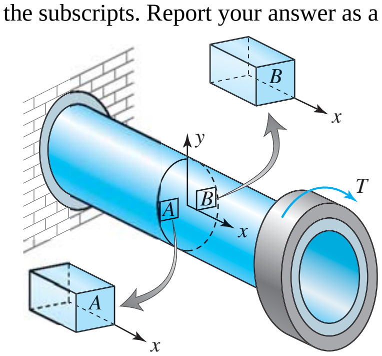


Figure P5.23

5.24 Determine the direction of torsional shear stress at points *A* and *B* in Figure P5.24 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}

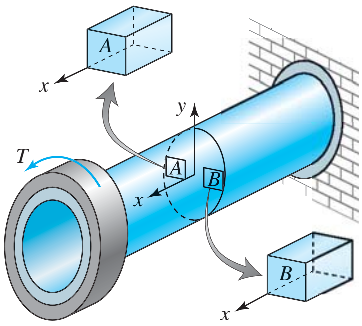


Figure P5.24

5.25 Determine the direction of torsional shear stress at points *A* and *B* in Figure P5.25 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

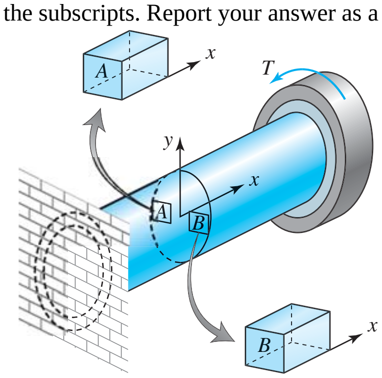


Figure P5.25

5.26 Determine the direction of torsional shear stress at points *A* and *B* in Figure P5.26 (a) by inspection; (b) by using the sign convention for internal torque and the subscripts. Report your answer as a positive or negative τ_{xy}.

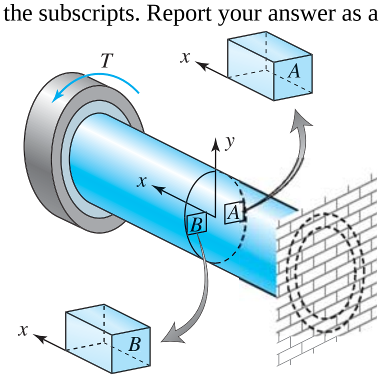
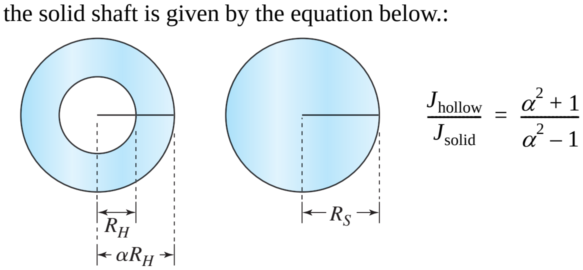


Figure P5.26

5.27 The two shafts shown in Figure P5.27 have the same cross sectional areas *A*. Show that the ratio of the polar moment of inertia of the hollow shaft to that of the solid shaft is given by the equation below.:



$$\frac{J_{\text{hollow}}}{J_{\text{solid}}} = \frac{\alpha^2 + 1}{\alpha^2 - 1}$$

Figure P5.27

5.28 Show that for a thin tube of thickness *t* and center-line radius *R* the polar moment of inertia can be approximated by $J = 2\pi R^3 t$. By thin tube we imply $t < R/10$.

5.29 (a) Draw the torque diagram in Figure P5.29. (b) Check the values of internal torque by making imaginary cuts and drawing free-body diagrams. (c) Determine the rotation of the rigid wheel *D* with respect to the rigid wheel *A* if the torsional rigidity of the shaft is 90,000 $\text{kips}\cdot\text{in.}^2$.

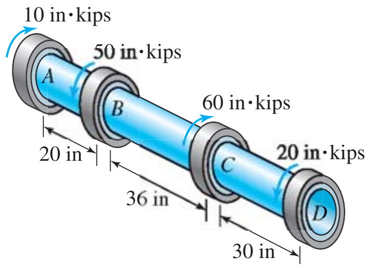


Figure P5.29

5.30 (a) Draw the torque diagram in Figure P5.30. (b) Check the values of internal torque by making imaginary cuts and drawing free-body diagrams. (c) Determine the rotation of the rigid wheel *D* with respect to the rigid wheel *A* if the torsional rigidity of the shaft is 1270 kN·m^2.

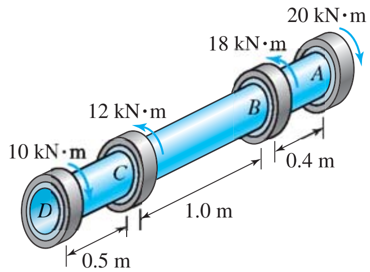


Figure P5.30

5.31 The shaft in Figure P5.31 is made of steel (G = 80 GPa) and has a diameter of 150 mm. Determine (a) the rotation of the rigid wheel *D*; (b) the magnitude of the torsional shear stress at point *E* and show it on a stress cube (Point *E* is on the top surface of the shaft.); (c) the magnitude of maximum torsional shear *strain* in the shaft.

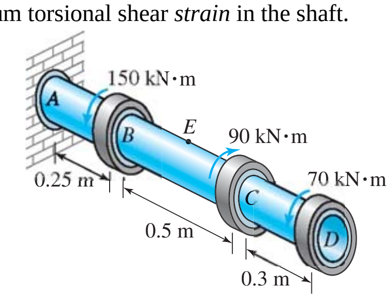


Figure P5.31

5.32 The shaft in Figure P5.32 is made of aluminum (G = 4000 ksi) and has a diameter of 4 in. Determine (a) the rotation of the rigid wheel *D*; (b) the magnitude of the torsional shear stress at point *E* and show it on a stress cube (Point *E* is on the bottom surface of the shaft.); (c) the magnitude of maximum torsional shear *strain* in the shaft.

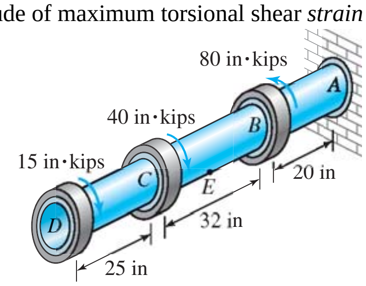


Figure P5.32

5.33 Two circular steel shafts (G =12,000 ksi) of diameter 2 in. are securely connected to an aluminum shaft (G =4,000 ksi) of diameter 1.5 in. as shown in Figure P5.33. Determine (a) the rotation of section at D with respect to the wall, and (b) the maximum shear stress in the shaft.

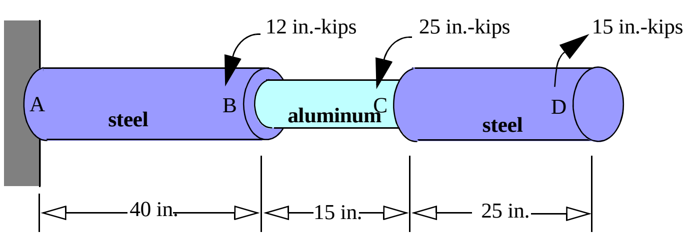


Figure P5.33

5.34 A solid circular steel shaft *BC* ($G_s = 12{,}000$ ksi) is securely attached to two hollow steel shafts *AB* and *CD*, as shown in Figure P5.34. Determine (a) the angle of rotation of the section at *D* with respect to the section at *A*; (b) the magnitude of maximum torsional shear stress in the shaft; (c) the torsional shear stress at point *E* and show it on a stress cube. (Point *E* is on the inside bottom surface of *CD*.)

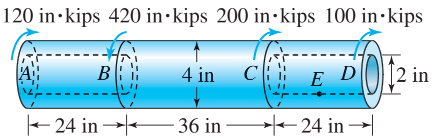


Figure P5.34

5.35 A steel shaft ($G = 80$ GPa) is subjected to the torques shown in Figure P5.35. Determine (a) the rotation of section *A* with respect to the no-load position; (b) the torsional shear stress at point *E* and show it on a stress cube. (Point *E* is on the surface of the shaft.)

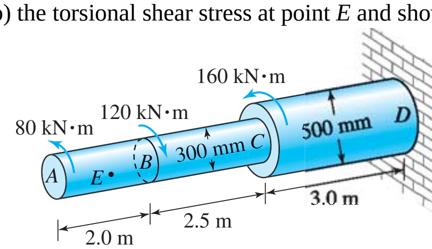


Figure P5.35

Tapered shafts

5.36 The radius of the tapered circular shaft shown in Figure P5.36 varies from 200 mm at *A* to 50 mm at *B*. The shaft between *B* and *C* has a constant radius of 50 mm. The shear modulus of the material is $G = 40$ GPa. Determine (a) the angle of rotation of wheel *C* with respect to the fixed end; (b) the maximum shear strain in the shaft

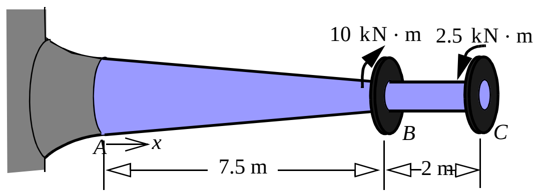


Figure P5.36

5.37 The radius of the tapered shaft in Figure P5.37 varies as $R = Ke^{-ax}$. Determine the rotation of the section at *B* in terms of the applied torque T_{ext}, length *L*, shear modulus of elasticity *G*, and geometric parameters *K* and *a*.

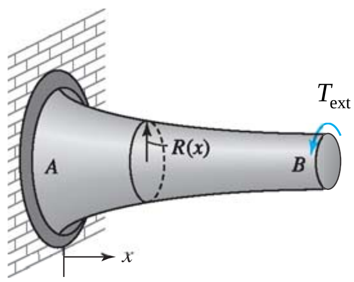


Figure P5.37

5.38 The radius of the tapered shaft shown in Figure P5.37 varies as $R = r\sqrt{(2 - 0.25x/L)}$. In terms of T_{ext}, *L*, *G*, and *r*, determine (a) the rotation of the section at *B*; (b) the magnitude of maximum torsional shear stress in the shaft.

Distributed torques

5.39 The external torque on a drill bit varies as a quadratic function to a maximum intensity of *q* in.·lb/in., as shown in Figure P5.39. If the drill bit diameter is *d*, its length *L*, and its modulus of rigidity *G*, determine (a) the maximum torsional shear stress on the drill bit; (b) the relative rotation of the end of the drill bit with respect to the chuck.

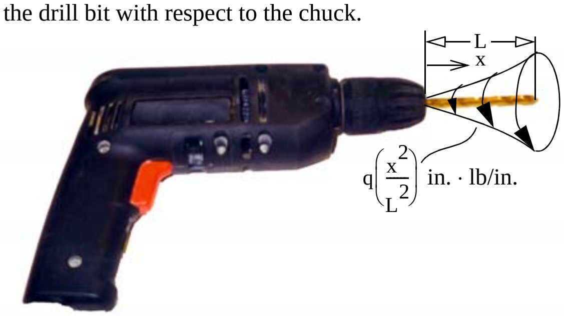


Figure P5.39

5.40 A circular solid shaft is acted upon by torques, as shown in Figure P5.40. Determine the rotation of the rigid wheel *A* with respect to the fixed end *C* in terms of *q*, *L*, *G*, and *J*.

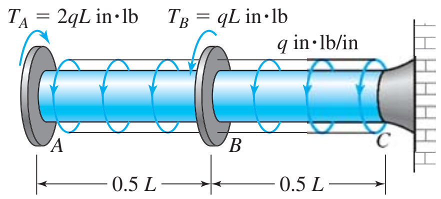


Figure P5.40

Design problems

5.41 A thin steel tube ($G = 12{,}000$ ksi) of $\frac{1}{8}$-in. thickness has a mean diameter of 6 in. and a length of 36 in. What is the maximum torque the tube can transmit if the allowable torsional shear stress is 10 ksi and the allowable relative rotation of the two ends is 0.015 rad?

5.42 Determine the maximum torque that can be applied on a 2-in. diameter solid aluminum shaft ($G = 4000$ ksi) if the allowable torsional shear stress is 18 ksi and the relative rotation over 4 ft of the shaft is to be limited to 0.2 rad.

5.43 A hollow steel shaft ($G = 80$ GPa) with an outside radius of 30 mm is to transmit a torque of 2700 N·m. The allowable torsional shear stress is 120 MPa and the allowable relative rotation over 1 m is 0.1 rad. Determine the maximum permissible inner radius to the nearest millimeter.

5.44 A 5-ft-long hollow shaft is to transmit a torque of 200 in.·kips. The outer diameter of the shaft must be 6 in. to fit existing attachments. The relative rotation of the two ends of the shaft is limited to 0.05 rad. The shaft can be made of steel or aluminum. The shear modulus of elasticity *G*, the allowable shear stress τ_{allow}, and the specific weight γ are given in Table P5.44. Determine the maximum inner diameter to the nearest $\frac{1}{8}$ in. of the lightest shaft that can be used for transmitting the torque and the corresponding weight.

TABLE P5.44

Material	G (ksi)	τ_{allow} (ksi)	γ (lb/in.^{3})
Steel	12,000	18	0.285
Aluminum	4,000	10	0.100

Transmission of power

Power P is the rate at which work dW/dt *is done; and work W done by a constant torque is equal to the product of torque T and angle of rotation* ϕ*. Noting that* $\omega = d\phi/dt$*, we obtain*

$$P = T\omega = 2\pi f T \tag{5.16}$$

where T is the torque transmitted, ω is the rotational speed in radians per second, and f is the frequency of rotation in hertz (Hz). Power is reported in units of horsepower in U.S. customary units or in watts. 1 horsepower (hp) is equal to 550 ft·lb/s = 6600 in·lb/s and 1 watt (W) is equal to 1 N·m/s. Use Equation (5.16) to solve Problems 5.38 through 5.40.

5.45 A 100-hp motor is driving a pulley and belt system, as shown in Figure P5.45. If the system is to operate at 3600 rpm, determine the minimum diameter of the solid shaft *AB* to the nearest $\frac{1}{8}$ in. if the allowable stress in the shaft is 10 ksi.

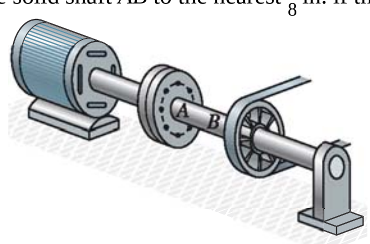


Figure P5.45

5.46 The bolts used in the coupling for transferring power in Problem 5.45 have an allowable strength of 12 ksi. Determine the minimum number (> 4) of $\frac{1}{4}$-in. diameter bolts that must be placed at a radius of $\frac{5}{8}$ in.

5.47 A 20-kW motor drives three gears, which are rotating at a frequency of 20 Hz. Gear *A* next to the motor transfers 8 kW of power. Gear *B*, which is in the middle, transfers 7 kW of power. Gear *C*, which is at the far end from the motor, transfers the remaining 5 kW of power. A single solid steel shaft connecting the motors to all three gears is to be used. The steel used has a yield strength in shear of 145

MPa. Assuming a factor of safety of 1.5, what is the minimum diameter of the shaft to the nearest millimeter that can be used if failure due to yielding is to be avoided? What is the magnitude of maximum torsional stress in the segment between gears *A* and *B*?

Stretch yourself

5.48 A circular shaft has a constant torsional rigidity *GJ* and is acted upon by a distributed torque $t(x)$. If at section *A* the internal torque is zero, show that the relative rotation of the section at *B* with respect to the rotation of the section at *A* is given by

$$\phi_B - \phi_A = \frac{1}{GJ}\left[\int_{x_A}^{x_B}(x - x_B)t(x)\,dx\right] \tag{5.17}$$

5.49 A composite shaft made from *n* materials is shown in Figure P5.49. G_i and J_i are the shear modulus of elasticity and polar moment of inertia of the i[th] material. (a) If Assumptions from 1 through 6 are valid, show that the stress $(\tau_{x\theta})_i$ in the i[th] material is given Equation (5.18a), where $\boldsymbol{T}$ is the total internal torque at a cross section. (b) If Assumptions 8 through 10 are valid, show that relative rotation $\phi_2 - \phi_1$ is given by Equation (5.18b). (c) Show that for $G_1=G_2=G_3....=G_n=G$ Equations (5.18a) and (5.18b) give the same results as Equations (5.10) and (5.12).

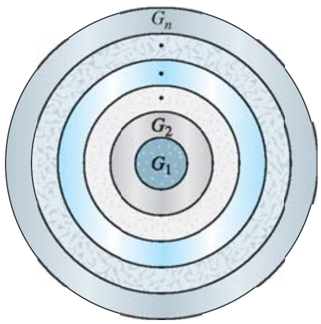


Figure P5.49

$$(\tau_{x\theta})_i = \frac{G_i\rho\boldsymbol{T}}{\sum_{j=1}^{n} G_jJ_j} \tag{5.18a}$$

$$\phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{\sum_{j=1}^{n} G_jJ_j} \tag{5.18b}$$

5.50 A circular solid shaft of radius *R* is made from a nonlinear material that has a shear stress-shear strain relationship given by $\tau = G\gamma^{0.5}$. Assume that the kinematic assumptions are valid and shear strain varies linearly with the radial distance across the cross-section. Determine the maximum shear stress and the rotation of section at B in terms of external torque T_{ext}, radius *R*, material constant *G*, and length *L*.

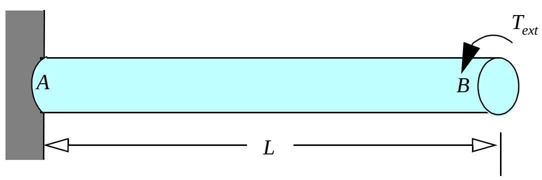


Figure P5.50

5.51 A hollow circular shaft is made from a non-linear materials that has the following shear stress--shear strain relation $\tau = G\gamma^2$. Assume that the kinematic assumptions are valid and shear strain varies linearly with the radial distance across the cross-section. In terms of internal torque $\boldsymbol{T}$, material constant *G*, and *R*, obtain formulas for (a) the maximum shear stress τ_{max} and (b) the relative rotation $\phi_2 - \phi_1$ of two cross-sections at x_1 and x_2.

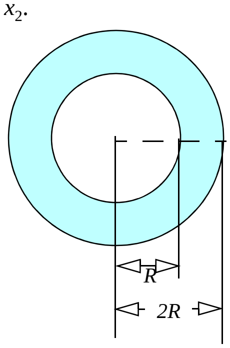


Figure P5.51

5.52 A solid circular shaft of radius *R* and length *L* is twisted by an applied torque *T*. The stress–strain relationship for a nonlinear material is given by the power law $\tau = G\gamma^n$. If Assumptions 1 through 4 are applicable, show that the maximum shear stress in the shaft and the relative rotation of the two ends are as follows:

$$\tau_{max} = \frac{\boldsymbol{T}(n+3)}{2\pi R^3} \qquad \Delta\phi = \left[\frac{(n+3)\boldsymbol{T}}{2\pi G R^{3+n}}\right]^{1/n} L$$

Substitute $n = 1$ in the formulas and show that we obtain the same results as from Equations 5.10 and 5.12.

5.53 The internal torque $\boldsymbol{T}$ and the displacements of a point on a cross section of a noncircular shaft shown in Figure P5.53 are given by the equations below

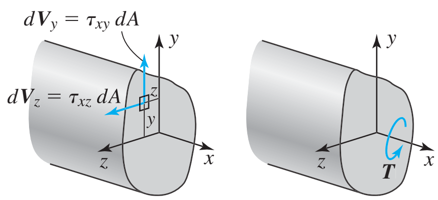


Figure P5.53 Torsion of noncircular shafts.

$$u = \psi(y,z)\frac{d\phi}{dx} \qquad (5.19a)$$

$$v = -xz\frac{d\phi}{dx} \qquad (5.19b)$$

$$w = xy\frac{d\phi}{dx} \qquad (5.19c)$$

$$\boldsymbol{T} = \int_A (y\tau_{xz} - z\tau_{xy})\, dA \qquad (5.20)$$

where u, v, and w are the displacements in the x, y, and z directions, respectively; $d\phi/dx$ is the rate of twist and is considered constant. $\psi(x, y)$ is called the *warping function*[4] and describes the movement of points out of the plane of cross section. Using Equations (2.12d) and (2.12f) and Hooke's law, show that the shear stresses for a noncircular shaft are given by

$$\tau_{xy} = G\left(\frac{\partial\psi}{\partial y} - z\right)\frac{d\phi}{dx} \qquad \tau_{xz} = G\left(\frac{\partial\psi}{\partial z} + y\right)\frac{d\phi}{dx} \qquad (5.21)$$

5.54 Show that for circular shafts, $\psi(x, y) = 0$, the equations in Problem 5.53 reduce to Equation (5.9).

5.55 Consider the dynamic equilibrium of the differential element shown in Figure P5.55, where $\boldsymbol{T}$ is the internal torque, γ is the material density, J is the polar area moment of inertia, and $\partial^2\phi/\partial t^2$ is the angular acceleration. Show that dynamic equilibrium results in Equation (5.22)

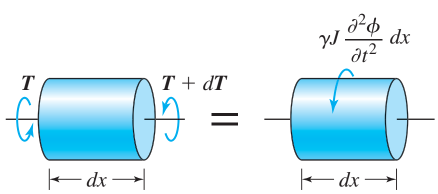


Figure P5.55 Dynamic equilibrium.

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} \quad \text{where } c = \sqrt{\frac{G}{\gamma}} \qquad (5.22)$$

5.56 Show by substitution that the solution of Equation (5.23) satisfies Equation (5.22):

$$\phi = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right) \times (C\cos\omega t + D\sin\omega t) \qquad (5.23)$$

where A, B, C, and D are constants that are determined from the boundary conditions and the initial conditions and ω is the frequency of vibration.

Computer problems

5.57 A hollow aluminum shaft of 5 ft in length is to carry a torque of 200 in.·kips. The inner radius of the shaft is 1 in. If the maximum torsional shear stress in the shaft is to be limited to 10 ksi, determine the minimum outer radius to the nearest $\frac{1}{8}$ in.

5.58 A 4-ft-long hollow shaft is to transmit a torque of 100 in.·kips. The relative rotation of the two ends of the shaft is limited to 0.06 rad. The shaft can be made of steel or aluminum. The shear modulus of rigidity G, the allowable shear stress τ_{allow}, and the specific weight γ are given in Table P5.58. The inner radius of the shaft is 1 in. Determine the outer radius of the lightest shaft that can be used for transmitting the torque and the corresponding weight.

TABLE P5.58

Material	G (ksi)	τ_{allow} (ksi)	γ (lb/in.^{3})
Steel	12,000	18	0.285
Aluminum	4000	10	0.100

5.59 Table P5.59 shows the measured radii of the solid tapered shaft shown in Figure P5.59, at several points along the axis of the shaft. The shaft is made of aluminum (G = 28 GPa) and has a length of 1.5 m. Determine: (a) the rotation of the free end with respect to the wall using numerical integration; (b) the maximum shear stress in the shaft.

[4]Equations of elasticity show that the warping function satisfies the Laplace equation, $\partial^2\psi/\partial y^2 + \partial^2\psi/\partial z^2 = 0$.

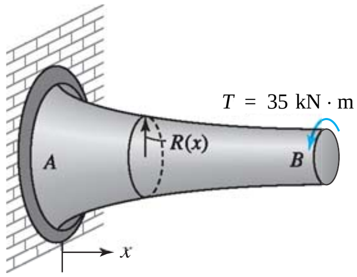


Figure P5.59

TABLE P5.59

x (m)	R(x) (mm)
0.0	100.6
0.1	92.7
0.2	82.6
0.3	79.6
0.4	75.9
0.5	68.8
0.6	68.0
0.7	65.9

TABLE P5.59

x (m)	R(x) (mm)
0.8	60.1
0.9	60.3
1.0	59.1
1.1	54.0
1.2	54.8
1.3	54.1
1.4	49.4
1.5	50.6

5.60 Let the radius of the tapered shaft in Problem 5.59 be represented by the equation $R(x) = a + bx$. Using the data in Table P5.59 determine the constants a and b by the least-squares method and then find the rotation of the section at B by analytical integration.

5.61 Table P5.61 shows the values of distributed torque at several points along the axis of the solid steel shaft (G = 12,000 ksi) shown in Figure P5.61. The shaft has a length of 36 in. and a diameter of 1 in. Determine (a) the rotation of end A with respect to the wall using numerical integration; (b) the maximum shear stress in the shaft.

TABLE P5.61

x (in.)	t(x) (in.·lb/in.)
0	93.0
3	146.0
6	214.1
9	260.4
12	335.0
15	424.7
18	492.6

TABLE P5.61

x (in.)	t(x) (in.·lb/in.)
21	588.8
24	700.1
27	789.6
30	907.4
33	1040.3
36	1151.4

5.62 Let the distributed torque $t(x)$ in Problem 5.61 be represented by the equation $t(x) = a + bx + cx^2$. Using the data in Table P5.61 determine the constants a, b, and c by the least-squares method and then find the rotation of the section at B by analytical integration.

QUICK TEST 5.1

Time: 20 minutes/Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E.

1. Torsional shear strain varies linearly across a homogeneous cross section.

2. Torsional shear *strain* is a maximum at the outermost radius for a homogeneous and a nonhomogeneous cross section.

3. Torsional shear *stress* is a maximum at the outermost radius for a homogeneous and a nonhomogeneous cross section.

4. The formula $\tau_{x\theta} = T\rho/J$ can be used to find the shear stress on a cross section of a tapered shaft.

5. The formula $\phi_2 - \phi_1 = T(x_2 - x_1)/GJ$ can be used to find the relative rotation of a segment of a tapered shaft.

6. The formula $\tau_{x\theta} = T\rho/J$ can be used to find the shear stress on a cross section of a shaft subjected to distributed torques.

7. The formula $\phi_2 - \phi_1 = T(x_2 - x_1)/GJ$ can be used to find the relative rotation of a segment of a shaft subjected to distributed torques.

8. The equation $T = \int_A \rho\tau_{x\theta}\, dA$ cannot be used for nonlinear materials.

9. The equation $T = \int_A \rho\tau_{x\theta}\, dA$ can be used for a nonhomogeneous cross section.

10. Internal torques jump by the value of the concentrated external torque at a section.

5.3 STATICALLY INDETERMINATE SHAFTS

In Chapter 4 we saw the solution of statically indeterminate axial problems require equilibrium equations and compatibility equations. This is equally true for statically indeterminate shafts. The primary focus in this section will be on the solution of statically indeterminate shafts that are on a single axis. However, equilibrium equations and compatibility equations can also be used for solution of shafts with composite cross section, as will be demonstrated in Example 5.14. The use of equilibrium equations and compatibility equations to shafts on multiple axis is left as exercises in Problem Set 5.3.

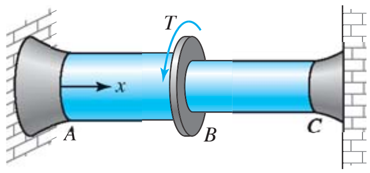


Figure 5.43 Statically indeterminate shaft.

Figure 5.43 shows a statically indeterminate shaft. In statically indeterminate shafts we have two reaction torques, one at the left and the other at the right end of the shaft. But we have only one static equilibrium equation, the sum of all torques in the x direction should be zero. Thus the degree of static redundancy is 1 and we need to generate 1 compatibility equation. We shall use the continuity of ϕ and the fact that the sections at the left and right walls have zero rotation. The compatibility equation state that the relative rotation of the right wall with respect to the left wall is zero. Once more we can use either the displacement method or the force method:

1. In the *displacement method*, we can use the rotation of the sections as the unknowns. If torque is applied at several sections along the shaft, then the rotation of each of the sections is treated as an unknown.
2. In the *force method*, we can use either the reaction torque as the unknown or the internal torques in the sections as the unknowns. Since the degree of static redundancy is 1, the simplest approach is often to take the left wall (or the right wall) reaction as the unknown variable. We can then apply the compatibility equation, as outlined next.

5.3.1 General Procedure for Statically Indeterminate Shafts.

Step 1 Make an imaginary cut in each segment and draw free-body diagrams by taking the left (or right) part if the left (right) wall reaction is carried as the unknown in the problem. Alternatively, draw the torque diagram in terms of the reaction torque.

Step 2 Write the internal torque in terms of the reaction torque.

Step 3 Using Equation (5.12) write the relative rotation of each segment ends in terms of the reaction torque.

Step 4 Add all the relative rotations. Obtain the rotation of the right wall with respect to the left wall and set it equal to zero to obtain the reaction torque.

Step 5 The internal torques can be found from equations obtained in Step 2, and angle of rotation and stresses calculated.

EXAMPLE 5.12

A solid circular steel shaft (G_s = 12,000 ksi, E_s = 30,000 ksi) of 4-in. diameter is loaded as shown in Figure 5.62. Determine the maximum shear stress in the shaft.

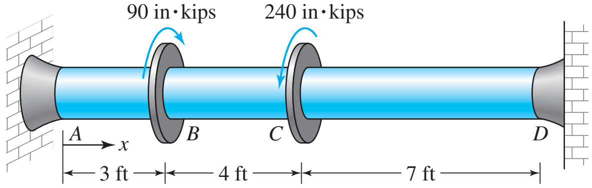


Figure P5.62 Shaft in Example 5.12.

PLAN

We follow the procedure outlined in Section 5.3.1 to determine the reaction torque T_A. For the uniform cross-section the maximum shear stress will occur in the segment that has the maximum internal torque.

SOLUTION

The polar moment of inertia and the torsional rigidity for the shaft can be found as

$$J = \frac{\pi(4\text{ in.})^4}{32} = 25.13\ \text{in.}^4 \qquad GJ = (12000\text{ ksi})(25.13\text{ in.}^4) = 301.6(10^3)\ \text{kips}\cdot\text{in.}^2 \tag{E1}$$

Step 1: We draw the reaction torques T_A and T_D as shown in Figure 5.44*a*. By making imaginary cuts in sections *AB*, *BC*, and *CD* and taking the left part we obtain the free body diagrams shown in Figures 5.44 *b*, *c*, and *d*.

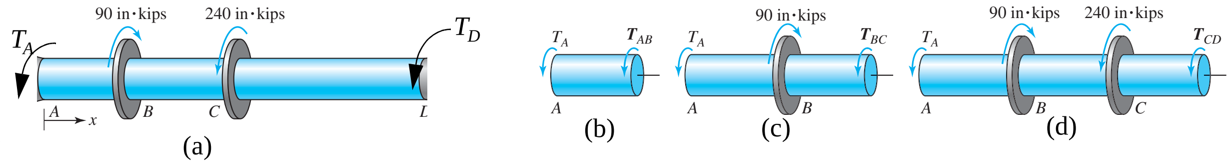


Figure 5.44 Free body diagrams of (a) entire shaft; (b) section *AB*; (c) section *BC*; (d) section *CD*.

Step 2: By equilibrium of moments in Figures 5.44 *b*, *c*, and *d*. or from Figure 5.45*b* we obtain the internal torques as

$$T_{AB} = -T_A \qquad \boldsymbol{T}_{BC} = (-T_A + 90\ )\text{in.}\cdot\text{kips} \qquad \boldsymbol{T}_{CD} = (-T_A - 150\ )\text{in.}\cdot\text{kip} \tag{E2}$$

Step 3: Using Equation (5.12), the relative rotation in each segment ends can be written as

$$\phi_B - \phi_A = \frac{\boldsymbol{T}_{AB}(x_B - x_A)}{G_{AB}J_{AB}} = \frac{-T_A(36)}{301.6(10^3)} = -0.1194(10^{-3})T_A \tag{E3}$$

$$\phi_C - \phi_B = \frac{\boldsymbol{T}_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{(-T_A + 90)48}{301.6\times 10^3} = (-0.1592T_A + 14.32)(10^{-3}) \tag{E4}$$

$$\phi_D - \phi_C = \frac{\boldsymbol{T}_{CD}(x_D - x_C)}{G_{CD}J_{CD}} = \frac{(-T_A - 150)84}{301.6\times 10^3} = (-0.2785T_A - 41.78)(10^{-3}) \tag{E5}$$

Step 4: We obtain $\phi_D - \phi_A$. by adding Equations (E3), (E4), and (E5), which we equate to zero to obtain T_A:

$$\phi_D - \phi_A = (-0.1194T_A - 0.1592T_A + 14.32 - 0.2785T_A - 41.78) = 0 \text{ or}$$

$$T_A = \frac{14.32 - 41.78}{0.1194 + 0.1592 + 0.2785} = -49.28\ \text{in.}\cdot\text{kips} \tag{E6}$$

Step 5: We obtain the internal torques by substituting Equation (E6) into Equation (E2):

$$T_{AB} = 49.28\ \text{in.}\cdot\text{kips} \qquad \boldsymbol{T}_{BC} = 139.28\ \text{in.}\cdot\text{kips} \qquad \boldsymbol{T}_{CD} = -100.72\ \text{in.}\cdot\text{kips} \tag{E7}$$

For the uniform cross-section, the maximum shear stress will occur in segment *BC* and can be found using Equation (5.10):

$$\tau_{max} = \frac{T_{BC}(\rho_{BC})_{max}}{J_{BC}} = \frac{(139.3\ \text{in.}\cdot\text{kips})(2\ \text{in.})}{25.13\ \text{in.}^4} \tag{E8}$$

ANS. $\tau_{\max} = 11.1\ \text{ksi}$

COMMENTS

1. We could have found the internal torques in terms of T_A using the template shown in Figure 5.45*a* and drawing the torque diagram in Figure 5.45*b*.

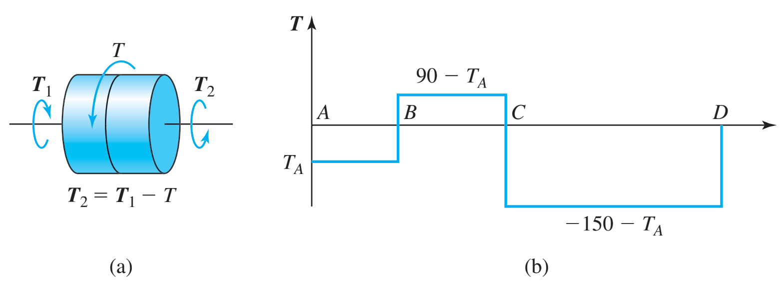


Figure 5.45 Template and torque diagram in Example

2. We can find the reaction torque at D from equilibrium of moment in the free body diagram shown in Figure 5.44*d* as:

$$T_D = 90 - 240 - T_A = -60.72\ \text{in.}\cdot\text{kips}$$

3. Because the applied torque at *C* is bigger than that at *B*, the reaction torques at *A* and *D* will be opposite in direction to the torque at *C*. In other words, the reaction torques at *A* and *D* should by clockwise with respect to the *x* axis. The sign of T_A and T_D confirms this intuitive reasoning.

EXAMPLE 5.13

A solid aluminum shaft (G_{al} = 27 GPa) and a solid bronze shaft (G_{br} = 45 GPa) are securely connected to a rigid wheel, as shown in Figure 5.46. The shaft has a diameter of 75 mm. The allowable shear stresses in aluminum and bronze are 100 MPa and 120 MPa, respectively. Determine the maximum torque that can be applied to wheel B.

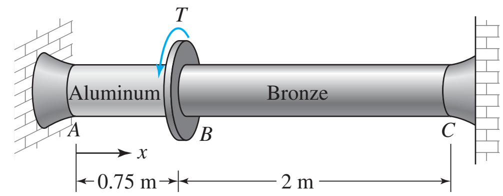


Figure 5.46 Shaft in Example 5.13.

PLAN

We will follow the procedure of Section 5.3.1 and solve for the maximum shear stress in aluminum and bronze in terms of T. We will obtain the two limiting values on T to meet the limitations on maximum shear stress and determine the maximum permissible value of T.

SOLUTION

We can find the polar moment of inertia and the torsional rigidities as

$$= \pi(0.075\text{ m})^4/32 = 3.106 \times 10^{-6}\text{ m}^4 \qquad G_{AB}J_{AB} = 83.87(10^3)\text{ N}\cdot\text{m}^2 \qquad G_{BC}J_{BC} = 139.8(10^3)\text{ N}\cdot\text{m} \tag{E1}$$

Step 1: Let T_A, the reaction torque at A, be clockwise with respect to the x axis. We can make imaginary cuts in AB and BC and draw the free-body diagrams as shown in Figure 5.47.

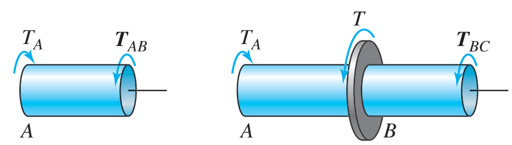


Figure 5.47 Free-body diagrams in Example 5.13.

Step 2: From equilibrium of moment about shaft axis in Figure 5.47 we obtain the internal torques in terms of T_A and T.

$$T_{AB} = T_A \qquad T_{BC} = T_A - T \tag{E2}$$

Step 3: Using Equation (5.12), we obtain the relative rotation in each segment ends as

$$\phi_B - \phi_A = \frac{T_{AB}(x_B - x_A)}{G_{AB}J_{AB}} = \frac{T_A(0.75)}{83.87(10^3)} = 8.942(10^{-6})T_A \tag{E3}$$

$$\phi_C - \phi_B = \frac{T_{BC}(x_C - x_B)}{G_{BC}J_{BC}} = \frac{(T_A - T)(2)}{139.8(10^3)} = (14.31T_A - 14.31T)(10^{-6}) \tag{E4}$$

Step 4: We obtain $\phi_C - \phi_A$ by adding Equations (E3) and (E4) and equate it to zero to find T_A in terms of T:

$$\phi_C - \phi_A = (8.942T_A + 14.31T_A - 14.31T)(10^{-6}) = 0 \qquad \text{or} \qquad T_A = \frac{14.31T}{8.942 + 14.31} = 0.6154T \tag{E5}$$

Step 5: We obtain the internal torques in terms of T by substituting Equation (E5) into Equation (E2):

$$T_{AB} = 0.6154T \qquad T_{BC} = -0.3846T \tag{E6}$$

The maximum shear stress in segment AB and BC can be found in terms of T using Equation (5.10) and noting that $\rho_{max} = 0.0375$ mm. Using the limits on shear stress we obtain the limits on T as

$$|(\tau_{AB})_{max}| = \left|\frac{T_{AB}(\rho_{AB})_{max}}{J_{AB}}\right| = \frac{(0.6154T)(0.0375\text{ m})}{3.106(10^{-6})\text{ m}^4} \le 100(10^6)\text{ N/m}^2 \qquad \text{or} \qquad T \le 13.46(10^3)\text{ N}\cdot\text{m} \tag{E7}$$

$$|(\tau_{BC})_{max}| = \left|\frac{T_{BC}(\rho_{BC})_{max}}{J_{BC}}\right| = \frac{(0.3846T)(0.0375\text{ m})}{3.106(10^{-6})\text{ m}^4} \le 120(10^6)\text{ N/m}^2 \qquad \text{or} \qquad T \le 25.84(10^3)\text{ N}\cdot\text{m} \tag{E8}$$

The value of T that satisfies Equations (E7) and (E8) is the maximum value we seek.

ANS. $T_{max} = 13.4\text{ kN}\cdot\text{m}$.

COMMENTS

1. The maximum torque is limited by the maximum shear stress in bronze. If we had a limitation on the rotation of the wheel, then we could easily incorporate it by calculating ϕ_B from Equation (E3) in terms of T.
2. We could have solved this problem by the displacement method. In that case we would carry the rotation of the wheel ϕ_B as the unknown.

3. We could have solved the problem by initially assuming that one of the materials reaches its limiting stress value, say aluminum. We can then do our calculations and find the maximum stress in bronze, which would exceed the limiting value of 120 MPa. We would then resolve the problem. The process, though correct, can become tedious as the number of limitations increases. Instead put off deciding which limitation dictates the maximum value of the torque toward the end. In this way we need to solve the problem only once, irrespective of the number of limitations.

EXAMPLE 5.14

A solid steel (G = 80 GPa) shaft is securely fastened to a hollow bronze (G = 40 GPa) shaft as shown in Figure 5.48. Determine the maximum value of shear stress in the shaft and the rotation of the right end with respect to the wall.

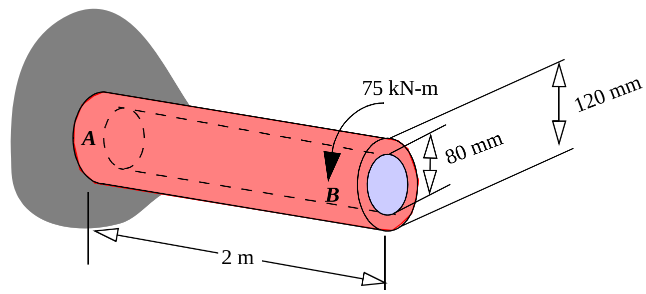


Figure 5.48 Composite shaft in Example 5.14.

PLAN

The steel shaft and the bronze shaft can be viewed as two independent shafts. At equilibrium the sum of the internal torques on each material is equal to the applied torque. The compatibility equation follows from the condition that a radial line on steel and bronze will rotate by the same amount. Hence, the relative rotation is the same for each length segment. Solving the equilibrium equation and the compatibility equation we obtain the internal torques in each material, from which the desired quantities can be found.

SOLUTION

We can find the polar moments and torsional rigidities as

$$J_S = \frac{\pi}{32}(0.08\ \text{m})^4 = 4.02(10^{-6})\ \text{m}^4 \qquad J_{Br} = \frac{\pi}{32}[(0.12\ \text{m})^4 - (0.08\ \text{m})^4] = 16.33(10^{-6})\ \text{m}^4 \tag{E1}$$

$$G_S J_S = 321.6(10^3)\ \ \text{N}\cdot\text{m}^2 \qquad G_{Br} J_{Br} = 653.2(10^3)\ \ \text{N}\cdot\text{m}^2 \tag{E2}$$

Figure 5.49a shows the free body diagram after making an imaginary cut in *AB*. Figure 5.49b shows the decomposition of a composite shaft as two homogenous shafts.

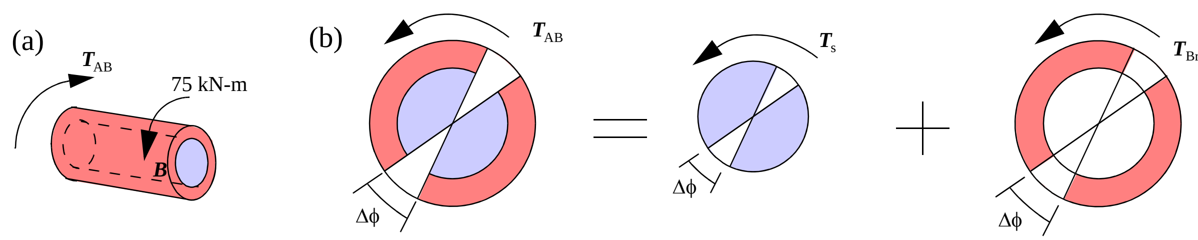


Figure 5.49 (a) Free body diagram (b) Composite shaft as two homogenous shafts in Example 5.14.

From Figure 5.49 we obtain the equilibrium equation,

$$\boldsymbol{T}_{AB} = \boldsymbol{T}_s + \boldsymbol{T}_{Br} = 75\ \ \text{kN}\cdot\text{m} = 75(10^3)\ \ \text{N}\cdot\text{m} \tag{E3}$$

Using Equation (5.12) we can write the relative rotation of section at *B* with respect to *A* for the two material as

$$\Delta\phi = \phi_B - \phi_A = \frac{\boldsymbol{T}_s(x_B - x_A)}{G_s J_s} = \frac{\boldsymbol{T}_s(2)}{321.6(10^3)} = 6.219(10^{-6})\boldsymbol{T}_s\ \text{rad} \tag{E4}$$

$$\Delta\phi = \phi_B - \phi_A = \frac{\boldsymbol{T}_{Br}(2)}{653.2(10^3)} = 3.0619(10^{-6})\boldsymbol{T}_{Br}\ \text{rad} \tag{E5}$$

Equating Equations (E4) and (E5) we obtain

$$\boldsymbol{T}_s = 2.03\boldsymbol{T}_{Br} \tag{E6}$$

Solving Equations (E3) and (E4) for the internal torques give

$$\boldsymbol{T}_s = 24.75(10^3)\ \text{N}\cdot\text{m} \qquad \boldsymbol{T}_{Br} = 50.25(10^3)\ \text{N}\cdot\text{m} \tag{E7}$$

Substituting Equation (7) into Equation (4), we find

$$\phi_B - \phi_A = 6.219(10^{-6})(24.75)(10^3) = 0.1538 \text{ rad} \tag{E8}$$

ANS. $\phi_B - \phi_A = 0.1538$ rad ccw

The maximum torsional shear stress in each material can be found using Equation (5.10):

$$(\tau_s)_{max} = \frac{T_s(\rho_s)_{max}}{J_s} = \frac{[24.75(10^3)\text{ N}\cdot\text{m}](0.04\text{ m})}{4.02(10^{-6})\text{ m}^4} = 246.3(10^6)\text{ N/m}^2 \tag{E9}$$

$$(\tau_{Br})_{max} = \frac{T_{Br}(\rho_{Br})_{max}}{J_{Br}} = \frac{[50.25(10^3)\text{ N}\cdot\text{m}](0.06\text{ m})}{16.33(10^{-6})\text{ m}^4} = 184.6(10^6)\text{ N/m}^2 \tag{E10}$$

The maximum torsional shear stress is the larger of the two.

ANS. $\tau_{max} = 246.3$ MPa

COMMENT

1. The kinematic condition that all radial lines must rotate by equal amount for a circular shaft had to be explicitly enforced to obtain Equations (E6). We could also have implicitly assumed this kinematic condition and developed formulas for composite shafts (see Problem 5.49) as we did for homogenous shaft. We can then use these formulas to solve statically determinate and indeterminate problems (see Problem 5.82) as we have done for homogenous shafts.

PROBLEM SET 5.3

Statically indeterminate shafts

5.63 A steel shaft (G_{st} = 12,000 ksi) and a bronze shaft (G_{br} = 5600 ksi) are securely connected at B, as shown in Figure P5.63. Determine the maximum torsional shear stress in the shaft and the rotation of the section at B if the applied torque T = 50 in.·kips.

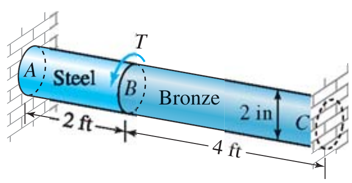


Figure P5.63

5.64 A steel shaft (G_{st} = 12,000 ksi) and a bronze shaft (G_{br} = 5600 ksi) are securely connected at B, as shown in Figure P5.63. Determine the maximum torsional shear strain and the applied torque T if the section at B rotates by an amount of 0.02 rad.

5.65 Two hollow aluminum shafts (G = 10,000 ksi) are securely fastened to a solid aluminum shaft and loaded as shown Figure P5.65. If T = 300 in.·kips, determine (a) the rotation of the section at C with respect to the wall at A; (b) the shear strain at point E. Point E is on the inner surface of the shaft.

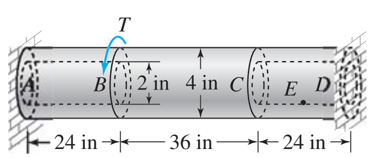


Figure P5.65

5.66 Two hollow aluminum shafts (G = 10,000 ksi) are securely fastened to a solid aluminum shaft and loaded as shown Figure P5.65. The torsional shear strain at point E which is on the inner surface of the shaft is –250 μ. Determine the rotation of the section at C and the applied torque T that produced this shear strain.

5.67 Two solid circular steel shafts (G_{st} = 80 GPa) and a solid circular bronze shaft (G_{br} = 40 GPa) are securely connected by a coupling at C as shown in Figure P5.67. A torque of T = 10 kN·m is applied to the rigid wheel B. If the coupling plates cannot rotate relative to one another, determine the angle of rotation of wheel B due to the applied torque.

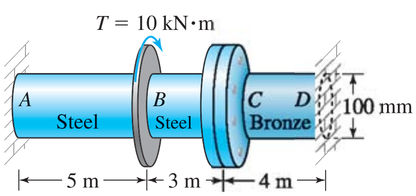


Figure P5.67

5.68 Two solid circular steel shafts (G_{st} = 80 GPa) and a solid circular bronze shaft (G_{br} = 40 GPa) are connected by a coupling at C as shown in Figure P5.67. A torque of T = 10 kN·m is applied to the rigid wheel B. If the coupling plates can rotate relative to one another by 0.5° before engaging, then what will be the angle of rotation of wheel B?

5.69 A solid steel shaft (G = 80 GPa) is securely fastened to a solid bronze shaft (G = 40 GPa) that is 2 m long, as shown in Figure P5.69. If T_{ext} = 10 kN · m, determine (a) the magnitude of maximum torsional shear stress in the shaft; (b) the rotation of the section at 1 m from the left wall.

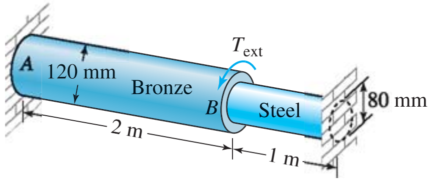


Figure P5.69

5.70 A solid steel shaft (G = 80 GPa) is securely fastened to a solid bronze shaft (G = 40 GPa) that is 2 m long, as shown in Figure P5.69. If the section at B rotates by 0.05 rad, determine (a) the maximum torsional shear strain in the shaft; (b) the applied torque T_{ext}.

5.71 Two shafts with shear moduli $G_1 = G$ and $G_2 = 2G$ are securely fastened at section B, as shown in Figure P5.71. In terms of T_{ext}, L, G, and d, find the magnitude of maximum torsional shear stress in the shaft and the rotation of the section at B.

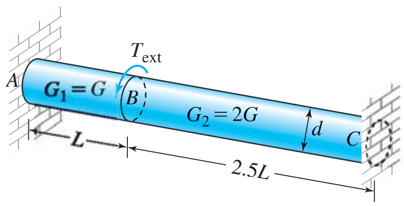


Figure P5.71

5.72 A uniformly distributed torque of q in.·lb/in. is applied to the entire shaft, as shown in Figure P5.72. In addition to the distributed torque a concentrated torque of $T = 3qL$ in.·lb is applied at section B. Let the shear modulus be G and the radius of the shaft r. In terms of q, L, G, and r determine (a) the rotation of the section at B; (b) the magnitude of maximum torsional shear stress in the shaft.

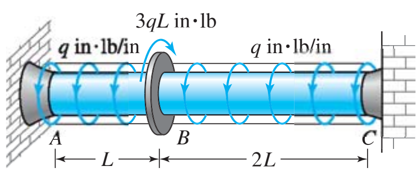


Figure P5.72

Design problems

5.73 A steel shaft (G_{st} = 80 GPa) and a bronze shaft (G_{br} = 40 GPa) are securely connected at B, as shown in Figure P5.69. The magnitude of maximum torsional shear stresses in steel and bronze are to be limited to 160 MPa and 60 MPa, respectively. Determine the maximum allowable torque T_{ext} to the nearest N·m that can act on the shaft.

5.74 A steel shaft (G_{st} = 80 GPa) and a bronze shaft (G_{br} = 40 GPa) are securely connected at B, as shown in Figure P5.74. The magnitude of maximum torsional shear stresses in steel and bronze are to be limited to 160 MPa and 60 MPa, respectively, and the rotation of section B is limited to 0.05 rad. (a) Determine the maximum allowable torque T to the nearest kN·m that can act on the shaft if the diameter of the shaft is d = 100 mm. (b) What are the magnitude of maximum torsional shear stress and the maximum rotation in the shaft corresponding to the answer in part (a)?

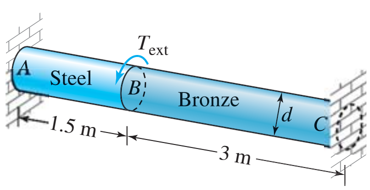


Figure P5.74

5.75 A steel shaft (G_{st} = 80 GPa) and a bronze shaft (G_{br} = 40 GPa) are securely connected at B, as shown in Figure P5.74. The magnitude of maximum torsional shear stresses in steel and bronze are to be limited to 160 MPa and 60 MPa, respectively, and the rotation of section B is limited to 0.05 rad. (a) Determine the minimum diameter d of the shaft to the nearest millimeter if the applied torque T = 20 kN · m. (b) What are the magnitude of maximum torsional shear stress and the maximum rotation in the shaft corresponding to the answer in part (a)?

5.76 The solid steel shaft shown in Figure P5.76 has a shear modulus of elasticity $G = 80$ GPa and an allowable torsional shear stress of 60 MPa. The allowable rotation of any section is 0.03 rad. The applied torques on the shaft are $T_1 = 10$ kN·m and $T_2 = 25$ kN·m. Determine (a) the minimum diameter d of the shaft to the nearest millimeter; (b) the magnitude of maximum torsional shear stress in the shaft and the maximum rotation of any section.

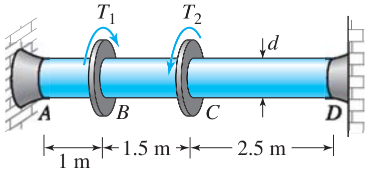


Figure P5.76

5.77 The diameter of the shaft shown in Figure P5.76 $d = 80$ mm. Determine the maximum values of the torques T_1 and T_2 to the nearest kN·m that can be applied to the shaft.

Composite Shafts

5.78 An aluminum tube and a copper tube, each having a thickness of 5 mm, are securely fastened to two rigid bars, as shown in Figure P5.78. The bars force the tubes to rotate by equal angles. The two tubes are 1.5 m long, and the mean diameters of the aluminum and copper tubes are 125 mm and 50 mm, respectively. The shear moduli for aluminum and copper are $G_{al} = 28$ GPa and $G_{cu} = 40$ GPa. Under the action of the applied couple section B of the two tubes rotates by an angle of 0.03 rad Determine (a) the magnitude of maximum torsional shear stress in aluminum and copper; (b) the magnitude of the couple that produced the given rotation.

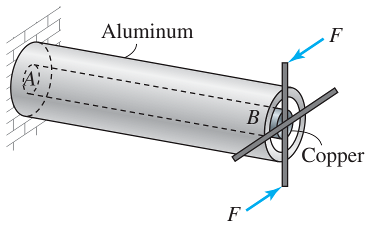


Figure P5.78

5.79 Solve Problem 5.78 using Equations (5.18a) and (5.18b).

5.80 An aluminum tube and a copper tube, each having a thickness of 5 mm, are securely fastened to two rigid bars, as shown in Figure P5.78. The bars force the tubes to rotate by equal angles. The two tubes are 1.5 m long and the mean diameters of the aluminum and copper tubes are 125 mm and 50 mm, respectively. The shear moduli for aluminum and copper are $G_{al} = 28$ GPa and $G_{cu} = 40$ GPa. The applied couple on the tubes shown in Figure P5.78 is 10 kN·m. Determine (a) the magnitude of maximum torsional shear stress in aluminum and copper; (b) the rotation of the section at B.

5.81 Solve Problem 5.80 using Equations (5.18a) and (5.18b).

5.82 Solve Example 5.14 using Equations (5.18a) and (5.18b).

5.83 The composite shaft shown in Figure P5.83 is constructed from aluminum ($G_{al} = 4000$ ksi), bronze ($G_{br} = 6000$ ksi), and steel ($G_{st} = 12{,}000$ ksi). (a) Determine the rotation of the free end with respect to the wall. (b) Plot the torsional shear strain and the shear stress across the cross section

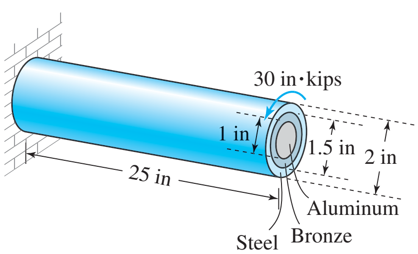


Figure P5.83

5.84 Solve Problem 5.83 using Equations (5.18a) and (5.18b).

5.85 If $T = 1500$ N·m in Figure P5.85, determine (a) the magnitude of maximum torsional shear stress in cast iron and copper; (b) the rotation of the section at D with respect to the section at A.

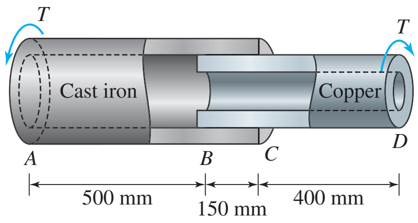


Figure P5.85

Shafts on multiple axis

5.86 Two steel (G = 80 GPa) shafts *AB* and *CD* of diameters 40 mm are connected with gears as shown in Figure P5.86. The radii of gears at *B* and *C* are 250 mm and 200 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. If an input torque of T_{ext} = 1.5 kN.m is applied at *D*, determine (a) the maximum torsional shear stress in *AB*; (b) the rotation of section at *D* with respect to the fixed section at *A*.

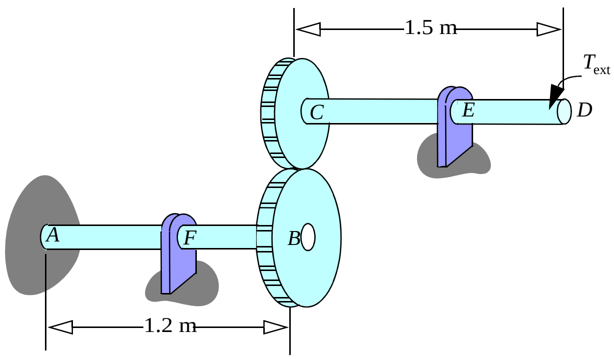


Figure P5.86

5.87 Two steel (G = 80 GPa) shafts *AB* and *CD* of diameters 40 mm are connected with gears as shown in Figure P5.86. The radii of gears at *B* and *C* are 250 mm and 200 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. The allowable shear stress in the shafts is 120 MPa. Determine the maximum torque *T* that can be applied at section *D*.

5.88 Two steel (G = 12,000 ksi) shafts *AB* and *CDE* of 1.5 in. diameters are connected with gears as shown in Figure P5.88. The radii of gears at *B* and *D* are 9 in. and 5 in., respectively. The bearings at *F, G*, and *H* offer no torsional resistance to the shafts. If an input torque of T_{ext} = 800 ft.lb is applied at D, determine (a) the maximum torsional shear stress in *AB*; (b) the rotation of section at *E* with respect to the fixed section at *C*.

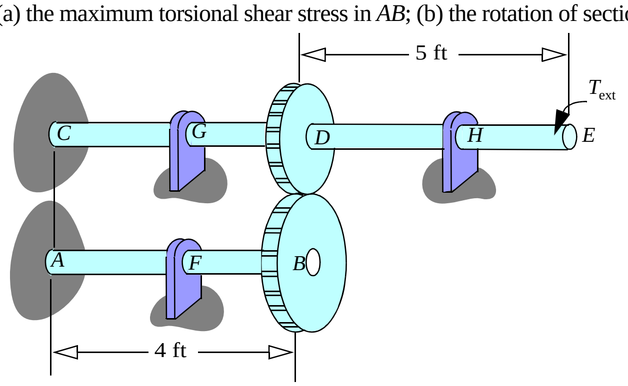


Figure P5.88

5.89 Two steel (G = 80 GPa) shafts *AB* and *CD* of 60 mm diameters are connected with gears as shown in Figure P5.89. The radii of gears at *B* and *D* are 175 mm and 125 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. If an input torque of T_{ext} = 2 kN.m is applied, determine (a) the maximum torsional shear stress in *AB*; (b) the rotation of section at *D* with respect to the fixed section at *C*.

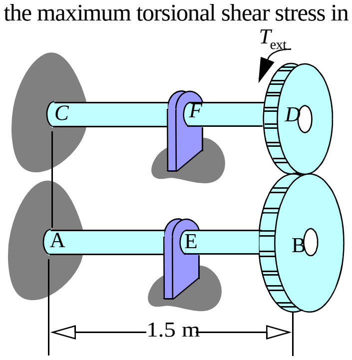


Figure P5.89

5.90 Two steel (G = 80 GPa) shafts *AB* and *CD* of 60 mm diameters are connected with gears as shown in Figure P5.89. The radii of gears at *B* and *D* are 175 mm and 125 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. The allowable shear stress in the shafts is 120 MPa. What is the maximum torque *T* that can be applied?

5.91 Two steel (G = 80 GPa) shafts *AB* and *CD* of equal diameters *d* are connected with gears as shown in Figure P5.89. The radii of gears at *B* and *D* are 175 mm and 125 mm, respectively. The bearings at *E* and *F* offer no torsional resistance to the shafts. The allowable shear stress in the shafts is 120 MPa and the input torque is *T* = 2 kN.m. Determine the minimum diameter *d* to the nearest millimeter.

Stress concentration

5.92 The allowable shear stress in the stepped shaft shown Figure P5.92 is 17 ksi. Determine the smallest fillet radius that can be used at section *B*. Use the stress concentration graphs given in Section C.4.3.

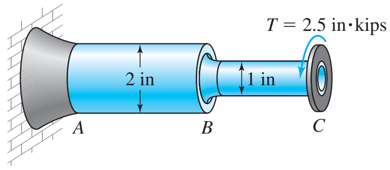


Figure P5.92

5.93 The fillet radius in the stepped shaft shown in Figure P5.93 is 6 mm. Determine the maximum torque that can act on the rigid wheel if the allowable shear stress is 80 MPa and the modulus of rigidity is 28 GPa. Use the stress concentration graphs given in Section C.4.3.

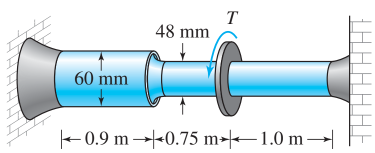


Figure P5.93

5.4* TORSION OF THIN-WALLED TUBES

The sheet metal skin on a fuselage, the wing of an aircraft, and the shell of a tall building are examples in which a body can be analyzed as a thin-walled tube. By thin wall we imply that the thickness *t* of the wall is smaller by a factor of at least 10 in comparison to the length *b* of the biggest line that can be drawn across two points on the cross section, as shown in Figure 5.50*a*. By a *tube* we imply that the length *L* is at least 10 times that of the cross-sectional dimension *b*.
We assume that this thin-walled tube is subjected to only torsional moments.

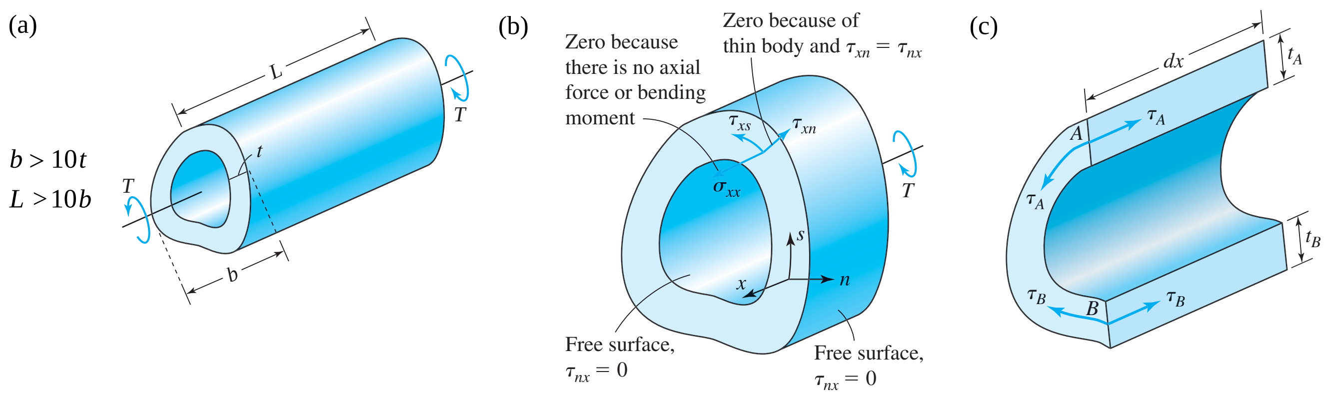


Figure 5.50 (a) Torsion of thin-walled tubes. (b) Deducing stress behavior in thin-walled tubes. (c) Deducing constant shear flow in thin-walled tubes.

The walls of the tube are bounded by two free surfaces, and hence by the symmetry of shear stresses the shear stress in the normal direction τ_{xn} must go to zero on these bounding surfaces, as shown in Figure 5.50*b*.This does not imply that τ_{xn} is zero in the interior. However, because the walls are thin, we approximate τ_{xn} as zero everywhere. The normal stress σ_{xx} would be equivalent to an internal axial force or an internal bending moment. Since there is no external axial force or bending moment, we approximate the value of σ_{xx} as zero.

Figure 5.50*b* shows that the only nonzero stress component is τ_{xs}. It can be assumed uniform in the *n* direction because the tube is thin. Figure 5.50*c* shows a free-body diagram of a differential element with an imaginary cut through points *A* and *B*. By equilibrium of forces in the *x* direction we obtain

$$\tau_A(t_A\, dx) = \tau_B(t_B\, dx) \tag{5.24a}$$

$$\tau_A t_A = \tau_B t_B \tag{5.24b}$$

$$q_A = q_B \tag{5.24c}$$

The quantity $q = \tau_{xs} t$ is called shear flow[5] and has units of force per unit length. Equation (5.24c) shows that shear flow is uniform at a given cross section.

We can replace the shear stresses (shear flow) by an equivalent internal torque, as shown in Figure 5.51. The line *OC* is perpendicular to the line of action of the force ***dV***, which is in the tangent direction to the arc at that point. Noting that the shear flow is a constant, we take it outside the integral sign,

$$\boldsymbol{T} = \oint dT = \oint q(h\, ds) = q\oint 2\, dA_E = 2qA_E \qquad \text{or} \qquad q = \frac{\boldsymbol{T}}{2A_E} \tag{5.25}$$

We thus obtain

$$\boxed{\tau_{xs} = \frac{\boldsymbol{T}}{2tA_E}} \tag{5.26}$$

where $\boldsymbol{T}$ is the internal torque at the section containing the point at which the shear stress is to be calculated, A_E is the area enclosed by the centerline of the tube, and *t* is the thickness at the point where the shear stress is to be calculated.

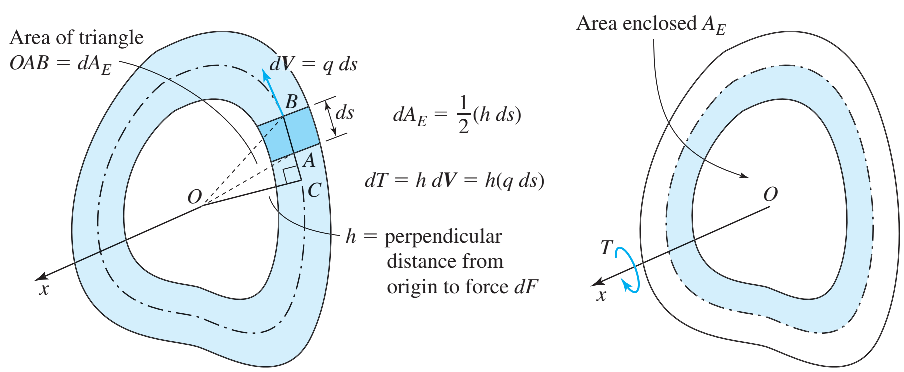


Figure 5.51 Equivalency of internal torque and shear stress (flow).

The thickness *t* can vary with different points on the cross section provided the assumption of thin-walled is not violated. If the thickness varies, then the shear stress will not be constant on the cross section, even though the shear flow is constant.

EXAMPLE 5.15

A semicircular thin tube is subjected to torques as shown in Figure 5.52. Determine: (a) The maximum torsional shear stress in the tube. (b) The torsional shear stress at point *O*. Show the results on a stress cube.

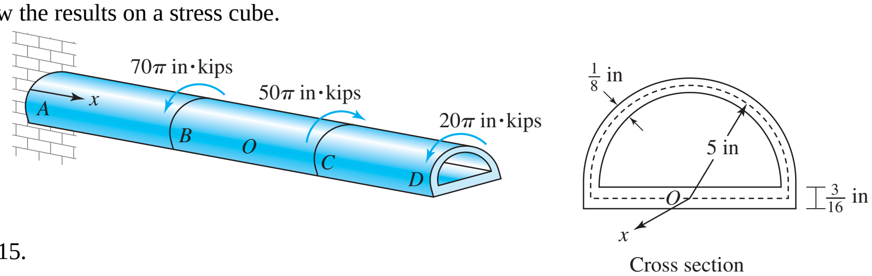


Figure 5.52 Thin-walled tube in Example 5.15.

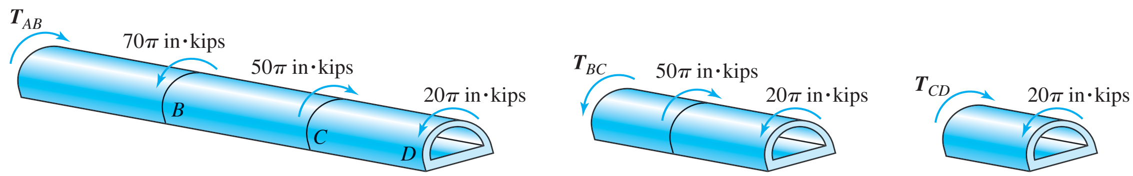
PLAN

From Equation 5.26 we know that the maximum torsional shear stress will exist in a section where the internal torque is maximum and the thickness minimum. To determine the maximum internal torque, we make cuts in *AB*, *BC*, and *CD* and draw free-body diagrams by taking the right part of each cut to avoid calculating the wall reaction.

[5]This terminology is from fluid mechanics, where an incompressible ideal fluid has a constant flow rate in a channel.

SOLUTION

Figure 5.53 shows the free-body diagrams after making an imaginary cut and taking the right part.



$T_{AB} + 50\pi - 70\pi - 20\pi = 0$ $-T_{BC} + 50\pi - 20\pi = 0$ $T_{CD} - 20\pi = 0$

$T_{AB} = 40\pi$ in·kips $T_{BC} = 30\pi$ in·kips $T_{CD} = 20\pi$ in·kips

Figure 5.53 Internal torque calculations in Example 5.15.

(a) The maximum torque is in *AB* and the minimum thickness is $\frac{1}{8}$ in. The enclosed area is $A_E = \pi(5\text{ in.})^2/2 = 12.5\pi\text{ in.}^2$. From Equation 5.26 we obtain

$$\tau_{max} = \frac{(40\pi\text{ in.}\cdot\text{kips})}{(12.5\pi\text{ in.}^2)(\frac{1}{8}\text{ in.})} \tag{E1}$$

ANS. $\tau_{max} = 25.6$ ksi

(b) At point *O* the internal torque is T_{BC} and $t = \frac{3}{16}$ in. We obtain the shear stress at *O* as

$$\tau_O = \frac{(30\pi\text{ in.}\cdot\text{kips})}{(12.5\pi\text{ in.}^2)(\frac{3}{16}\text{ in.})} \tag{E2}$$

ANS. $\tau_O = 12.8\ ksi$

Figure 5.54 shows part of the tube between sections *B* and *C*. Segment *BO* would rotate counterclockwise with respect to segment *OC*. The shear stress must be opposite to this possible motion and hence in the clockwise direction, as shown. The direction on the other surfaces can be drawn using the observation that the symmetric pair of shear stress components either point toward the corner or away from it.

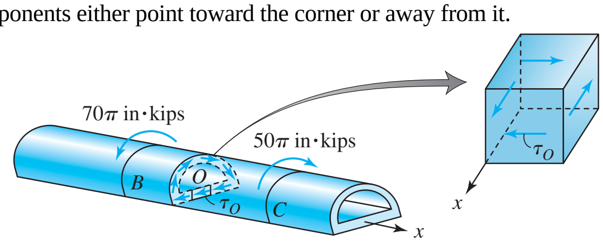


Figure 5.54 Direction of shear stress in Example 5.15.

COMMENT

1. The shear flow in the cross-section containing point *O* is a constant over the entire cross-section. The magnitude of torsional shear stress at point *O* however will be two-thirds that of the value of the shear stress in the circular part of the cross-section because of the variation in wall thickness.

PROBLEM SET 5.4

Torsion of thin-walled tubes

5.94 Calculate the magnitude of the maximum torsional shear stress if the cross section shown in Figure P5.94 is subjected to a torque $T = 100$ in.·kips.

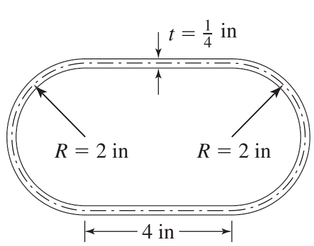


Figure P5.94

5.95 Calculate the magnitude of the maximum torsional shear stress if the cross section shown in Figure P5.95 is subjected to a torque $T = 900\ \text{N} \cdot \text{m}$.

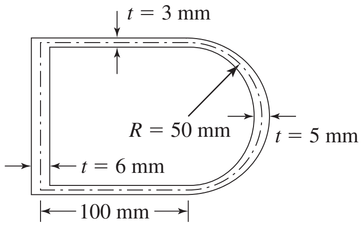


Figure P5.95

5.96 Calculate the magnitude of the maximum torsional shear stress if the cross section shown in Figure P5.96 is subjected to a torque $T = 15\ \text{kN} \cdot \text{m}$.

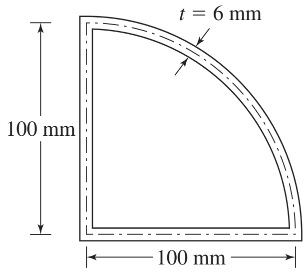


Figure P5.96

5.97 A tube of uniform thickness *t* and cross section shown in Figure P5.97 has a torque *T* applied to it. Determine the maximum torsional shear stress in terms of *t*, *a*, and *T*.

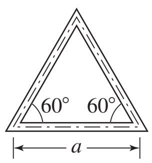


Figure P5.97

5.98 A tube of uniform thickness *t* and cross section shown in Figure P5.98 has a torque *T* applied to it. Determine the maximum torsional shear stress in terms of *t*, *a*, and *T*.

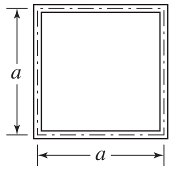


Figure P5.98

5.99 A tube of uniform thickness *t* and cross section shown in Figure P5.99 has a torque *T* applied to it. Determine the maximum torsional shear stress in terms of *t*, *a*, and *T*.

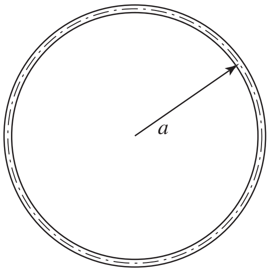


Figure P5.99

5.100 The tube of uniform thickness *t* shown in Figure P5.100 has a torque *T* applied to it. Determine the maximum torsional shear stress in terms of *t*, *a*, *b*, and *T*.

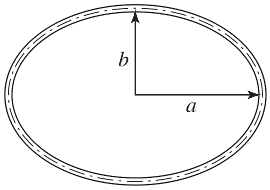


Figure P5.100

5.101 A hexagonal tube of uniform thickness is loaded as shown in Figure P5.101. Determine the magnitude of the maximum torsional shear stress in the tube

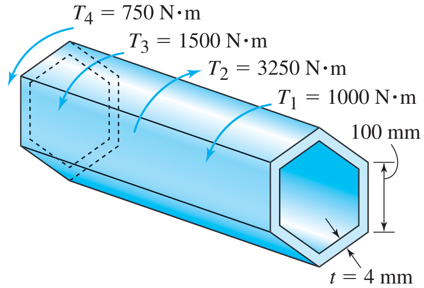


Figure P5.101

5.102 A rectangular tube is loaded as shown in Figure P5.102. Determine the magnitude of the maximum torsional shear stress.

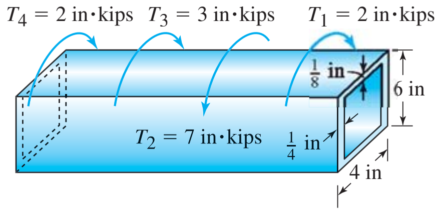


Figure P5.102

5.103 The three tubes shown in Problems 5.97 through 5.99 are to be compared for the maximum torque-carrying capability, assuming that all tubes have the same thickness t, the maximum torsional shear stress in each tube can be τ, and the amount of material used in the cross section of each tube is A. (a) Which shape would you use? (b) What is the percentage torque carried by the remaining two shapes in terms of the most efficient structural shape?

5.5* CONCEPT CONNECTOR

Like so much of science, the theory of torsion in shafts has a history filled with twists and turns. Sometimes experiments led the way; sometimes logic pointed to a solution. As so often, too, serendipity guided developments. The formulas were developed empirically, to meet a need—but not in the mechanics of materials. Instead, a scientist had a problem to solve in electricity and magnetism, and torsion helped him measure the forces. It was followed with an analytical development of the theory for circular and non-circular shaft cross sections that stretched over a hundred years. The description of the history concludes with an experimental technique used in the calculation of torsional rigidity, even for shafts of arbitrary shapes.

5.5.1 History: Torsion of Shafts

It seems fitting that developments begin with Charles-Augustin Coulomb (Figure 5.55). Coulomb, who first differentiated shear stress from normal stress (see Section 1.6.1), also studied torsion, in which shear stress is the dominant component. In 1781 Coulomb started his research in electricity and magnetism. To measure the small forces involved, he devised a very sensitive *torsion balance*. A weight was suspended by a wire, and a pointer attached to the weight indicated the wire's angular rotation.

Figure 5.55 Charles-Augustin Coulomb.

The design of this torsion balance led Coulomb to investigate the resistance of a wire in torsion. He assumed that the resistance torque (or internal torque $\boldsymbol{T}$) in a twisted wire is proportional to the angle of twist ϕ. To measure the changes, he twisted the wire by a small angle and set it free to oscillate, like a pendulum. After validating his formula experimentally, thus confirming his assumption, he proceeded to conduct a parametric study with regard to the length L and the diameter D of the wire and developed the following formula $\boldsymbol{T} = (\mu D^4/L)\phi$, where μ is a material constant. If we substitute $d\phi/dx = \phi/L$ and $J = \pi D^4/32$ into Equation 5.9 and compare our result with Coulomb's formula, we see that Coulomb's material constant is $\mu = \pi G/32$.

Coulomb's formula, although correct, was so far only an empirical relationship. The analytical development of the theory for circular shafts is credited to Alphonse Duleau. Duleau, born in Paris the year of the French revolution, was commissioned in 1811 to design a forged iron bridge over the Dordogne river, in the French city of Cubzac. Duleau had graduated from the École Polytechnique, one of the early engineering schools. Founded in 1794, it had many pioneers in the mechanics of materials among its faculty and students. At the time, there was little or no data on the behavior of bars under the loading conditions needed for bridge design. Duleau therefore conducted extensive experiments on tension, compression, flexure, torsion, and elastic stability. He also compared bars of circular, triangular, elliptical, and rectangular cross section. In 1820 he published his results.

In this paper Duleau developed Coulomb's torsion formula analytically, starting with our own Assumptions 1 and 3 (see page 215), that is, cross sections remain planes and radial lines remain straight during small twists to circular bars. He also established that these assumptions are not valid for noncircular shafts.

Augustin-Louis Cauchy, whose contributions to the mechanics of materials we have encountered in several chapters, was also interested in the torsion of rectangular bars. Cauchy showed that the cross section of a rectangular bar does not remain a plane. Rather, it warps owing to torsional loads.

Jean Claude Saint-Venant proposed in 1855 the displacement behavior we encountered in Problem 5.53. Building on the observations of Coulomb, Duleau, and Cauchy, he developed torsion formulas for a variety of shapes. Saint-Venant's assumed a displacement function that incorporates some features based on experience and empirical information but containing sufficient unknown parameters to satisfy equations of elasticity, an approach now called *Saint-Venant's semi-inverse method.*

Ludwig Prandtl (1875-1953) is best known for his work in aerodynamics, but the German physicist's interests ranged widely in engineering design. He originated boundary-layer theory in fluid mechanics. He also invented the wind tunnel and its use in airplane design. In 1903 Prandtl was studying the differential equations that describe the equilibrium of a soap film, a thin-walled membrane. He found that these are similar to torsion equations derived using Saint-Venant's semi-inverse method. Today, Prandtl's *membrane analogy* is used to obtain torsional rigidities for complex cross sections simply from experiments on soap films. Handbooks list torsional rigidities for variety of shapes, many of which were obtained from membrane analogy.

We once more see that theory is the outcome of a serendipitous combination of experimental and analytical thinking.

5.6 CHAPTER CONNECTOR

In this chapter we established formulas for torsional deformation and stresses in circular shafts. We saw that the calculation of stresses and relative deformation requires the calculation of the internal torque at a section. For statically determinate shafts, the internal torque can be calculated in either of two ways. In the first, we make an imaginary cut and draw an appropriate free-body diagram. In the second, we draw a torque diagram. In statically indeterminate single-axis shafts, the internal torque expression contains an unknown reaction torque, which has to be determined using the compatibility equation. For single-axis shafts, the relative rotation of a section at the right wall with respect to the rotation at the left wall is zero. This result is the compatibility equation.

We also saw that torsional shear stress should be drawn on a stress element. This approach will be important in studying stress and strain transformation in future chapters. In Chapter 8, on *stress transformation*, we will first find torsional shear stress using the stress formula from this chapter. We then find stresses on inclined planes, including planes with maximum normal stress. In Chapter 9, on *strain transformation*, we will find the torsional shear strain and then consider strains in different coordinate systems, including coordinate systems in which the normal strain is maximum. In Section 10.1, we will consider the combined loading problems of axial, torsion, and bending. This will lead to the design of simple structures that may be either determinate or indeterminate.

POINTS AND FORMULAS TO REMEMBER

- Our theory describing the torsion of shafts is limited to: (1) slender shafts of circular cross sections; and (2) regions away from the neighborhood of stress concentration. The variation in cross sections and external torques is gradual.

$$T = \int_A \rho \tau_{x\theta}\, dA \quad (5.1) \qquad \phi = \phi(x) \quad (5.2) \qquad \text{small strain} \quad \gamma_{x\theta} = \rho \frac{d\phi}{dx} \quad (5.3)$$

- where T is the internal torque that is positive counterclockwise with respect to the outward normal to the imaginary cut surface, ϕ is the angle of rotation of the cross section that is positive counterclockwise with respect to the x axis, $\tau_{x\theta}$ and $\gamma_{x\theta}$ are the torsional shear stress and strain in polar coordinates, and ρ is the radial coordinate of the point where shear stress and shear strain are defined.
- Equations (5.1), (5.2), and (5.3) are independent of material model.
- Torsional shear strain varies linearly with radial coordinate across the cross section.
- Torsional shear strain is maximum at the outer surface of the shaft.
- The formulas below are valid for shafts with material that is linear, elastic, and isotropic and has a homogeneous cross section:

$$\frac{d\phi}{dx} = \frac{T}{GJ} \quad (5.9) \qquad \tau_{x\theta} = \frac{T\rho}{J} \quad (5.10) \qquad \phi_2 - \phi_1 = \frac{T(x_2 - x_1)}{GJ} \quad (5.12)$$

- where G is the shear modulus of elasticity, and J is the polar moment of the cross section given by $J = (\pi/2)(R_o^4 - R_i^4)$, R_o and R_i being the outer and inner radii of a hollow shaft.
- The quantity GJ is called torsional rigidity.
- If T, G, or J *change* with x, we find the relative rotation of a cross section by integration of Equation (5.9).
- If T, G, and J *do not change* between x_1 and x_2, we use Equation (5.12) to find the relative rotation of a cross section.
- Torsional shear stress varies linearly with radial coordinate across the homogeneous cross section, reaching a maximum value on the outer surface of the shaft.

CHAPTER SIX
SYMMETRIC BENDING OF BEAMS

Learning objectives

1. Understand the theory of symmetric bending of beams, its limitations, and its applications for a strength-based design and analysis.
2. Visualize the direction of normal and shear stresses and the surfaces on which they act in the symmetric bending of beams.

On April 29th, 2007 at 3:45 AM, a tanker truck crashed into a pylon on interstate 80 near Oakland, California, spilling 8600 gallons of fuel that ignited. Fortunately no one died. But the heat generated from the ignited fuel, severely reduced the strength and stiffness of the steel beams of the interchange, causing it to collapse under its own weight (Figure 6.1*a*). In this chapter we will study the stresses, hence strength of beams. In Chapter 7 we will discuss deflection, hence stiffness of the beams.

Which structural member can be called a beam? Figure 6.1*b* shows a bookshelf whose length is much greater than its width or thickness, and the weight of the books is perpendicular to its length. Girders, the long horizontal members in bridges and highways transmit the weight of the pavement and traffic to the columns anchored to the ground, and again the weight is perpendicular to the member. Bookshelves and girders can be modeled as **beams**—long structural member on which loads act perpendicular to the longitudinal axis. The mast of a ship, the pole of a sign post, the frame of a car, the bulkheads in an aircraft, and the plank of a seesaw are among countless examples of beams.

The simplest theory for symmetric bending of beams will be developed rigorously, following the logic described in Figure 3.15, but subject to the limitations described in Section 3.13.

(a) (b)

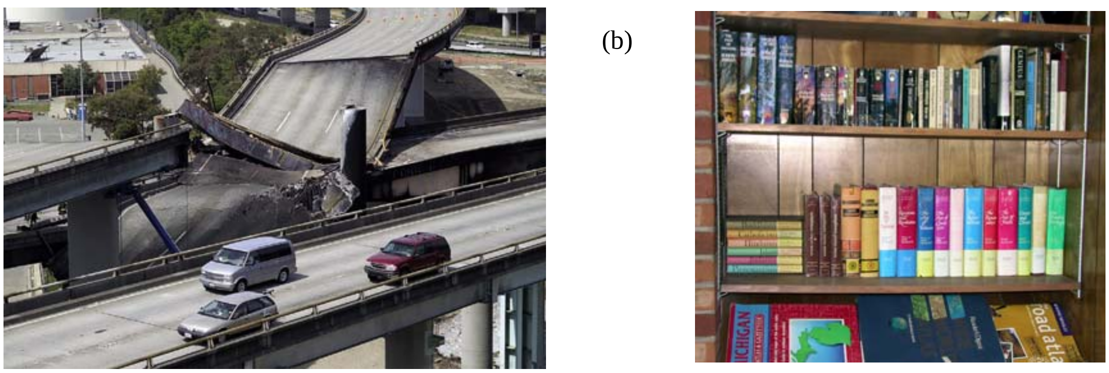


Figure 6.1 (a) I-80 interchange collapse. (b) Beam example.

6.1 PRELUDE TO THEORY

As a prelude to theory, we consider several examples, all solved using the logic discussed in Section 3.2. They highlight observations and conclusions that will be formalized in Section 6.2.

- Example 6.1, discrete bars welded to a rigid plate, illustrates how to calculate the bending normal strains from geometry.
- Example 6.2 shows the similarity of Example 6.1 to the calculation of normal strains for a continuous beam.
- Example 6.3 applies the logic described in Figure 3.15 to beam bending.
- Example 6.4 shows how the choice of a material model alters the calculation of the internal bending moment. As we saw in Chapter 5 for shafts, the material model affects only the stress distribution, leaving all other equations unaffected. Thus, the kinematic equation describing strain distribution is not affected. Neither are the static equivalency equations

between stress and internal moment and the equilibrium equations relating internal forces and moments. Although we shall develop the simplest theory using Hooke's law, most of the equations will apply to complex material models as well.

EXAMPLE 6.1

The left ends of three bars are built into a rigid wall, and the right ends are welded to a rigid plate, as shown in Figure 6.2. The undeformed bars are straight and perpendicular to the wall and the rigid plate. The rigid plate is observed to rotate due to the applied moment by an angle of 3.5° from the vertical plane. If the normal strain in bar 2 is zero, determine the normal strains in bars 1 and 3.

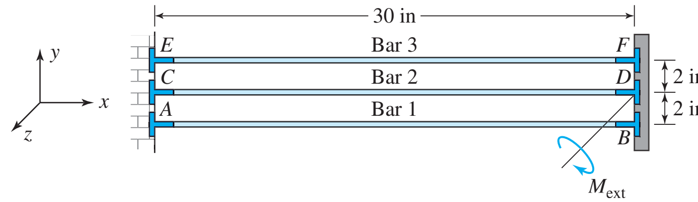


Figure 6.2 Geometry in Example 6.1.

METHOD 1: PLAN

The tangent to a circular arc is perpendicular to the radial line. If the bars are approximated as circular arcs and the wall and the rigid plate are in the radial direction, then the kinematic restriction of bars remaining perpendicular to the wall and plate is satisfied by the deformed shape. We can relate the angle subtended by the arc to the length of arc formed by *CD*, as we did in Example 2.3. From the deformed geometry, the strains of the remaining bars can be found.

SOLUTION

Figure 6.3 shows the deformed bars as circular arcs with the wall and the rigid plate in the radial direction. We know that the length of arc CD_1 is still 30 in., since it does not undergo any strain. We can relate the angle subtended by the arc to the length of arc formed by *CD* and calculate the radius of the arc *R* as

$$\psi = \left(\frac{3.5°}{180°}\right)(3.142 \text{ rad}) = 0.0611 \text{ rad} \qquad CD_1 = R\psi = 30 \text{ in.} \qquad \text{or} \qquad R = 491.1 \text{ in.} \tag{E1}$$

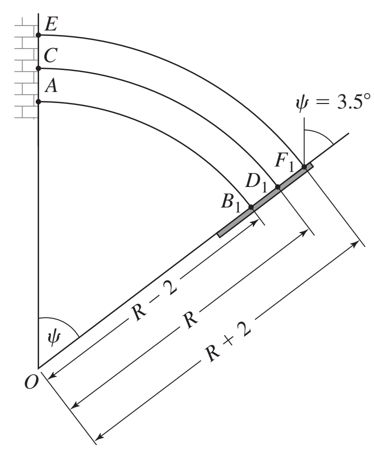


Figure 6.3 Normal strain calculations in Example 6.1.

The arc length AB_1 and EF_1 can be found using Figure 6.3 and the strains in bars 1 and 3 calculated.

$$AB_1 = (R-2)\psi = 29.8778 \text{ in.} \qquad \varepsilon_1 = \frac{AB_1 - AB}{AB} = \frac{-0.1222 \text{ in.}}{30 \text{ in.}} = -0.004073 \text{ in./in} \tag{E2}$$

ANS. $\varepsilon_1 = -4073\ \mu\text{in./in.}$

$$EF_1 = (R+2)\psi = 30.1222 \text{ in.} \qquad \varepsilon_3 = \frac{EF_1 - EF}{EF} = \frac{0.1222 \text{ in.}}{30 \text{ in.}} = 0.004073 \text{ in./in.} \tag{E3}$$

ANS. $\varepsilon_3 = 4073\ \mu\text{in./in.}$

COMMENT

1. In developing the theory for beam bending, we will view the cross section as a rigid plate that rotates about the *z* axis but stays perpendicular to the longitudinal lines. The longitudinal lines will be analogous to the bars, and bending strains can be calculated as in this example.

METHOD 2: PLAN

We can use small-strain approximation and find the deformation component in the horizontal (original) direction for bars 1 and 3. The normal strains can then be found.

SOLUTION

Figure 6.4 shows the rigid plate in the deformed position. The horizontal displacement of point D is zero as the strain in bar 2 is zero. Points B, D, and F move to B_1, D_1, and F_1 as shown. We can use point D_1 to find the relative displacements of points B and F as shown in Equations (E4) and (E5). We make use of small strain approximation to the sine function by its argument:

$$\Delta u_3 = DF_2 = D_2F_1 = D_1F_1 \sin\psi \approx 2\psi = 0.1222 \text{ in.} \tag{E4}$$

$$\Delta u_1 = B_2D = D_3D_1 = B_1D_1 \sin\psi \approx 2\psi = 0.1222 \text{ in.} \tag{E5}$$

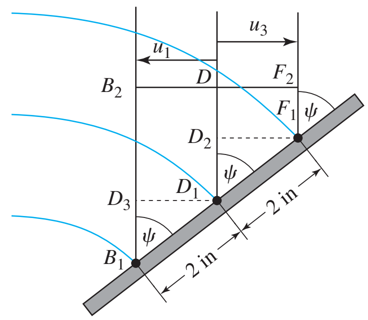


Figure 6.4 Alternate method for normal strain calculations in Example 6.1.

The normal strains in the bars can be found as

$$\varepsilon_1 = \frac{\Delta u_1}{30 \text{ in.}} = \frac{-0.1222 \text{ in.}}{30 \text{ in.}} = -0.004073 \text{ in./in.} \qquad \varepsilon_3 = \frac{\Delta u_3}{30 \text{ in.}} = \frac{0.1222 \text{ in.}}{30 \text{ in.}} = 0.004073 \text{ in./in.} \tag{E6}$$

ANS. $\varepsilon_1 = -4073 \ \mu\text{in./in.} \qquad \varepsilon_3 = 4073 \ \mu\text{in./in.}$

COMMENTS

1. Method 1 is intuitive and easier to visualize than method 2. But method 2 is computationally simpler. We will use both methods when we develop the kinematics in beam bending in Section 6.2.
2. Suppose that the normal strain of bar 2 was not zero but $\varepsilon_2 = 800\ \mu$in/in. What would be the normal strains in bars 1 and 3? We could solve this new problem as in this example and obtain $R = 491.5$ in., $\varepsilon_1 = -3272\ \mu$in./in, and $\varepsilon_3 = 4872\ \mu$in./in. Alternatively, we view the assembly was subjected to axial strain before the bending took place. We could then superpose the axial strain and bending strain to obtain $\varepsilon_1 = -4073 + 800 = -3273\ \mu$in./in. and $\varepsilon_3 = 4073 + 800 = 4873\ \mu$in./in. The superposition principle can be used only for linear systems, which is a consequence of small strain approximation, as observed in Chapter 2.

EXAMPLE 6.2

A beam made from hard rubber is built into a rigid wall at the left end and attached to a rigid plate at the right end, as shown in Figure 6.5. After rotation of the rigid plate the strain in line CD at $y = 0$ is zero. Determine the strain in line AB in terms of y and R, where y is the distance of line AB from line CD, and R is the radius of curvature of line CD.

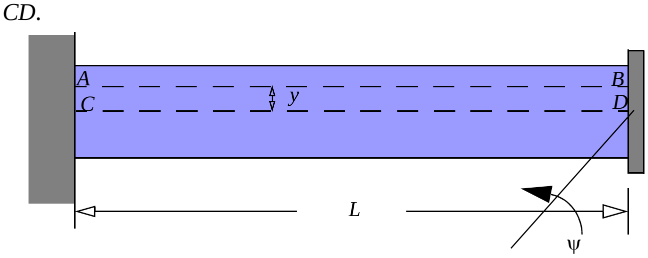


Figure 6.5 Beam geometry in Example 6.2.

PLAN

We visualize the beam as made up of bars, as in Example 6.1, but of infinitesimal thickness. We consider two such bars, AB and CD, and analyze the deformations of these two bars as we did in Example 6.1.

SOLUTION

Because of deformation, point B moves to point B_1 and point D moves to point D_1, as shown in Figure 6.6. We calculate the strain in AB:

$$_D = \frac{CD_1 - CD}{CD} = 0 \qquad \text{or} \qquad CD_1 = CD = R\psi = L \qquad \psi = \tag{E1}$$

$$AB_1 = (R - y)\psi = \frac{(R-y)L}{R} \qquad \varepsilon_{AB} = \frac{AB_1 - AB}{AB} = \frac{(R-y)L/R - L}{L} = \frac{L - yL/R - L}{L} \tag{E2}$$

ANS. $\varepsilon_{AB} = \dfrac{-y}{R}$

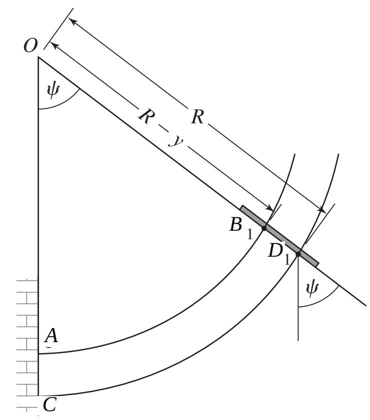


Figure 6.6 Exaggerated deformed geometry in Example 6.2.

COMMENTS

1. In Example 6.1, $R = 491.1$ and $y = +2$ for bar 3, and $y = -2$ for bar 1. On substituting these values into the preceding results, we obtain the results of Example 6.1.
2. Suppose the strain in *CD* were ε_{CD}. Then the strain in *AB* can be calculated as in comment 2 of Method 2 in Example 6.1 to obtain $\varepsilon_{AB} = \varepsilon_{CD} - y/R$. The strain ε_{CD} is the axial strain, and the remaining component is the normal strain due to bending.

EXAMPLE 6.3

The modulus of elasticity of the bars in Example 6.1 is 30,000 ksi. Each bar has a cross-sectional area $A = \frac{1}{2}$ in.^{2}. Determine the external moment M_{ext} that caused the strains in the bars in Example 6.1.

PLAN

Using Hooke's law, determine the stresses from the strains calculated in Example 6.1. Replace the stresses by equivalent internal axial forces. Draw the free-body diagram of the rigid plate and determine the moment M_{ext}.

SOLUTION

1. *Strain calculations:* The strains in the three bars as calculated in Example 6.1 are

$$\varepsilon_1 = -4073\ \mu\text{in./in.} \qquad \varepsilon_2 = 0 \qquad \varepsilon_3 = 4073\ \mu\text{in./in.} \tag{E1}$$

2. *Stress calculations:* From Hooke's law we obtain the stresses

$$\sigma_1 = E\varepsilon_1 = (30{,}000\ \text{ksi})(-4073)(10^{-6}) = 122.19\ \text{ksi (C)} \tag{E2}$$

$$\sigma_2 = E\varepsilon_2 = 0 \tag{E3}$$

$$\sigma_3 = E\varepsilon_3 = (30{,}000\ \text{ksi})(4073)(10^{-6}) = 122.19\ \text{ksi (T)} \tag{E4}$$

3. *Internal forces calculations:* The internal normal forces in each bar can be found as

$$\boldsymbol{N}_1 = \sigma_1 A = 61.095\ \text{kips (C)} \qquad \boldsymbol{N}_3 = \sigma_3 A = 61.095\ \text{kips (T)} \tag{E5}$$

4. *External moment calculations:* Figure 6.7 is the free body diagram of the rigid plate. By equilibrium of moment about point O we can find M_z:

$$M_z = \boldsymbol{N}_1(y) + \boldsymbol{N}_3(y) = (61.095\ \text{kips})(2\ \text{in.}) + (61.095\ \text{kips})(2\ \text{in.}) \tag{E6}$$

ANS. $M_z = 244.4\ \text{in.} \cdot \text{kips}$

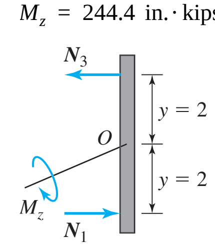


Figure 6.7 Free-body diagram in Example 6.3.

COMMENTS

1. The sum in Equation (E6) can be rewritten $\sum_{i=1}^{2} y\sigma\Delta A_i$, where σ is the normal stress acting at a distance y from the zero strain bar, and ΔA_i is the cross-sectional area of the i^{th} bar. If we had n bars attached to the rigid plate, then the moment would be given by $\sum_{i=1}^{n} y\sigma\Delta A_i$. As we increase the number of bars n to infinity, the cross-sectional area ΔA_i tends to zero, becoming the infinitesimal area dA and the summation is replaced by an integral. In effect, we are fitting an infinite number of bars to the plate, resulting in a continuous body.
2. The total axial force in this example is zero because of symmetry. If this were not the case, then the axial force would be given by the summation $\sum_{i=1}^{n} \sigma\Delta A_i$. As in comment 1, this summation would be replaced by an integral as n tends to infinity, as will be shown in Section 6.1.1.

6.1.1 Internal Bending Moment

In this section we formalize the observation made in Example 6.3: that is, the normal stress σ_{xx} can be replaced by an equivalent bending moment using an integral over the cross-sectional area. Figure 6.8 shows the normal stress distribution σ_{xx} to be replaced by an equivalent internal bending moment $\boldsymbol{M_z}$. Let y represent the coordinate at which the normal stress acts. Static equivalency in Figure 6.8 results in

$$\boxed{\boldsymbol{M}_z = -\int_A y\sigma_{xx}\, dA} \tag{6.1}$$

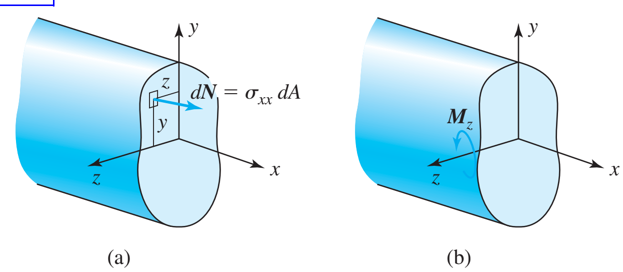


Figure 6.8 Statically equivalent internal moment.

Figure 6.8*a* suggests that for static equivalency there should be an axial force $\boldsymbol{N}$ and a bending moment about the y axis $\boldsymbol{M}_y$. However, the requirement of symmetric bending implies that the normal stress σ_{xx} is symmetric about the axis of symmetry—that is, the y axis. Thus $\boldsymbol{M}_y$ is implicitly zero owing to the limitation of symmetric bending. Our desire to study bending independent of axial loading requires that the stress distribution be such that the internal axial force $\boldsymbol{N}$ should be zero. Thus we must explicitly satisfy the condition

$$\boxed{\int_A \sigma_{xx}\, dA = 0} \tag{6.2}$$

Equation (6.2) implies that the stress distribution across the cross section must be such that there is no net axial force. That is, the compressive force must equal the tensile force on a cross section in bending. If stress is to change from compression to tension, then there must be a location of zero normal stress in bending. The line on the cross section where the bending normal stress is zero is called **neutral axis**.

Equations (6.1) and (6.2) are independent of the material model. That is because they represent static equivalency between the normal stress on the entire cross section and the internal moment. If we were to consider a composite beam cross section or a nonlinear material model, then the value and distribution of σ_{xx} would change across the cross section yet Equation (6.1) relating σ_{xx} to $\boldsymbol{M_z}$ would remain unchanged. Example 6.4 elaborates on this idea. The origin of the y coordinate is located at the neutral axis irrespective of the material model. Hence, determining the location of the neutral axis is critical in all bending problems. The location of the origin will be discussed in greater detail for a homogeneous, linearly elastic, isotropic material in Section 6.2.4.

EXAMPLE 6.4

Figure 6.9 shows a homogeneous wooden cross section and a cross section in which the wood is reinforced with steel. The normal strain for both cross sections is found to vary as $\varepsilon_{xx} = -200y\ \mu$. The moduli of elasticity for steel and wood are $E_{steel} = 30{,}000$ ksi and $E_{wood} = 8000$ ksi. (a) Write expressions for normal stress σ_{xx} as a function of y, and plot the σ_{xx} distribution for each of the two cross sections shown. (b) Calculate the equivalent internal moment $\boldsymbol{M_z}$ for each cross section.

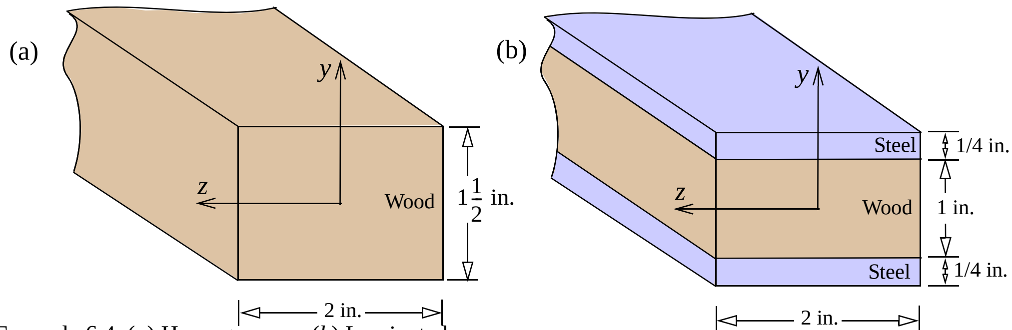


Figure 6.9 Cross sections in Example 6.4. (*a*) Homogeneous. (*b*) Laminated.

PLAN

(a) From the given strain distribution we can find the stress distribution by Hooke's law. We note that the problem is symmetric and stresses in each region will be linear in y. (b) The integral in Equation (6.1) can be written as twice the integral for the top half since the stress distribution is symmetric about the center. After substituting the stress as a function of y in the integral, we can perform the integration to obtain the equivalent internal moment.

SOLUTION

(a) From Hooke's law we can write the stress in each material as

$$(\sigma_{xx})_{wood} = (8000\text{ ksi})(-200y)10^{-6} = -1.6y\text{ ksi} \tag{E1}$$

$$(\sigma_{xx})_{steel} = (30000\text{ ksi})(-200y)10^{-6} = -6y\text{ ksi} \tag{E2}$$

For the homogeneous cross section the stress distribution is given in Equation (E1), but for the laminated case it switches from Equation (E1) to Equation (E2), depending on the value of y. We can write the stress distribution for both cross sections as a function of y.

Homogeneous cross section:

$$\sigma_{xx} = -1.6y\text{ ksi} \qquad -0.75\text{ in.} \le y < 0.75\text{ in.} \tag{E3}$$

Laminated cross section:

$$\sigma_{xx} = \begin{cases} -6y\text{ ksi} & 0.5\text{ in.} < y \le 0.75\text{ in.} \\ -1.6y\text{ ksi} & -0.5\text{ in.} < y < 0.5\text{ in.} \\ -6y\text{ ksi} & -0.75\text{ in.} \le y < -0.5\text{ in.} \end{cases} \tag{E4}$$

Using Equations (E3) and (E4) the strains and stresses can be plotted as a function of y, as shown in Figure 6.10.

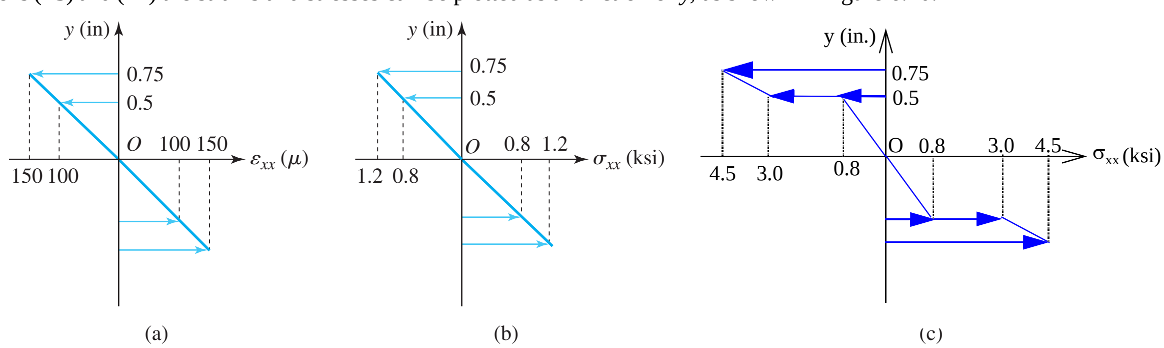


Figure 6.10 Strain and stress distributions in Example 6.4: (*a*) strain distribution; (*b*) stress distribution in homogeneous cross section; (*c*) stress distribution in laminated cross section.

(b) The thickness (dimension in the z direction) is 2 in. Hence we can write $dA = 2dy$. Noting that the stress distribution is symmetric, we can write the integral in Equation (6.1) as

$$M_z = -\int_{-0.75}^{0.75} y\sigma_{xx}(2dy) = -2\left[\int_0^{0.75} y\sigma_{xx}(2dy)\right] \tag{E5}$$

Homogeneous cross section: Substituting Equation (E3) into Equation (E5) and integrating, we obtain the equivalent internal moment.

$$M_z = -2\left[\int_0^{0.75} y(-1.6y \text{ ksi})(2dy)\right] = 6.4\frac{y^3}{3}\bigg|_0^{0.75} = 6.4\frac{0.75^3}{3} \tag{E6}$$

ANS. $\boldsymbol{M_z} = 0.9 \text{ in.}\cdot\text{kips}$

Laminated cross section: Substituting Equation (E4) into Equation (E5) and integrating, we obtain the equivalent internal moment.

$$M_z = -2\left[\int_0^{0.5} y(-1.6y)(2dy) + \int_{0.5}^{0.75} y(-6y)(2dy)\right] = 4\left(1.6\frac{y^3}{3}\bigg|_0^{0.5} + 6\frac{y^3}{3}\bigg|_{0.5}^{0.75}\right) \tag{E7}$$

ANS. $\boldsymbol{M_z} = 2.64 \text{ in.}\cdot\text{kips}$

COMMENTS

1. As this example demonstrates, although the strain varies linearly across the cross section, the stress may not. In this example we considered material nonhomogeneity. In a similar manner we can consider other models, such as elastic–perfectly plastic model, or material models that have nonlinear stress–strain curves.
2. Figure 6.11 shows the stress distribution on the surface. The symmetry of stresses about the center results in a zero axial force.

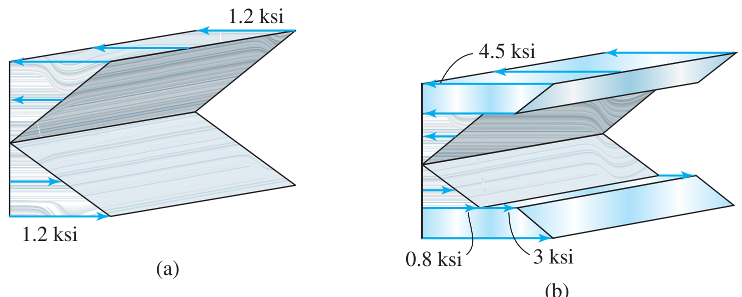


Figure 6.11 Surface stress distributions in Example 6.4 for (*a*) homogeneous cross section; (*b*) laminated cross section.

3. We can obtain the equivalent internal moment for a homogeneous cross section by replacing the triangular load by an equivalent load at the centroid of each triangle. We then find the equivalent moment, as shown in Figure 6.12. This approach is very intuitive. However, as the stress distribution becomes more complex, such as in a laminated cross section, or for more complex cross-sectional shapes, this intuitive approach becomes very tedious. The generalization represented by Equation (6.1) and the resulting formula can then simplify the calculations.

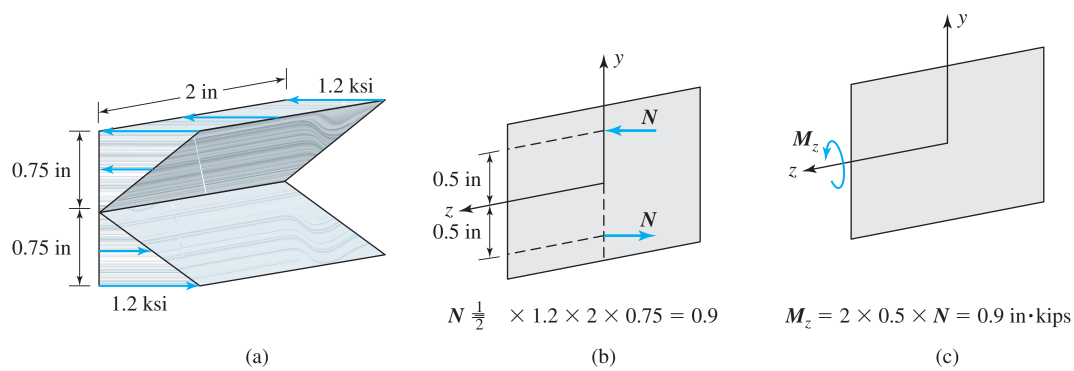


Figure 6.12 Statically equivalent internal moment in Example 6.4.

4. The relationship between the internal moment and the external loads can be established by drawing the appropriate free-body diagram for a particular problem. The relationship between internal and external moments depends on the free-body diagram and is independent of the material homogeneity.

PROBLEM SET 6.1

6.1 The rigid plate that is welded to the two bars in Figure P6.1 is rotated about the z axis, causing the two bars to bend. The normal strains in bars 1 and 2 were found to be $\varepsilon_1 = 2000\ \mu$in./in. and $\varepsilon_2 = -1500\ \mu$in./in. Determine the angle of rotation ψ from the vertical plane. .

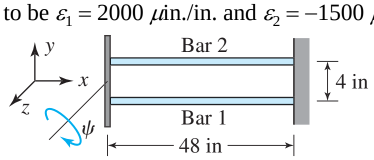


Figure P6.1

6.2 Determine the location h in Figure P6.2 at which a third bar in Problem 6.1 must be placed so that there is no normal strain in the third bar.

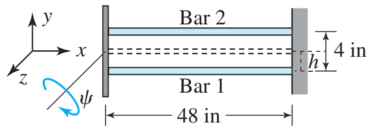


Figure P6.2

6.3 The two rigid plates that are welded to six bars in Figure P6.3 are rotated about the z axis, causing the six bars to bend. The normal strains in bars 2 and 5 were found to be zero. What are the strains in the remaining bars?

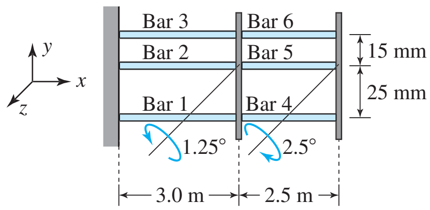


Figure P6.3

6.4 The strains in bars 1 and 3 in Figure P6.4 were found to be $\varepsilon_1 = 800\ \mu$ and $\varepsilon_3 = 500\ \mu$. Determine the strains in the remaining bars.

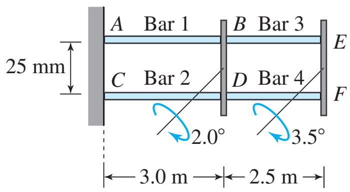


Figure P6.4

6.5 The rigid plate shown in Figure P6.5 was observed to rotate by 2° from the vertical plane due to the action of the external moment M_{ext} and force P, and the normal strain in bar 1 was found to be $\varepsilon_1 = 2000\ \mu$in./in. Both bars have a cross-sectional area $A = \frac{1}{2}$ in.^{2} and a modulus of elasticity $E = 30{,}000$ ksi. Determine the applied moment M_{ext} and force P.

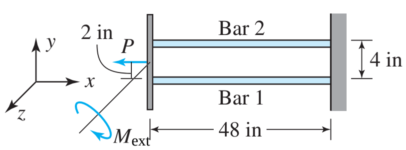


Figure P6.5

6.6 The rigid plate shown in Figure P6.6 was observed to rotate 1.25° from the vertical plane due to the action of the external moment M_{ext} and the force P. All three bars have a cross-sectional area $A = 100$ mm^2 and a modulus of elasticity $E = 200$ GPa. If the strain in bar 2 was measured as zero, determine the external moment M_{ext} and the force P.

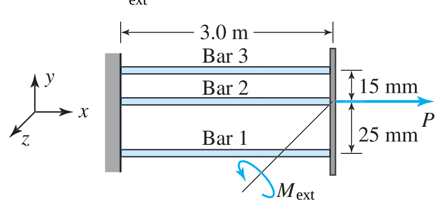


Figure P6.6

6.7 The rigid plates BD and EF in Figure P6.7 were observed to rotate by 2° and 3.5° from the vertical plane in the direction of applied moments. All bars have a cross-sectional area of $A = 125$ mm^2. Bars 1 and 3 are made of steel $E_S = 200$ GPa, and bars 2 and 4 are made of aluminum $E_{al} = 70$ GPa. If the strains in bars 1 and 3 were found to be $\varepsilon_1 = 800\ \mu$ and $\varepsilon_3 = 500\ \mu$ determine the applied moment M_1 and M_2 and the forces P_1 and P_2 that act at the center of the rigid plates.

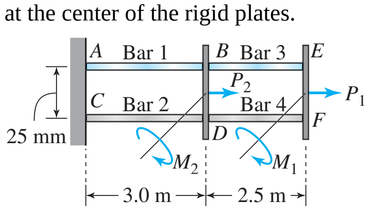


Figure P6.7

6.8 Three wooden beams are glued to form a beam with the cross-section shown in Figure 6.8. The normal strain due to bending about the z axis is ε_{xx} = -0.012y, where y is measured in meters. The modulus of elasticity of wood is 10 GPa. Determine the equivalent internal moment acting at the cross-section. Use t_W =20 mm, h =250 mm, t_F = 20 mm, and d= 125 mm.

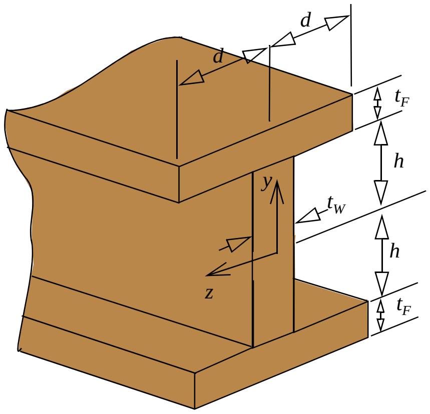


Figure P6.8

6.9 Three wooden beams are glued to form a beam with the cross-section shown in Figure 6.8. The normal strain at the cross due to bending about the z axis is ε_{xx} = -0.15y, where y is measured in meters. The modulus of elasticity of wood is 10 GPa. Determine the equivalent internal moment acting at the cross-section. Use t_W =10 mm, h =50 mm, t_F = 10 mm, and d= 25 mm.

6.10 Three wooden beams are glued to form a beam with the cross-section shown in Figure 6.8. The normal strain at the cross due to bending about the z axis is ε_{xx} = 0.02y, where y is measured in meters. The modulus of elasticity of wood is 10 GPa. Determine the equivalent internal moment acting at the cross-section. Use t_W =15 mm, h =200 mm, t_F = 20 mm, and d= 150 mm.

6.11 Steel strips (E_S = 30,000 ksi) are securely attached to wood (E_W = 2000 ksi) to form a beam with the cross section shown in Figure P6.11. The normal strain at the cross section due to bending about the z axis is $\varepsilon_{xx} = -100y\ \mu$, where y is measured in inches. Determine the equivalent internal moment $\boldsymbol{M}_z$. Use d = 2 in., h_W = 4 in., and $h_S = \frac{1}{8}$ in.

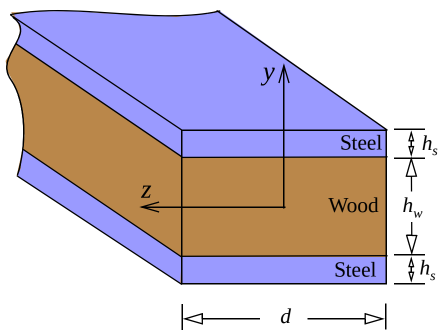


Figure P6.11

6.12 Steel strips (E_S = 30,000 ksi) are securely attached to wood (E_W = 2000 ksi) to form a beam with the cross section shown in Figure P6.11 . The normal strain at the cross section due to bending about the z axis is $\varepsilon_{xx} = -50y\ \mu$, where y is measured in inches. Determine the equivalent internal moment $\boldsymbol{M}_z$. Use d = 1 in., h_W = 6 in., and $h_S = \frac{1}{4}$ in.

6.13 Steel strips (E_S = 30,000 ksi) are securely attached to wood (E_W = 2000 ksi) to form a beam with the cross section shown in Figure P6.11 . The normal strain at the cross section due to bending about the z axis is $\varepsilon_{xx} = 200y\ \mu$, where y is measured in inches. Determine the equivalent internal moment $\boldsymbol{M}_z$. Use d = 1 in., h_W = 2 in., and $h_S = \frac{1}{16}$ in.

6.14 Steel strips (E_S = 200 GPa) are securely attached to wood (E_W = *10 GPa)* to form a beam with the cross section shown in Figure P6.14. The normal strain at the cross section due to bending about the *z axis is* $\varepsilon_{xx} = 0.02y$, where *y* is measured in meters. Determine the equivalent internal moment $\boldsymbol{M}_z$ .Use t_W = 15 mm, h_W = 200 mm, t_F = 20 mm, and d_F = 150 mm.

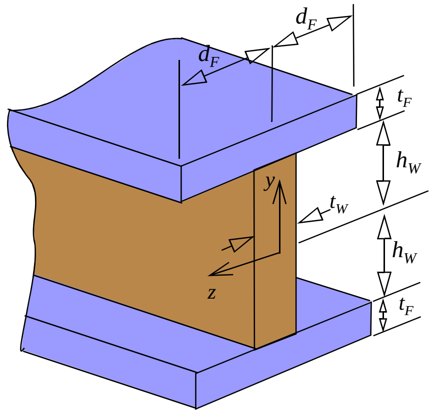


Figure P6.14

Stretch Yourself

6.15 A beam of rectangular cross section shown in Figure 6.15 is made from elastic-perfectly plastic material. If the stress distribution across the cross section is as shown determine the equivalent internal bending moment.

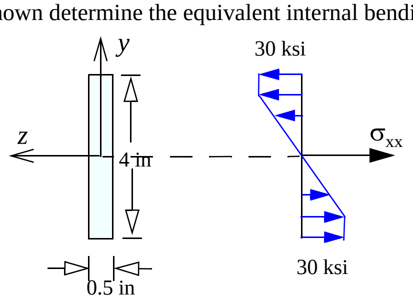


Figure P6.15

6.16 A rectangular beam cross section has the dimensions shown in Figure 6.16. The normal strain due to bending about the *z* axis was found to vary as $\varepsilon_{xx} = -0.01y$, with *y* measured in meters. Determine the equivalent internal moment that produced the given state of strain. The beam is made from elastic-perfectly plastic material that has a yield stress of σ_{yield}= 250 MPa and a modulus of elasticity E = 200 GPa. Assume material that the behaves in a similar manner in tension and compression (see Problem 3.152)

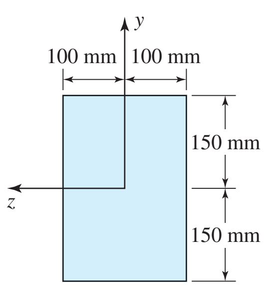


Figure P6.16

6.17 A rectangular beam cross section has the dimensions shown in Figure 6.16. The normal strain due to bending about the *z* axis was found to vary as $\varepsilon_{xx} = -0.01y$, with *y* measured in meters. Determine the equivalent internal moment that would produce the given strain. The beam is made from a bi-linear material that has a yield stress of σ_{yield}= 200 MPa, modulus of elasticity E_1 = 250 GPa, and E_2= 80 GPa. Assume that the material behaves in a similar manner in tension and compression (see Problem 3.153).

6.18 A rectangular beam cross section has the dimensions shown in Figure 6.16. The normal strain due to bending about the *z* axis was found to vary as $\varepsilon_{xx} = -0.01y$, with *y* measured in meters. Determine the equivalent internal moment that would produce the given strain. The beam material has a stress strain relationship given by $\sigma = 952\varepsilon^{0.2}$ MPa . Assume that the material behaves in a similar manner in tension and compression (see Problem 3.154).

6.2 THEORY OF SYMMETRIC BEAM BENDING

In this section we develop formulas for beam deformation and stress. We follow the procedure in Section 6.1 with variables in place of numbers. The theory will be subject to the following limitations:

1. The length of the member is significantly greater than the greatest dimension in the cross section.
2. We are away from the regions of stress concentration;
3. The variation of external loads or changes in the cross-sectional areas are gradual except in regions of stress concentration.
4. The cross section has a plane of symmetry. This limitation separates bending about the z axis from bending about the y axis. (See Problem 6.135 for unsymmetric bending.)
5. The loads are in the plane of symmetry. Load P_1 in Figure 6.13 would bend the beam as well as twist (rotate) the cross section. Load P_2, which lies in the plane of symmetry, will cause only bending. Thus, this limitation decouples the bending problem from the torsion problem[1].
6. The load direction does not change with deformation. This limitation is required to obtain a linear theory and works well as long as the deformations are small.
7. The external loads are not functions of time; that is, we have a static problem. (See Problems 7.50 and 7.51 for dynamic problems.)

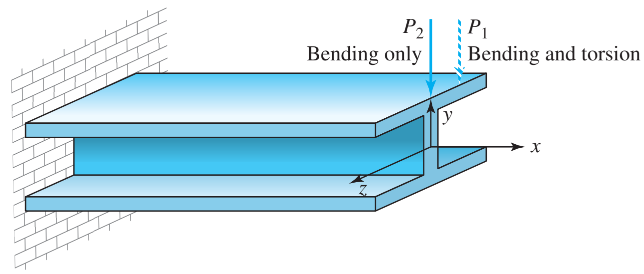


Figure 6.13 Loading in plane of symmetry.

Figure 6.14 shows a segment of a beam with the x–y plane as the plane of symmetry. The beam is loaded by transverse forces P_1 and P_2 in the y direction, moments M_1 and M_2 about the z axis, and a transverse distributed force $p_y(x)$. The distributed force $p_y(x)$ has units of force per unit length and is considered positive in the positive y direction. Because of external loads, a line on the beam deflects by v in the y direction.

The objectives of the derivation are:

1. To obtain a formula for bending normal stress σ_{xx} and bending shear stress τ_{xy} in terms of the internal moment $\boldsymbol{M_z}$ and the internal shear force $\boldsymbol{V_y}$.
2. To obtain a formula for calculating the beam deflection v(x).

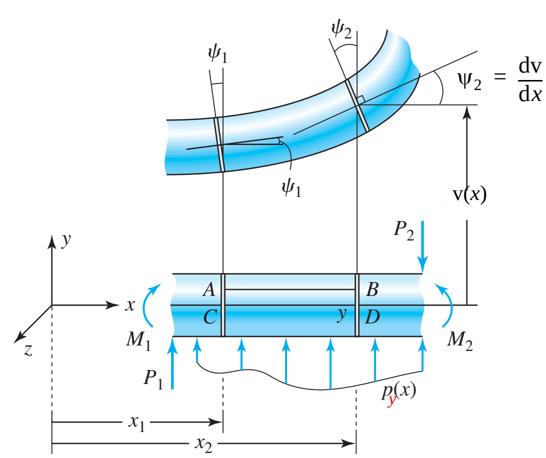


Figure 6.14 Beam segment.

[1]The separation of torsion from bending requires that the load pass through the *shear center*, which always lies on the axis of symmetry.

To account for the gradual variation of $p_y(x)$ and the cross-sectional dimensions, we will take $\Delta x = x_2 - x_1$ as infinitesimal distance in which these quantities can be treated as constants. The logic shown in Figure 6.15 and discussed in Section 3.2 will be used to develop the simplest theory for the bending of beams. Assumptions will be identified as we move from one step to the next. The assumptions identified as we move from each step are also points at which complexities can later be added, as discussed in examples and Stretch Yourself problems.

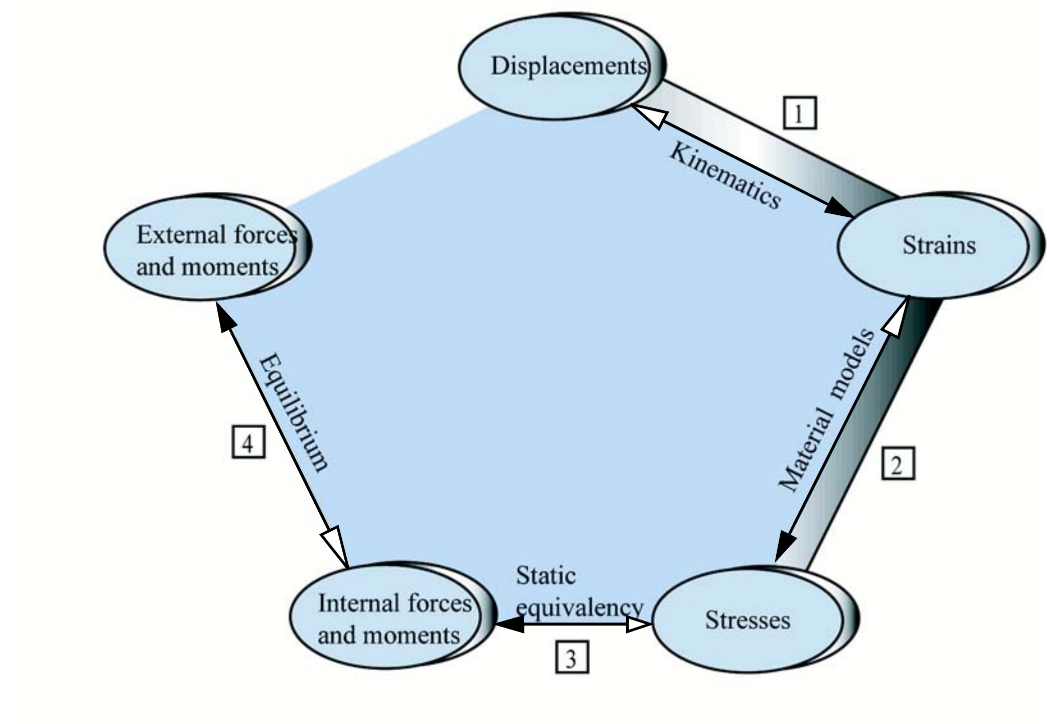


Figure 6.15 Logic in mechanics of materials.

6.2.1 Kinematics

In Example 6.1 we found the normal strains in bars welded to rigid plates rotating about the *z* axis. Here we state assumptions that will let us simulate the behavior of a cross section like that of the rigid plate. We will consider the experimental evidence justifying our assumptions and the impact of these assumptions on the theory.

Assumption 1: Squashing—that is, dimensional changes in the *y* direction—is significantly smaller than bending.
Assumption 2: Plane sections before deformation remain planes after deformation.
Assumption 3: Plane sections perpendicular to the beam axis remain *nearly* perpendicular after deformation.

Figure 6.16 shows a rubber beam with a grid on its surface that is bent by hand. Notice that the dimensional changes in the *y* direction are significantly smaller than those in the *x* direction, the basis of Assumption 1. The longer the beam, the better is the validity of Assumption 1. Neglecting dimensional changes in the *y* direction implies that the normal strain in the *y* direction is small[2] and can be neglected in the kinematic calculations; that is, $\varepsilon_{yy} = \partial v/\partial y \approx 0$. This implies that deflection of the beam v cannot be a function of *y*:

$$v = v(x) \tag{6.3}$$

Equation (6.3) implies that if we know the curve of one longitudinal line on the beam, then we know how all other longitudinal lines on the beam bend. The curve described by v(*x*), called the **elastic curve**, will be discussed in detail in the next chapter.

[2]It is accounted for as the *Poisson effect*. However the normal strain in the *y* direction is not an independent variable and hence is negligible in kinematics.

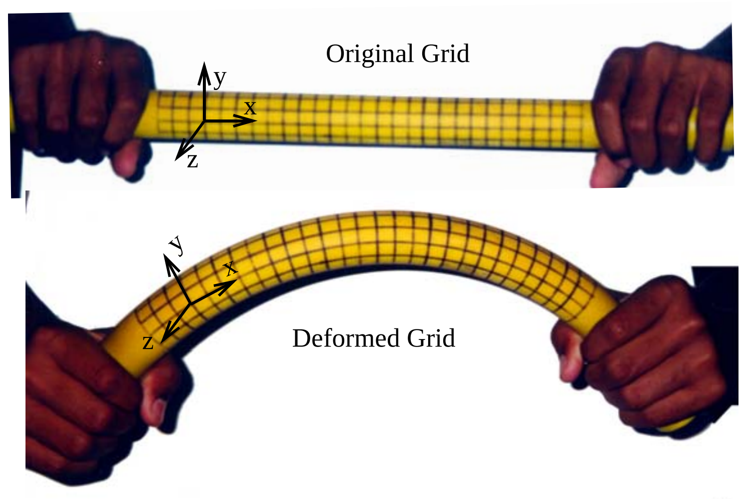


Figure 6.16 Deformation in bending. (Courtesy Professor J. B. Ligon.)

Figure 6.16 shows that lines initially in the *y* direction continue to remain straight but rotate about the *z* axis, validating Assumption 2. This implies that the displacement *u* varies linearly, as shown in Figure 6.17. In other words, the equation for *u* is

$$u = u_0 - \psi y \tag{6.4}$$

where u_0 is the axial displacement at $y = 0$ and ψ is the slope of the plane. (We accounted for uniform axial displacement u_0 in Chapter 4.) In order to study each problem independently, we will assume $u_0 = 0$. (See Problem 6.133 for $u_0 \neq 0$.)

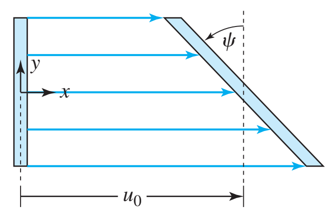


Figure 6.17 Linear variation of axial displacement *u*.

Figure 6.16 also shows that the right angle between the *x* and *y* directions is nearly preserved during bending, validating Assumption 3. This implies that the shear strain γ_{xy} is nearly zero. We cannot use this assumption in building theoretical models of beam bending if shear is important, such as in *sandwich beams* (see comment 3 in Example 6.7) and *Timoshenko beams* (see Problem 7.49). But Assumption 3 helps simplify the theory as it eliminates the variable ψ by imposing the constraint that the angle between the longitudinal direction and the cross section be always 90°. This is accomplished by relating ψ to v as described next.

6.2.2 Strain Distribution

Assumption 4: Strains are small.

The bending curve is defined by v(*x*). As shown in Figure 6.14, the angle of the tangent to the curve v(*x*) is equal to the rotation of the cross section when Assumption 3 is valid. For small strains, the tangent of an angle can be replaced by the angle itself, that is, $\tan\psi \approx \psi = dv/dx$. Substituting ψ and $u_0 = 0$ in Equation (6.4), we obtain

$$\boxed{u = -y\frac{dv}{dx}(x)} \tag{6.5}$$

Figure 6.18 shows the exaggerated deformed shape of a segment of the beam. The rotation of the right cross section is taken relative to the left. Thus, the left cross section is viewed as a fixed wall, as in Examples 6.1 and 6.2. We assume that line *CD* representing $y = 0$ has zero bending normal strain. The calculations of Example 6.2 show that the bending normal strain for line *AB* is given by

$$\varepsilon_{xx} = -\frac{y}{R} \tag{6.6a}$$

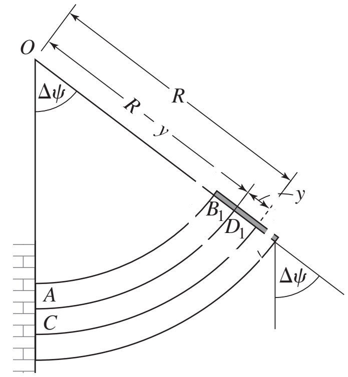


Figure 6.18 Normal strain calculations in symmetric bending.

We can also obtain the equation of bending normal strain by substituting Equation (6.5) into Equation (2.12a) to obtain

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-y\frac{dv}{dx}(x)\right) \qquad \text{or}$$

$$\boxed{\varepsilon_{xx} = -y\frac{d^2 v}{dx^2}(x)} \tag{6.6b}$$

Equations 6.6a and 6.6b show that the bending normal strain ε_{xx} varies linearly with y and has a maximum value at either the top or the bottom of the beam. d^2v/dx^2 is the **curvature** of the beam, and its magnitude is equal to $1/R$, where R is the *radius of curvature*.

6.2.3 Material Model

In order to develop a simple theory for the bending of symmetric beams, we shall use the material model given by Hooke's law. We therefore make the following assumptions regarding the material behavior.

Assumption 5: The material is isotropic.
Assumption 6: The material is linearly elastic.[3]
Assumption 7: There are no inelastic strains.[4]

Substituting Equation (6.6b) into Hooke's law $\sigma_{xx} = E\varepsilon_{xx}$, we obtain

$$\sigma_{xx} = -Ey\frac{d^2 v}{dx^2} \tag{6.7}$$

Though the strain is a linear function of y, we cannot say the same for stress. The modulus of elasticity E could change across the cross section, as in laminated structures.

6.2.4 Location of Neutral Axis

Equation (6.7) shows that the stress σ_{xx} is a function of y, and its value must be zero at $y = 0$. That is, the origin of y must be at the neutral axis. But where is the neutral axis on the cross section? Section 6.1.1 noted that the distribution of σ_{xx} is such that the total tensile force equals the total compressive force on a cross section, given by Equation (6.2). d^2v/dx^2 is a function of x only, whereas the integration is with respect to y and z ($dA = dy\,dz$). Substituting Equation (6.7) into Equation (6.2), we obtain

[3]See Problems 6.57 and 6.58 for nonlinear material behavior.
[4]Inelastic strains could be due to temperature, humidity, plasticity, viscoelasticity, etc. See Problem 6.134 for including thermal strains.

$$-\int_A Ey\frac{d^2\text{v}}{dx^2}(x)\,dA = -\frac{d^2\text{v}}{dx^2}(x)\int_A Ey\;dA = 0 \tag{6.8a}$$

The integral in Equation (6.8a) must be zero as shown in Equation (6.8b), because a zero value of $d^2\text{v}/dx^2$ would imply that there is no bending.

$$\int_A yE\;dA = 0 \tag{6.8b}$$

Equation (6.8b) is used for determining the origin (and thus the neutral axis) in composite beams. Consistent with the motivation of developing the simplest possible formulas, we would like to take E outside the integral. In other words, E should not change across the cross section, as implied in Assumption 8:

Assumption 8: The material is homogeneous across the cross section[5] of a beam.

Equation (6.8b) can be written as

$$E\int_A y\;dA = 0 \tag{6.8c}$$

In Equation (6.8c) either E or $\int_A y\;dA$ must be zero. As E cannot be zero, we obtain

$$\boxed{\int_A y\;dA = 0} \tag{6.9}$$

Equation (6.9) is satisfied if y is measured from the centroid of the cross section. That is, the origin must be at the centroid of the cross section of a linear, elastic, isotropic, and homogeneous material. Equation (6.9) is the same as Equation (4.12a) in axial members. However, in axial problems we required that the internal bending moment that generated Equation (4.12a) be zero. Here it is zero axial force that generates Equation (6.9). Thus by choosing the origin to be the centroid, we decouple the axial problem from the bending problem.

From Equations (6.7) and (6.9) two conclusion follow for cross sections constructed from linear, elastic, isotropic, and homogeneous material:

- The bending normal stress σ_{xx} varies linearly with y.
- The bending normal stress σ_{xx} has maximum value at the point farthest from the centroid of the cross section.

The point farthest from the centroid is the top surface or the bottom surface of the beam. Example 6.5 demonstrates the use of our observations.

EXAMPLE 6.5

The maximum bending normal strain on a homogeneous steel (E = 30,000 ksi) cross section shown in Figure 6.19 was found to be ε_{xx} = +1000 μ. Determine the bending normal stress at point A.

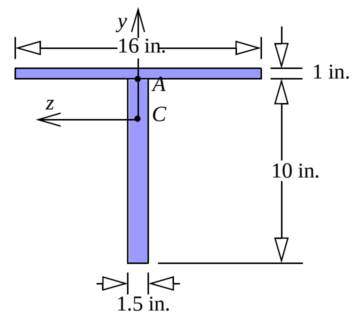


Figure 6.19 T cross section in Example 6.5.

[5]See Problems 6.55 and 6.56 on composite beams for nonhomogeneous cross sections.

PLAN

The centroid C of the cross section can be found where the bending normal stress is zero. The maximum bending normal stress will be at the point farthest from the centroid. Its value can be found from the given strain and Hooke's law. Knowing the normal stress at two points of a linear distribution, we can find the normal stress at point A.

SOLUTION

Figure 6.20a can be used to find the centroid η_c of the cross section.

$$\eta_c = \frac{\sum_i \eta_i A_i}{\sum_i A_i} = \frac{(5\text{ in.})(10\text{ in.})(1.5\text{ in.}) + (10.5\text{ in.})(16\text{ in.})(1\text{ in.})}{(10\text{ in.})(1.5\text{ in.}) + (16\text{ in.})(1\text{ in.})} = 7.84\text{ in.} \tag{E1}$$

The maximum bending normal stress will be at point B, which is farthest from centroid, and its value can be found as

$$\sigma_B = E\varepsilon_{max} = (30{,}000\text{ ksi})(1000)(10^{-6}) = 30\text{ ksi} \tag{E2}$$

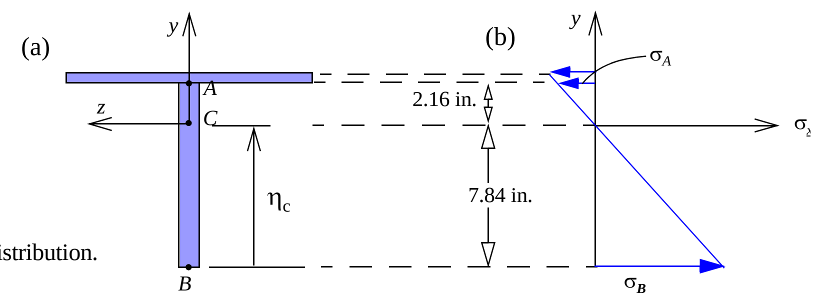


Figure 6.20 (a) Centroid location (b) Linear stress distribution.

The linear distribution of bending normal stress across the cross section can be drawn as shown in Figure 6.20b. By similar triangles we obtain

$$\frac{\sigma_A}{2.16\text{ in.}} = \frac{30\text{ ksi}}{7.84\text{ in.}} \tag{E3}$$

ANS. $\sigma_A = 8.27\text{ ksi (C)}$

COMMENT

1. The stress distribution in Figure 6.20b can be represented as $\sigma_{xx} = -3.82y$ ksi. The equivalent internal moment can be found using Equation (6.1).

6.2.5 Flexure Formulas

Note that d^2v/dx^2 is a function of x only, while integration is with respect to y and z ($dA = dy\,dz$). Substituting σ_{xx} from Equation (6.8b) into Equation (6.1), we therefore obtain

$$M_z = \int_A Ey^2 \frac{d^2v}{dx^2}\,dA = \frac{d^2v}{dx^2}\int_A Ey^2\,dA \tag{6.10}$$

With material homogeneity (Assumption 8), we can take E outside the integral in Equation (6.10) to obtain

$$M_z = E\frac{d^2v}{dx^2}\int_A y^2\,dA \quad \text{or}$$

$$\boxed{M_z = EI_{zz}\frac{d^2v}{dx^2}} \tag{6.11}$$

where $I_{zz} = \int_A y^2\,dA$ is the second area moment of inertia about the z axis passing through the centroid of the cross section. The quantity EI_{zz} is called the **bending rigidity** of a beam cross section. The higher the value of EI_{zz}, the smaller will be the deformation (curvature) of the beam; that is, the beam rigidity increase. A beam can be made more rigid either by choosing a stiffer material (a higher value of E) or by choosing a cross sectional shape that has a large area moment of inertia (see Example 6.7).

Solving for d^2v/dx^2 in Equation (6.11) and substituting into Equation (6.7), we obtain the *bending* stress formula or *flexure* stress formula:

$$\boxed{\sigma_{xx} = -\frac{M_z y}{I_{zz}}} \tag{6.12}$$

The subscript z emphasizes that the bending occurs about the z axis. If bending occurs about the y axis, then y and z in Equation (6.12) are interchanged, as elaborated in Section 10.1 on combined loading.

6.2.6 Sign Conventions for Internal Moment and Shear Force

Equation (6.1) allowed us to replace the normal stress σ_{xx} by a statically equivalent internal bending moment. The normal stress σ_{xx} is positive on two surfaces; hence the equivalent internal bending moment is positive on two surfaces, as shown in Figure 6.21. If we want the formulas to give the correct signs, then we must follow a sign convention for the internal moment when we draw a free body diagram: At the imaginary cut the internal bending moment must be drawn in the positive direction.

> **Sign Convention:** The direction of positive internal moment $\boldsymbol{M}_z$ on a free-body diagram must be such that it puts a point in the positive y direction into compression.

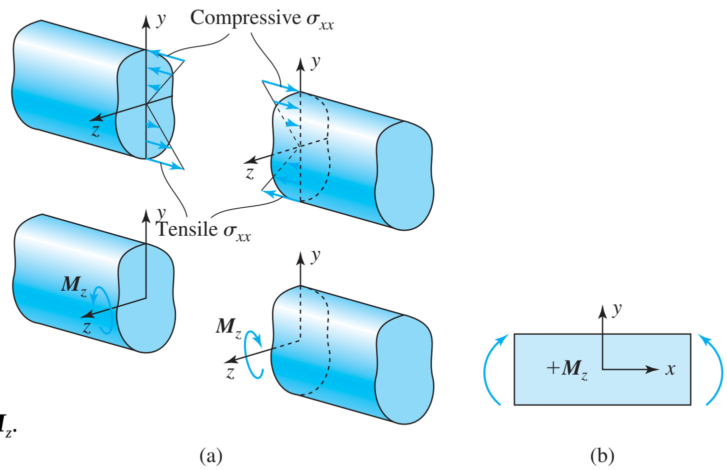


Figure 6.21 Sign convention for internal bending moment $\boldsymbol{M}_z$.

$\boldsymbol{M}_z$ may be found in either of two possible ways as described next (see also Example 6.8).

1. In one method, on a free-body diagram $\boldsymbol{M}_z$ is always drawn according to the sign convention. The equilibrium equation is then used to get a positive or negative value for $\boldsymbol{M}_z$. Positive values of stress σ_{xx} from Equation (6.12) are tensile, and negative values of σ_{xx} are compressive.
2. Alternatively, $\boldsymbol{M}_z$ is drawn at the imaginary cut in a direction that equilibrates the external loads. Since inspection is being used in determining the direction of $\boldsymbol{M}_z$, Equation (6.12) can determine only the magnitude. The tensile and compressive nature of σ_{xx} must be determined by inspection.

Figure 6.22 shows a cantilever beam loaded with a transverse force *P*. An imaginary cut is made at section *AA*, and a free-body diagram is drawn. For equilibrium it is clear that we need an internal shear force $\boldsymbol{V}_y$, which is possible only if there is a nonzero shear stress τ_{xy}. By Hooke's law this implies that the shear strain γ_{xy} cannot be zero. Assumption 3 implied that shear strain was small but not zero. In beam bending, a check on the validity of the analysis is to compare the maximum shear stress τ_{xy} to the maximum normal stress σ_{xx} for the entire beam. If the two stress components are comparable, then the shear strain cannot be neglected in kinematic considerations, and our theory is not valid.

- The maximum normal stress σ_{xx} in the beam should be nearly an order of magnitude greater than the maximum shear stress τ_{xy}.

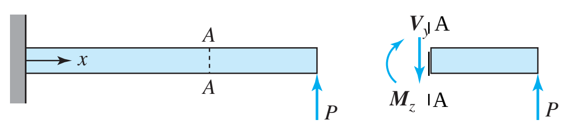


Figure 6.22 Internal forces and moment necessary for equilibrium.

The internal shear force is defined as

$$\boxed{V_y = \int_A \tau_{xy}\, dA} \tag{6.13}$$

In Section 1.3 we studied the use of subscripts to determine the direction of a stress component, which we can now use to determine the positive direction of τ_{xy}. According to this second sign convention, the equivalent shear force V_y is in the same direction as the shear stress τ_{xy}.

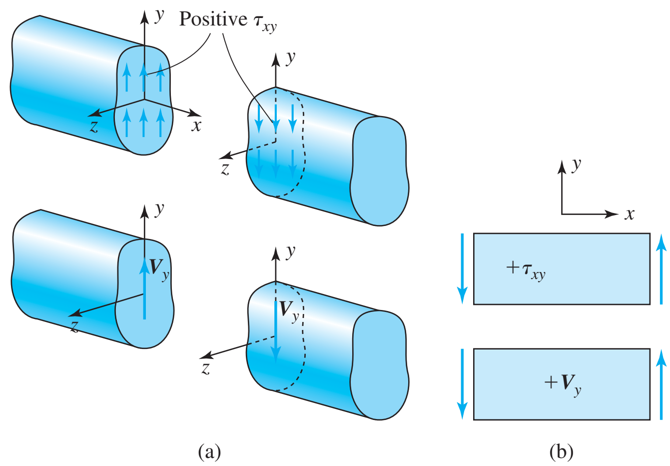


Figure 6.23 Sign convention for internal shear force V_y.

Sign Convention: The direction of positive internal shear force V_y on a free-body diagram is in the direction of the positive shear stress on the surface.[6]

Figure 6.23 shows the positive direction for the internal shear force V_y. The sign conventions for the internal bending moment and the internal shear force are tied to the coordinate system because the sign convention for stresses is tied to the coordinate system. But we are free to choose the directions for our coordinate system. Example 6.6 elaborates this comment further.

EXAMPLE 6.6

Figure 6.24 shows a beam and loading in three different coordinate systems. Determine for the three cases the internal shear force and bending moment at a section 36 in. from the free end using the sign conventions described in Figures 6.21 and 6.23.

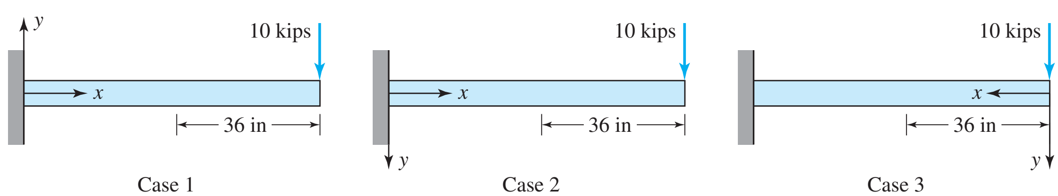


Figure 6.24 Example 6.6 on sign convention.

PLAN

We make an imaginary cut at 36 in. from the free end and take the right-hand part in drawing the free-body diagram. We draw the shear force and bending moment for each of the three cases as per our sign convention. By writing equilibrium equations we obtain the values of the shear force and the bending moment.

SOLUTION

We draw three rectangles and the coordinate axes corresponding to each of the three cases, as shown in Figure 6.25. Point A is on the surface that has an outward normal in the positive x direction, and hence the force will be in the positive y direction to produce a positive shear stress. Point B is on the surface that has an outward normal in the negative x direction, and hence the force will be in the negative y direction to produce a positive shear stress. Point C is on the surface where the y coordinate is positive. The moment direction is shown to put this surface into compression.

[6]Some mechanics of materials books use an opposite direction for a positive shear force. This is possible because Equation (6.13) is a definition, and a minus sign can be incorporated into the definition. Unfortunately positive shear force and positive shear stress are then opposite in direction, causing problems with intuitive understanding.

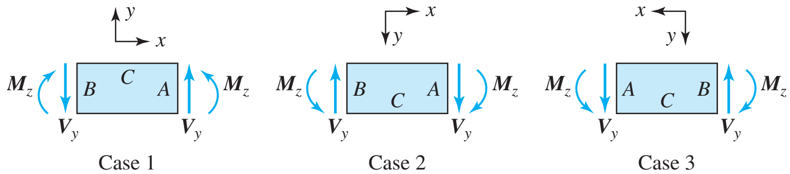


Figure 6.25 Positive shear forces and bending moments in Example 6.6.

Figure 6.26 shows the free body diagram for the three cases with the shear forces and bending moments drawn on the imaginary cut as shown in Figure 6.25. By equilibrium of forces in the y direction we obtain the shear force values. By equilibrium of moment about point O we obtain the bending moments for each of the three cases as shown in Table 6.1.

TABLE 6.1 **Results for Example 6.6.**

Case 1	*Case 2*	*Case 3*
$V_y = -10$ kips	$V_y = 10$ kips	$V_y = -10$ kips
$M_z = -360$ in.·kips	$M_z = 360$ in.·kips	$M_z = 360$ in.·kips

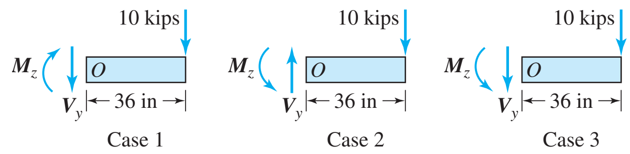


Figure 6.26 Free-body diagrams in Example 6.6.

COMMENTS

1. In Figure 6.26 we drew the shear force and bending moment directions without consideration of the external force of 10 kips. The equilibrium equations then gave us the correct signs. When we substitute these internal quantities, with the proper signs, into the respective stress formulas, we will obtain the correct signs for the stresses.
2. Suppose we draw the shear force and the bending moment in a direction such that it satisfies equilibrium. Then we shall always obtain positive values for the shear force and the bending moment, irrespective of the coordinate system. In such cases the sign for the stresses will have to be determined intuitively, and the stress formulas should be used only for the magnitude. To reap the benefit of both approaches, the internal quantities should be drawn using the sign convention, and the answers should be checked intuitively.
3. All three cases show that the shear force acts upward and the bending moment is counterclockwise, which are the directions for equilibrium.

EXAMPLE 6.7

The two square beam cross sections shown in Figure 6.27 have the same material cross-sectional area A. Show that the hollow cross section has a higher area moment of inertia about the z axis than the solid cross section.

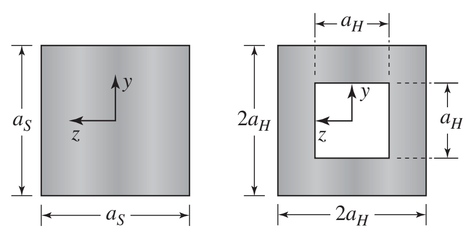


Figure 6.27 Cross sections in Example 6.7.

PLAN

We can find dimensions a_S and a_H in terms of the cross-sectional area A. Then we can find the area moments of inertia in terms of A and compare.

SOLUTION

The dimensions a_S and a_H in terms of area can be found as

$$A_S = a_S^2 = A \quad \text{or} \quad a_S = \sqrt{A} \quad \text{and} \quad A_H = (2a_H)^2 - a_H^2 = 3a_H^2 = A \quad \text{or} \quad a_H = \sqrt{A/3} \tag{E1}$$

Let I_S and I_H represent the area moments of inertia about the z axis for the solid cross section and the hollow cross section, respectively. We can find I_S and I_H in terms of area A as

$$I_S = \frac{1}{12}a_S a_S^3 = \frac{1}{12}A^2 \quad \text{and} \quad I_H = \frac{1}{12}(2a_H)(2a_H)^3 - \frac{1}{12}a_H a_H^3 = \frac{15}{12}a_H^4 = \frac{15}{12}\left(\frac{A}{3}\right)^2 = \frac{5}{36}A^2 \tag{E2}$$

Dividing I_H by I_S we obtain

$$\frac{I_H}{I_S} = \frac{5}{3} = 1.677 \tag{E3}$$

ANS. As $I_H > I_S$ the area moment of inertia for the hollow beam is greater than that of the solid beam for the same amount of material.

COMMENTS

1. The hollow cross section has a higher area moment of inertia for the same cross-sectional area. From Equations (6.11) and (6.12) this implies that the hollow cross section will have lower stresses and deformation. Alternatively, a hollow cross section will require less material (and be lighter in weight) giving the same area moment of inertia. This observation plays a major role in the design of beam shapes. Figure 6.28 shows some typical steel beam cross sections used in structures. Notice that in each case material from the region near the centroid is removed. Cross sections so created are thin near the centroid. This thin region near the centroid is called the **web**, while the wide material near the top or bottom is referred to as the **flange**. Section C.6 in Appendix has tables showing the geometric properties of some structural steel members.

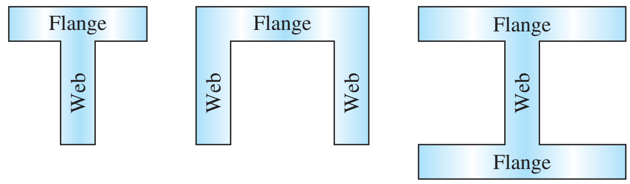


Figure 6.28 Metal beam cross sections.

2. We know that the bending normal stress is zero at the centroid and maximum at the top or bottom surfaces. We take material near the centroid, where it is not severely stressed, and move it to the top or bottom surface, where stress is maximum. In this way, we use material where it does the most good in terms of carrying load. This phenomenological explanation is an alternative explanation for the design of the cross sections shown in Figure 6.28. It is also the motivation in design of **sandwich beams**, in which two stiff panels are separated by softer and lighter core material. Sandwich beams are common in the design of lightweight structures such as aircrafts and boats.
3. Wooden beams are usually rectangular as machining costs do not offset the saving in weight.

EXAMPLE 6.8

An S180 × 30 steel beam is loaded and supported as shown in Figure 6.29. Determine: (a) The bending normal stress at a point *A* that is 20 mm above the bottom of the beam. (b) The maximum compressive bending normal stress in a section 0.5 m from the left end.

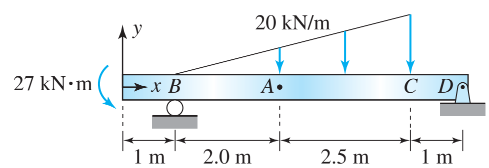


Figure 6.29 Beam in Example 6.8.

PLAN

From Section C.6 we can find the cross section, the centroid, and the moment of inertia. Using free body diagram for the entire beam, we can find the reaction force at B. Making an imaginary cut through *A* and drawing the free body diagram, we can determine the internal moment. Using Equation (6.12) we determine the bending normal stress at point *A* and the maximum bending normal stress in the section.

SOLUTION

From Section C.6 we obtain the cross section of S180 × 30 shown in Figure 6.30*a* and the area moment of inertia:

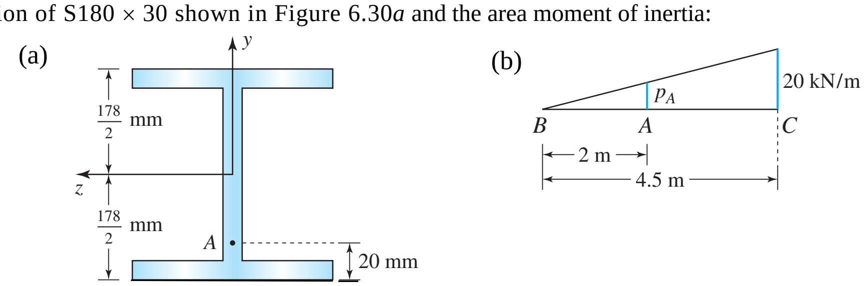


Figure 6.30 (a) S180 x 30 cross section in Example 6.8. (b) Intensity of distributed force at point *A* in Example 6.8

$$I_{zz} = 17.65(10^6)\ \text{mm}^4 \tag{E1}$$

The coordinates of point *A* can be found from Figure 6.30*a*, as shown in Equation (E2). The maximum bending normal stress will occur at the top or at the bottom of the cross section.The *y* coordinates are

$$y_A = -\left(\frac{178\ \text{mm}}{2} - 20\ \text{mm}\right) = -69\ \text{mm} \qquad y_{\max} = \pm\frac{178\ \text{mm}}{2} = \pm 89\ \text{mm} \tag{E2}$$

We draw the free-body diagram of the entire beam with distributed load replaced by a statically equivalent load placed at the centroid of the load as shown in Figure 6.31a. By equilibrium of moment about point *D* we obtain R_B

$$R_B\ (5.5\ \text{m}) - (27\ \text{kN}\cdot\text{m}) - (45\ \text{kN})\ (2.5\ \text{m}) = 0 \qquad \text{or} \qquad R_B = 25.36\ \text{kN} \tag{E3}$$

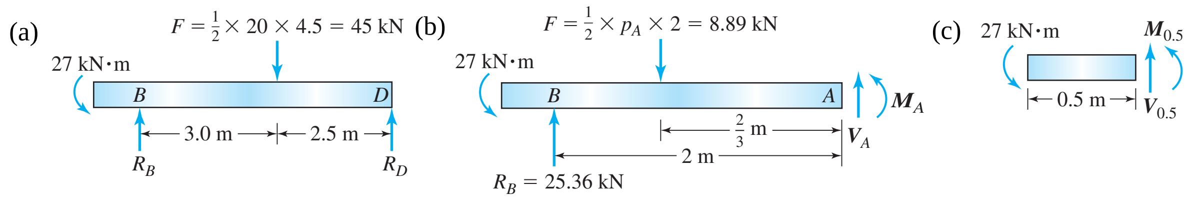


Figure 6.31 Free-body diagrams in Example 6.8 for (a) entire beam (b) calculation of $\boldsymbol{M}_A$ (c) calculation of $\boldsymbol{M}_{0.5}$.

Figure 6.30*b* shows the variation of distributed load. The intensity of the distributed load acting on the beam at point *A* can be found from similar triangles,

$$\frac{p_A}{2\text{ m}} = \frac{20\text{ kN/m}}{4.5\text{ m}} \qquad \text{or} \qquad p_A = 8.89\text{ kN/m} \tag{E4}$$

We make an imaginary cut through point *A* in Figure 6.29 and draw the internal bending moment and the shear force using our sign convention. We also replace that portion of the distributed load acting at left of *A* by an equivalent force to obtain the free-body diagram shown in Figure 6.31b. By equilibrium of moment at point *A* we obtain the internal moment.

$$\boldsymbol{M}_A + (27\text{ kN}\cdot\text{m}) - (25.36\text{ kN})(2\text{ m}) + (8.89\text{ kN})\left(\frac{2}{3}\text{ m}\right) = 0 \qquad \text{or} \qquad \boldsymbol{M}_A = 17.8\text{ kN}\cdot\text{m} \tag{E5}$$

(a) Using Equations (6.12) we obtain the bending normal stress at point *A*.

$$\sigma_A = -\frac{\boldsymbol{M}_A y_A}{I_{zz}} = -\frac{[17.8(10^3)\text{ N}\cdot\text{m}][-69(10^{-3})\text{ m}]}{17.65(10^{-6})\text{ m}^4} = 69.6(10^6)\text{ N/m}^2 \tag{E6}$$

ANS. $\sigma_A = 69.6\text{ MPa (T)}$

(b) We make an imaginary cut at 0.5 m from the left, draw the internal bending moment and the shear force using our sign convention to obtain the free body diagram shown in Figure 6.30c. By equilibrium of moment we obtain

$$M_{0.5} = -27\text{ kN}\cdot\text{m} \tag{E7}$$

The maximum compressive bending normal stress will be at the bottom of the beam, where $y = -88.9\ (10^{-3})$ m. Its value can be calculated as

$$\sigma_{0.5} = -\frac{-[27(10^3)\text{ N}\cdot\text{m}][-89(10^{-3})\text{ m}]}{17.65(10^{-6})\text{ m}^4} \tag{E8}$$

ANS. $\sigma_{0.5} = 136.1\text{ MPa (C)}$

COMMENT

1. For an *intuitive check* on the answer, we can draw an approximate deformed shape of the beam, as shown in Figure 6.32. We start by drawing the approximate shape of the bottom surface (or the top surface). At the left end the beam deflects downward owing to the applied moment. At the support point *B* the deflection must be zero. Since the slope of the beam must be continuous (otherwise a corner will be formed), the beam has to deflect upward as one crosses *B*. Now the externally distributed load pushes the beam downward. Eventually the beam will deflect downward, and finally it must have zero deflection at the support point *D*. The top surface is drawn parallel the bottom surface.

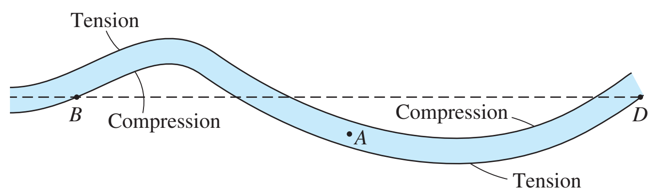


Figure 6.32 Approximate deformed shape of beam in Example 6.8.

2. By inspection of Figure 6.32 we see that point *A* is in the region where the bottom surface is in tension and the top surface in compression. If point *A* were closer to the inflection point, then we would have greater difficulty in assessing the situation. This once more emphasizes that intuitive checks are valuable but their conclusions must be viewed with caution.

Consolidate your knowledge

1. Identity five examples of beams from your daily life.
2. With the book closed, derive Equations (6.11) and (6.12), listing all the assumptions as you go along.

MoM in Action: Suspension Bridges

The Golden Gate Bridge (Figure 6.33*a*) opened May 27, 1937, spanning the opening of San Francisco Bay. More than 100,000 vehicles cross it every day, and more than 9 million visitors come to see it each year. The first bridge to span the Tacoma Narrows, between the Olympic peninsula and the Washington State mainland, opened just three years later, on July 1, 1940. It quickly acquired the name *Galloping Gertie* (Figure 6.33*b*) for its vertical undulations and twisting of the bridge deck in even moderate winds. Four months later, on November 7, it fell. The two suspension bridges, one famous, the other infamous, are a story of pushing design limits to cut cost.

Bridges today frequently have spans of up to 7000 ft and high clearances, for large ships to pass through. But suspension bridges are as old as the vine and rope bridges (Figure 6.33*c*) used across the world to ford rivers and canyons. Simply walking on rope bridges can cause them to sway, which can be fun for a child on a playground but can make a traveler very uncomfortable crossing a deep canyon. In India in the 4th century C.E., cables were introduced – first of plaited bamboo and later iron chains – to increase rigidity and decrease swaying. But the modern form, in which a roadway is suspended by cables, came about in the early nineteenth century in England, France, and America to bridge navigable streams. Still, early bridges were susceptible to stability and strength failures from wind, snow, and droves of cattle. John Augustus Roebling solved the problem, first in bridging Niagara Falls Gorge and again with his masterpiece—the Brooklyn Bridge, completed in 1883. Roebling increased rigidity and strength by adding on either side, a truss underneath the roadway.

(a)

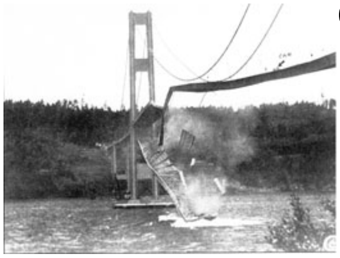
(b)

(c)


Figure 6.33 Suspension bridges: (a) Golden Gate (Courtesy Mr. Rich Niewiroski Jr.); (b) Galloping Gertie collapse; (c) Inca's rope bridge

Clearly engineers have long been aware of the impact of wind and traffic loads on the strength and motion of suspension bridges. Galloping Gertie was strong enough to withstanding bending stresses from winds of 120 mph. However, the cost of public works is always a serious consideration, and in case of Galloping Gertie it led to design decisions with disastrous consequences. The six-lane Golden Gate Bridge is 90 feet wide, has a bridge-deck depth of 25 feet and a center-span length to width ratio of 47:1. Galloping Gertie's two lanes were only 27 feet wide, a bridge-deck depth of only 8 feet, and center-span length to width ratio of 72:1. Thus, the bending rigidity (*EI)* and torsional rigidity (*GJ*) per unit length of Galloping Gertie were significantly less than the Golden Gate bridge. To further save on construction costs, the roadway was supported by solid I-beam girders, which unlike the open lattice of Golden Gate did not allow wind to pass through it but rather over and under it— that is, the roadway behaved like a wing of a plane. The bridge collapsed in a wind of 42 mph, and torsional and bending rigidity played a critical role.

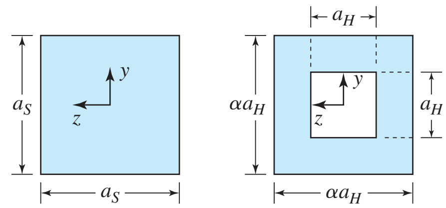
There are two kinds of aerodynamic forces: *lift*, which makes planes rise into air, and *drag*, a dissipative force that helps bring the plane back to the ground. Drag and lift forces depend strongly on the wind direction relative to the structure. If the structure twists, then the relative angle of the wind changes. The structure's rigidity resists further deformation due to changes in torsional and bending loads. However, when winds reach the *flutter speed*, torsional and bending deformation couple, with forces and deformations feeding each other till the structure breaks. This aerodynamic instability, known as *flutter*, was not understood in bridge design in 1940.

Today, wind-tunnel tests of bridge design are mandatory. A Tacoma Narrows Bridge with higher bending and torsional rigidity and an open lattice roadway support was built in 1950. Suspension bridges are as popular as ever. The Pearl Bridge built in 1998, linking Kobe, Japan, with Awaji-shima island has the world's longest center span at 6532 ft. Its mass dampers swing to counter earthquakes and wind. Galloping Gertie, however, will be remembered for the lesson it taught in design decisions that are penny wise but pound foolish.

PROBLEM SET 6.2

Second area moments of inertia

6.19 A solid and a hollow square beam have the same cross-sectional area A, as shown in Figure P6.19. Show that the ratio of the second area moment of inertia for the hollow beam I_H to that of the solid beam I_S is given by the equation below.



$$\frac{I_H}{I_S} = \frac{\alpha^2 + 1}{\alpha^2 - 1}$$

Figure P6.19

6.20 Figure P6.20*a* shows four separate wooden strips that bend independently about the neutral axis passing through the centroid of each strip. Figure 6.15*b* shows the four strips glued together and bending as a unit about the centroid of the glued cross section. (a) Show that $I_G = 16I_S$, where I_G is the area moment of inertia for the glued cross section and I_S is the total area moment of inertia of the four separate beams. (b) Also show that $\sigma_G = \sigma_S/4$, where σ_G and σ_S are the maximum bending normal stresses at any cross section for the glued and separate beams, respectively.

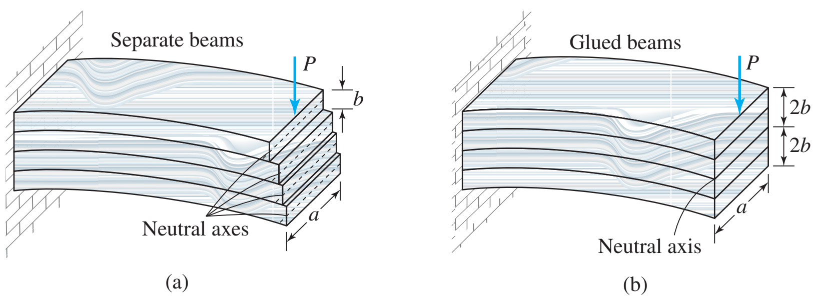


Figure P6.20

6.21 The cross sections of the beams shown in Figure P6.21 is constructed from thin sheet metal of thickness t. Assume that the thickness $t \ll a$. Determine the second area moments of inertia about an axis passing through the centroid in terms of a and t.

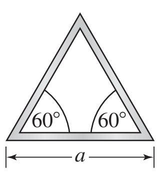


Figure P6.21

6.22 The cross sections of the beams shown in Figure P6.22 is constructed from thin sheet metal of thickness t. Assume that the thickness $t \ll a$. Determine the second area moments of inertia about an axis passing through the centroid in terms of a and t.

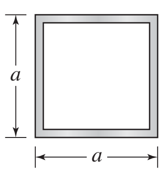


Figure P6.22

6.23 The cross sections of the beams shown in Figure P6.23 is constructed from thin sheet metal of thickness t. Assume that the thickness $t \ll a$. Determine the second area moments of inertia about an axis passing through the centroid in terms of a and t.

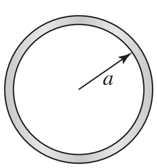


Figure P6.23

6.24 The same amount of material is used for constructing the cross sections shown in Figures P6.21, P6.22, and P6.23. Let the maximum bending normal stresses be σ_T, σ_S, and σ_C for the triangular, square, and circular cross sections, respectively. For the same moment-carrying capability determine the proportional ratio of the maximum bending normal stresses; that is, σ_T: σ_S: σ_C. What is the proportional ratio of the section moduli?

Normal stress and strain variations across a cross section

6.25 Due to bending about the *z* axis the normal strain at point A on the cross section shown in Figures P6.25 is $\varepsilon_{xx} = 200\ \mu$. The modulus of elasticity of the beam material is E = 8000 ksi. Determine the maximum tensile and compressive normal stress on the cross-section.

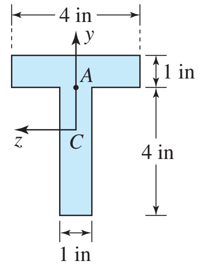


Figure P6.25

6.26 Due to bending about the z axis the maximum bending normal stress on the cross section shown in Figures P6.26 was found to be 40 ksi (C). The modulus of elasticity of the beam material is E = 30,000 ksi. Determine (a) the bending normal strain at point A. (b) the maximum bending tensile stress.

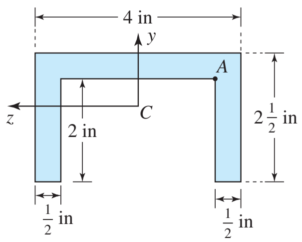


Figure P6.26

6.27 A composite beam cross section is shown in Figure 6.27. The bending normal strain at point *A* due to bending about the *z* axis was found to be $\varepsilon_{xx} = -200\ \mu$. The modulus of elasticity of the two materials are E_1 = 200 GPa, E_2 = 70 GPa. Determine the maximum bending stress in each of the two materials.

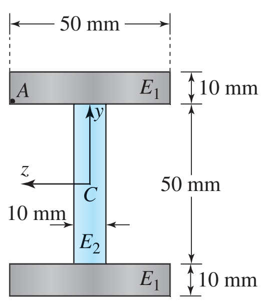


Figure P6.27

6.28 A composite beam cross section is shown in Figure 6.28. The bending normal strain at point *A* due to bending about the *z* axis was found to be $\varepsilon_{xx} = 300\ \mu$. The modulus of elasticity of the two materials are E_1 = 30,000 ksi, E_2 = 20,000 ksi. Determine the maximum bending stress in each of the two materials.

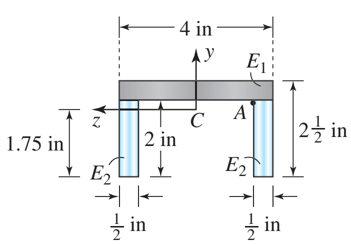


Figure P6.28

6.29 The internal moment due to bending about the *z* axis, at a beam cross section shown in Figures P6.29 is M_z = 20 in.·kips. Determine the bending normal stresses at points *A*, *B*, and *D*.

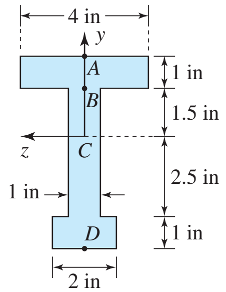


Figure P6.29

6.30 The internal moment due to bending about the *z* axis, at a beam cross section shown in Figures P6.30 is M_z = 10 kN·m. Determine the bending normal stresses at points *A*, *B*, and *D*.

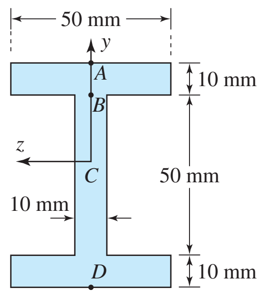


Figure P6.30

6.31 The internal moment due to bending about the *z* axis, at a beam cross section shown in Figures P6.31 is M_z = –12 kN·m. Determine the bending normal stresses at points *A*, *B*, and *D*.

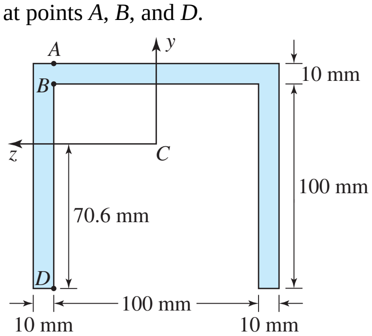


Figure P6.31

Sign convention

6.32 A beam and loading in three different coordinate systems is shown in Figures P6.32. Determine the internal shear force and bending moment at the section containing point *A* for the three cases shown using the sign convention described in Section 6.2.6.

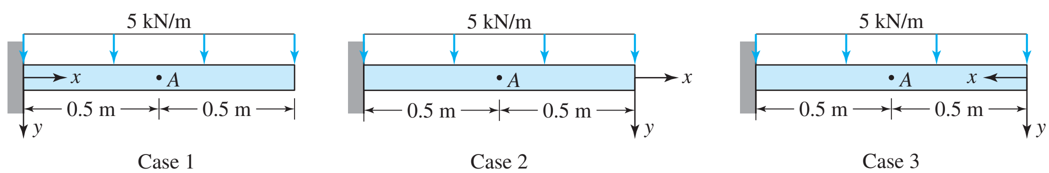


Figure P6.32

6.33 A beam and loading in three different coordinate systems is shown in Figures P6.33. Determine the internal shear force and bending moment at the section containing point *A* for the three cases shown using the sign convention described in Section 6.2.6.

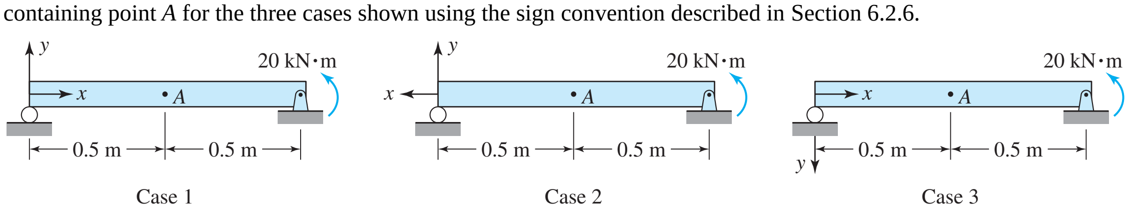


Figure P6.33

6.34 A beam and loading in three different coordinate systems is shown in Figures P6.34. Determine the internal shear force and bending moment at the section containing point *A* for the three cases shown using the sign convention described in Section 6.2.6.

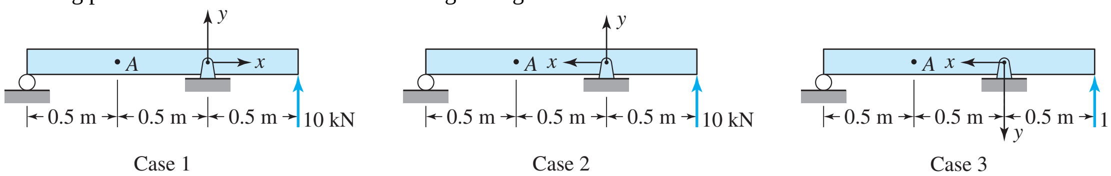


Figure P6.34

Sign of stress by inspection

6.35 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.35. By inspection determine whether the bending normal stress is tensile or compressive at points *A* and *B*.

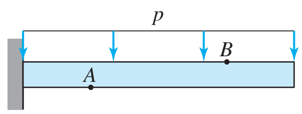


Figure P6.35

6.36 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.36. By inspection determine whether the bending normal stress is tensile or compressive at points *A* and *B*.

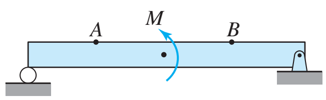


Figure P6.36

6.37 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.37. By inspection determine whether the bending normal stress is tensile or compressive at points *A* and *B*.

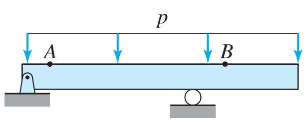


Figure P6.37

6.38 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.38. By inspection determine whether the bending normal stress is tensile or compressive at points *A* and *B*.

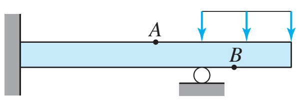


Figure P6.38

6.39 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.39. By inspection determine whether the bending normal stress is tensile or compressive at points *A* and *B*.

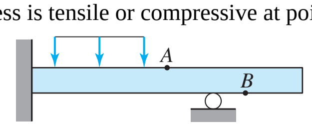


Figure P6.39

6.40 Draw an approximate deformed shape of the beam for the beam and loading shown in Figure P6.40. By inspection determine whether the bending normal stress is tensile or compressive at points *A* and *B*.

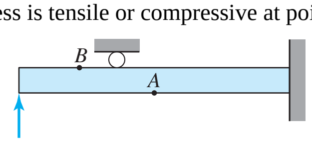


Figure P6.40

Bending normal stress and strain calculations

6.41 A W150 × 24 steel beam is simply supported over a length of 4 m and supports a distributed load of 2 kN/m. At the midsection of the beam, determine (a) the bending normal stress at a point 40 mm above the bottom surface; (b) the maximum bending normal stress.

6.42 A W10 × 30 steel beam is simply supported over a length of 10 ft and supports a distributed load of 1.5 kips/ft. At the midsection of the beam, determine (a) the bending normal stress at a point 3 in below the top surface; (b) the maximum bending normal stress.

6.43 An S12 × 35 steel cantilever beam has a length of 20 ft. At the free end a force of 3 kips acts downward. At the section near the built-in end, determine (a) the bending normal stress at a point 2 in above the bottom surface; (b) the maximum bending normal stress.

6.44 An S250 × 52 steel cantilever beam has a length of 5 m. At the free end a force of 15 kN acts downward. At the section near the built-in end, determine (a) the bending normal stress at a point 30 mm below the top surface; (b) the maximum bending normal stress.

6.45 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.45.

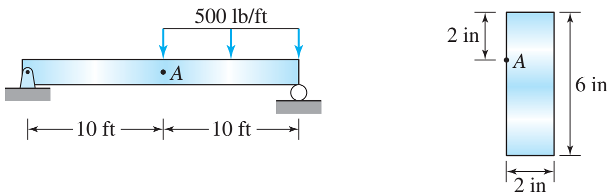


Figure P6.45

6.46 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.46.

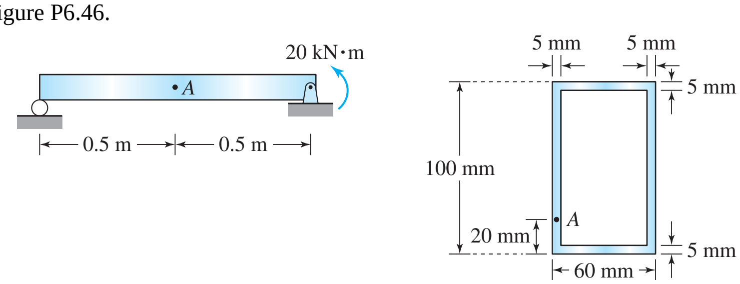


Figure P6.46

6.47 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.47.

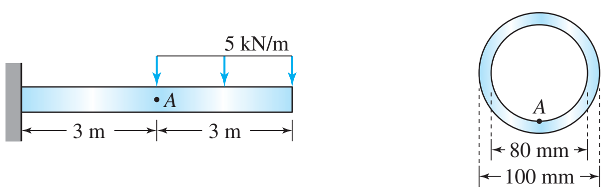


Figure P6.47

6.48 A simply supported beam with its cross section is shown in Figure P6.48. The intensity of distributed load reaches a maximum value of 5 kN/m. Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A*.

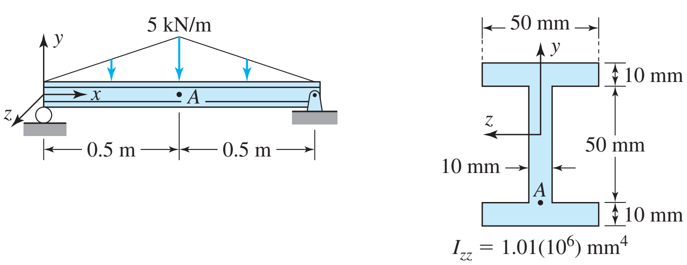


Figure P6.48

6.49 A cantilever beam with cross section is shown in Figure P6.49. The distributed load reaches its maximum intensity of 300 lb/in. Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading

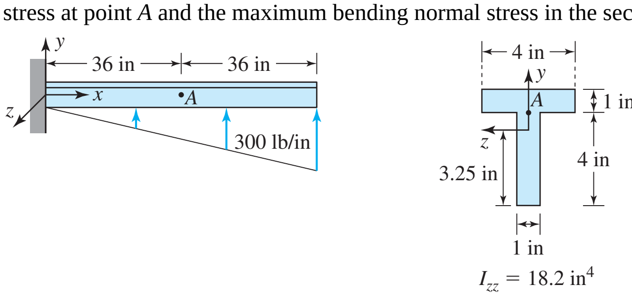


Figure P6.49

6.50 Determine the bending normal stress at point *A* and the maximum bending normal stress in the section containing point *A* for the beam and loading shown in Figure P6.50.

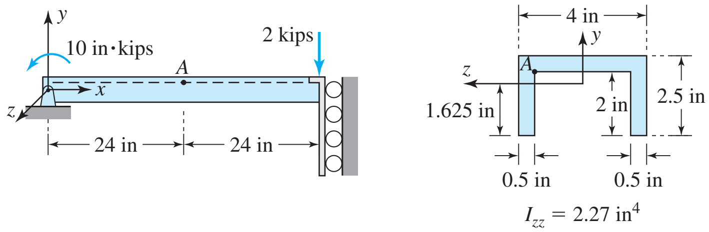


Figure P6.50

6.51 A wooden rectangular beam (E = 10 GPa), its loading, and its cross section are as shown in Figure P6.51. If the distributed force w = 5 kN/m, determine the normal strain ε_{xx} at point *A* on the bottom of the beam.

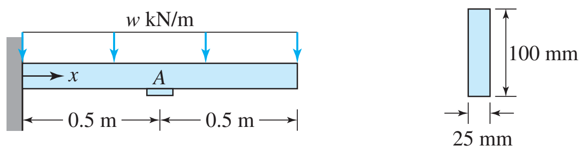


Figure P6.51

6.52 A wooden rectangular beam (E = 10 GPa), its loading, and its cross section are as shown in Figure P6.51. The normal strain at point *A* was measured as $\varepsilon_{xx} = -600\ \mu$. Determine the distributed force w that is acting on the beam.

6.53 A wooden beam (E = 8000 ksi), its loading, and its cross section are as shown in Figure P6.53. If the applied load P = 6 kips, determine the normal strain ε_{xx} at point *A*.

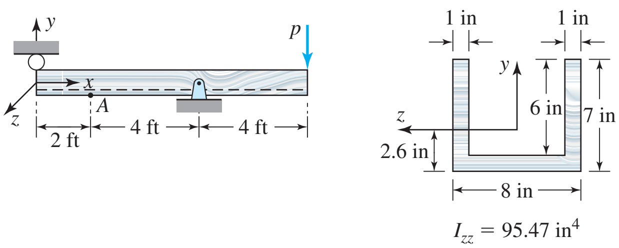


Figure P6.53

6.54 A wooden beam (E = 8000 ksi), its loading, and its cross section are as shown in Figure P6.53. The normal strain at point *A* was measured as $\varepsilon_{xx} = -250\ \mu$. Determine the load *P*.

Stretch Yourself

6.55 A composite beam made from *n* materials is shown in Figure 6.55. If Assumptions 1 through 7 are valid, show that the location of neutral axis η_c is given by Equation (6.14), where η_j, E_j, and A_j are location of the centroid, the modulus of elasticity, and cross sectional area of the j^{th} material.

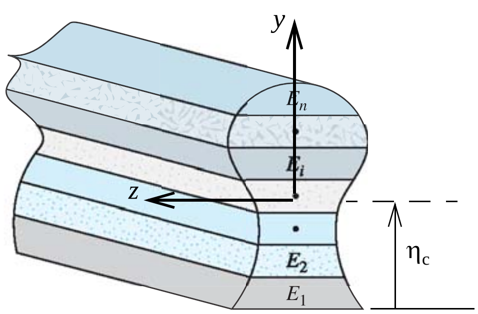


Figure P6.55

$$\eta_c = \frac{\sum_{j=1}^{n} \eta_j E_j A_j}{\sum_{j=1}^{n} E_j A_j} \tag{6.14}$$

6.56 A composite beam made from n materials is shown in Figure 6.55. If Assumptions 1 through 7 are valid, show that the moment curvature relationship and the equation for bending normal stress $(\sigma_{xx})_i$ in the i[th] material are as given by

$$\boldsymbol{M_z} = \frac{d^2v}{dx^2}\left[\sum_{j=1}^{n} E_j (I_{zz})_j\right] \qquad (6.15)$$

$$(\sigma_{xx})_i = -E_i y \frac{\boldsymbol{M_z}}{\left[\sum_{j=1}^{n} E_j (I_{zz})_j\right]} \qquad (6.16)$$

where E_j and $(I_{zz})_j$ are the modulus of elasticity and cross sectional area, and second area moment of inertia of the j[th] material. Show that if $E_1=E_2=...=E_n=E$ then Equations (6.15) and (6.16) reduce to Equations (6.11) and (6.12).

6.57 The stress–strain curve in tension for a material is given by $\sigma = K\varepsilon^{0.5}$. For the rectangular cross section shown in Figure P6.57, show that the bending normal stress is given by the equations below.

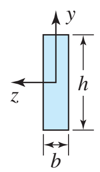


Figure P6.57

$$\sigma_{xx} = \begin{cases} \dfrac{-5\sqrt{2}}{bh^2}\left(\dfrac{y}{h}\right)^{0.5}\boldsymbol{M_z} & y > 0 \\[2ex] \dfrac{5\sqrt{2}}{bh^2}\left(-\dfrac{y}{h}\right)^{0.5}\boldsymbol{M_z}, & y < 0 \end{cases}$$

6.58 The hollow square beam shown in Figure P6.58 is made from a material that has a stress–strain relation given by $\sigma = K\varepsilon^{0.4}$. Assume the same behavior in tension and in compression. In terms of K, L, a, and M_{ext} determine the bending normal strain and stress at point A.

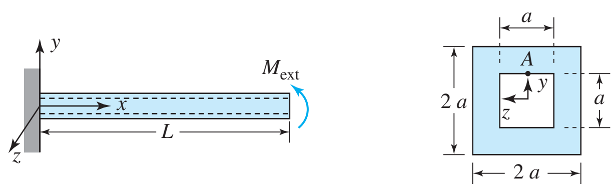


Figure P6.58

6.3 SHEAR AND MOMENT BY EQUILIBRIUM

Equilibrium equations at a point on the beam are differential equations relating the distributed force p_y, the shear force $\boldsymbol{V_y}$, and the bending moment $\boldsymbol{M_z}$. The differential equations can be integrated analytically or graphically to obtain $\boldsymbol{V_y}$ and $\boldsymbol{M_z}$ as a function of x. These in turn can be used to determine the maximum values of $\boldsymbol{V_y}$ and $\boldsymbol{M_z}$, and hence the maximum values of the bending normal stress from Equation (6.12) and the maximum bending shear stress, as discussed in Section 6.6. $\boldsymbol{M_z}$ as a function of x is also needed when integrating Equation (6.11) to find the deflection of the beam, as we will discuss in Chapter 7.

Consider a differential element Δx of the beam shown at left in Figure 6.34. Recall that a positive distributed force p_y acts in the positive y direction, as shown in Figure 6.14. Internal shear forces and the internal moment change as one moves across the element, as shown in Figure 6.34. By replacing the distributed force by an equivalent force, we obtain the diagram on the right of Figure 6.34.

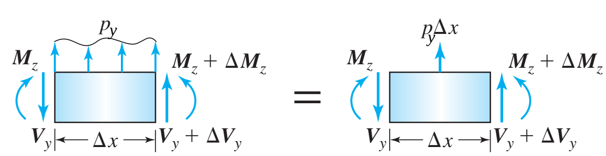


Figure 6.34 Differential beam element.

By equilibrium of forces in the y direction, we obtain

$$-\boldsymbol{V_y} + (\boldsymbol{V_y} + \Delta\boldsymbol{V_y}) + p_y\Delta x = 0 \qquad \text{or} \qquad \frac{\Delta\boldsymbol{V_y}}{\Delta x} = -p_y$$

As $\Delta x \to 0$, we obtain

$$\boxed{\frac{d\boldsymbol{V_y}}{dx} = -p_y} \qquad (6.17)$$

By equilibrium of moment in the z direction about an axis passing through the right side, we obtain

$$-M_z + (M_z + \Delta M_z) + V_y \Delta x + (p_y \Delta x)\frac{\Delta x}{2} = 0 \qquad \text{or} \qquad \frac{\Delta M_z}{\Delta x} + \frac{p\Delta x}{2} = -V_y$$

As $\Delta x \to 0$, we obtain

$$\boxed{\frac{dM_z}{dx} = -V_y} \tag{6.18}$$

Equations (6.17) and (6.18) are differential equilibrium equations that are applicable at every point on the beam, except where V_y and M_z are discontinuous. In Example 6.10 we shall see that V_y and M_z are discontinuous at the points where concentrated (point) external forces or moments are applied. We shall consider two methods for finding V_y and M_z as a function of x:

1. We can integrate Equation (6.17) to obtain V_y and then integrate Equation (6.18) to obtain M_z. The integration constants can be found from the values of V_y and M_z at the end of the beam, as illustrated in Example 6.9.
2. Alternatively, we can make an imaginary cut at some location defined by the variable x and draw the free-body diagram. We then determine V_y and M_z in terms of x by writing equilibrium equations. We can check our results by substituting the expressions of V_y and M_z in Equations (6.17) and (6.18), respectively.

The first approach, by integration, is a general approach. This is particularly useful if p_y is represented by a complicated function. But for uniform and linear variations of p_y the free-body diagram method is simpler. Example 6.9 compares the two methods, and Example 6.10 elaborates the use of the free-body diagram approach further.

EXAMPLE 6.9

Figure 6.35 shows two models of wind pressure on a light pole. Find V_y and M_z as a function of x for the two distributions shown. Neglect the weights of the light and the pole.

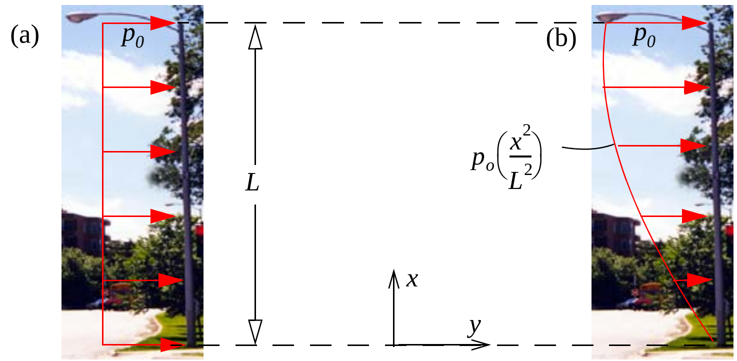


Figure 6.35 Light pole in Example 6.9. (*a*) Uniform distribution. (*b*) Quadratic distribution.

PLAN

For uniform distribution we can find V_y and M_z as a function of x by making an imaginary cut at a distance x from the bottom and drawing the free-body diagram of the top part. For the quadratic distribution we can first integrate Equation (6.17) to find V_y and then integrate Equation (6.18) to find M_z. To find the integration constants, we can construct a free-body diagram of infinitesimal length at the top ($x = L$) and obtain the boundary conditions on V_y and M_z. Using boundary conditions and integrated expressions, we can obtain V_y and M_z as a function of x for the quadratic distribution.

SOLUTION

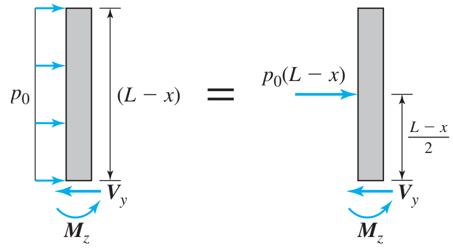


Figure 6.36 Shear force and bending moment by free-body diagram.

Uniform distribution: We can make an imaginary cut at location x and draw the free-body diagram of the top part, as shown in Figure 6.36.We then replace the distributed load by an equivalent force and write the equilibrium equations.

$$V_y = p_0(L-x) \qquad M_z = p_0(L-x)\left(\frac{L-x}{2}\right) = \frac{p_0}{2}(x^2 - 2xL + L^2) \tag{E1}$$

Check: Differentiating Equation (E1), we obtain

$$\frac{dV_y}{dx} = -p_0 = -p_y \qquad \frac{dM_z}{dx} = -p_0(L-x) = -V_y \tag{E2}$$

Equation (E2) shows that the equilibrium Equations (6.17) and (6.18) are satisfied.

Quadratic distribution: Substituting $p_y = p_0(x^2/L^2)$ into Equation (6.17) and integrating, we obtain

$$V_y = -\left(\frac{p_0}{3L^2}\right)x^3 + C_1 \tag{E3}$$

Substituting Equation (E3) into Equation (6.18) and integrating, we obtain

$$M_z = \frac{p_0}{3L^2}\left(\frac{x^4}{4}\right) - C_1 x + C_2 \tag{E4}$$

We make an imaginary cut at a distance Δx from the top and draw the free-body diagram shown in Figure 6.37.By equilibrium of forces in the y direction and equilibrium of moment about point O and letting Δx tend to zero we obtain the boundary conditions:

$$\lim_{\Delta x \to 0}[V_y(x = L) + p\Delta x] = 0 \qquad \text{or} \qquad V_y(x = L) = 0 \tag{E5}$$

$$\lim_{\Delta x \to 0}\left[M_z(x = L) + \frac{p\Delta x^2}{2}\right] = 0 \qquad \text{or} \qquad M_z(x = L) = 0 \tag{E6}$$

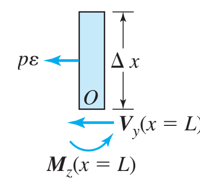


Figure 6.37 Boundary conditions on shear force and bending moments.

Substituting $x = L$ into Equation (E3) and using the condition Equation (E5), we obtain

$$-\left(\frac{p_0}{3L^2}\right)L^3 + C_1 = 0 \qquad \text{or} \qquad C_1 = \frac{p_0 L}{3} \tag{E7}$$

Substituting Equation (E7) into Equation (E3), we obtain the shear force,

$$V_y = -\frac{p_0}{3L^2}x^3 + \frac{p_0 L}{3} \tag{E8}$$

ANS. $V_y = \frac{p_0}{3L^2}(L^3 - x^3)$

Substituting $x = L$ into Equation (E4), and using Equations (E6) and (E7), we obtain

$$\frac{p_0}{3L^2}\left(\frac{L^4}{4}\right) - \frac{p_0 L}{3}(L) + C_2 = 0 \qquad \text{or} \qquad C_2 = \frac{p_0 L^2}{4} \tag{E9}$$

From Equation (E4) we obtain the moment

$$M_z = \frac{p_0 x^4}{12L^2} - \frac{p_0}{3}xL + \frac{p_0 L^2}{4} \tag{E10}$$

ANS. $M_z = \frac{p_0}{12L^2}(x^4 - 4xL^3 + 3L^4)$

COMMENTS

1. Suppose that for the uniform distribution we integrate Equation (6.17) after substituting $p_y = p_0$. We would obtain $V_y = -p_0 x + C_3$. On substituting this into Equation (6.18) and integrating, we would obtain $M_z = p_0(x^2/2) - C_3 x + C_4$. Substituting $x = L$ in the expressions of V_y and M_z and equating the results to zero, we obtain $C_3 = p_0 L$ and $C_4 = p_0 L^2/2$. Substituting these in the expressions of V_y and M_z, we obtain Equation (E1).
2. The free-body diagram approach is simpler than the integration approach for uniform distribution for two reasons. First, we did not have to perform any integration to obtain the equivalent load $p_0 L$ or to determine its location when we constructed the free-body diagram in Figure 6.36. Second, we do not have to impose zero boundary conditions on the shear force and bending moments at $x = L$, because these conditions are implicitly included in the free-body diagram in Figure 6.36.

3. The free-body diagram approach would present difficulties for the quadratic distribution, as we would need to find the equivalent load and its location. Both involve the same integrals as obtained from Equations (6.17) and (6.18). Thus for simple distributions the free-body diagram approach is preferred, whereas the integration approach is better for more complex loading.

EXAMPLE 6.10

(a) Write the equations for the internal shear force V_y and the internal bending moments M_z as a function of x for the entire beam shown in Figure 6.38. (b) Determine the values of V_y and M_z just before and after point B.

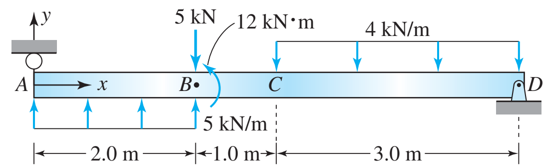


Figure 6.38 Beam in Example 6.10.

PLAN

By considering the free-body diagram of the entire beam we can determine the reactions at supports A and D. (a) The loading changes at points B and C. Thus shear force and bending moment will be represented by different functions in AB, BC, and CD. We draw free body diagrams after making imaginary cuts in AB, BC, and CD and determine shear force and bending moment by equilibrium. We can use Equations (6.17) and (6.18) to check our answers. (b) By substituting $x = 2$ m in the expressions for V_y and M_z in segment AB we can find the values just before B, and by substituting $x = 2$ in segment BC we find the values just after B.

SOLUTION

(a) We replace the distributed loads by equivalent forces and draw the free-body diagram of the entire beam as shown in Figure 6.39.By equilibrium of moment about point D and equilibrium of forces in the y direction we obtain the reaction forces.

$$R_A(6\text{ m}) - (10\text{ kN})(5\text{ m}) + (12\text{ kN}\cdot\text{m}) + (5\text{ kN})(4\text{ m}) + (12\text{ kN})(1.5\text{ m}) = 0 \qquad \text{or} \qquad R_A = 0 \tag{E1}$$

$$-R_A + 10\text{ kN} - 5\text{ kN} - 12\text{ kN} + R_D = 0 \qquad \text{or} \qquad R_D = 7\text{ kN} \tag{E2}$$

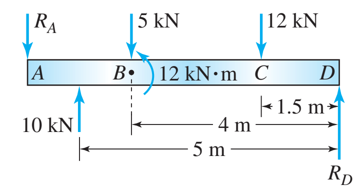


Figure 6.39 Free-body diagram of entire beam in Example 6.10.

Segment AB, $0 \le x < 2$: We make an imaginary cut at some location x in segment AB. We take the left part of the cut and draw the free-body diagram after replacing the distributed force over the distance x by a statically equivalent force, as shown in Figure 6.40a. We write the equilibrium equations to obtain V_y and M_z as a function of x.

$$V_y + 5x = 0 \qquad \text{or} \qquad V_y = -5x\text{ kN} \tag{E3}$$

$$M_z - 5x\left(\frac{x}{2}\right) = 0 \qquad \text{or} \qquad M_z = \frac{5}{2}x^2\text{ kN}\cdot\text{m} \tag{E4}$$

ANS. $V_y = -5x\text{ kN} \quad M_z = \frac{5}{2}x^2\text{ kN}\cdot\text{m}$

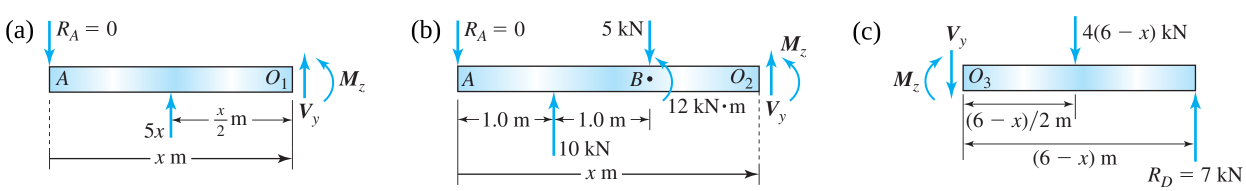


Figure 6.40 Free body diagrams in Example 6.10 after imaginary cut in (a) AB (b) BC (c) CD.

Check: Differentiating the shear force and bending moment, we obtain

$$\frac{dV_y}{dx} = -5 = -p_y, \qquad \frac{dM_z}{dx} = 5x = -V_y \tag{E5}$$

Equation (E5) shows that Equations (6.17) and (6.18) are satisfied.

Segment BC, $2 < x < 3$: We make an imaginary cut at some location x in segment BC. We take the left part of the cut and draw the free-body diagram after replacing the distributed force by a statically equivalent force, as shown in Figure 6.40b. We write the equilibrium equations to obtain V_y and M_z as a function of x.

$$V_y + 10\text{ kN} - 5\text{ kN} = 0 \qquad \text{or} \qquad V_y = -5\text{ kN} \tag{E6}$$

$$M_z - (10\text{ kN})(x-1) + (12\text{ kN}\cdot\text{m}) + (5\text{ kN})(x-2) = 0 \qquad \text{or} \qquad M_z = (5x-12)\text{ kN}\cdot\text{m} \tag{E7}$$

ANS. $V_y = -5\text{ kN}$ $\quad M_z = (5x-12)\text{ kN}\cdot\text{m}$

Check: Differentiating shear force and bending moment, we obtain

$$\frac{dV_y}{dx} = 0 = -p_y \qquad \frac{dM_z}{dx} = 5 = -V_y \tag{E8}$$

Equation (E8) shows that Equations (6.17) and (6.18) are satisfied.

Segment CD, $3 < x < 6$: We make an imaginary cut at some location x in segment CD. We take the right part of the cut and note that left part is x m long and the right part hence is $6 - x$ m long. We draw the free-body diagram after replacing the distributed force by a statically equivalent force, as shown in Figure 6.40c.We write the equilibrium equations to obtain V_y and M_z as a function of x,

$$V_y + (4)(6-x)\text{ kN} - (7\text{ kN}) = 0 \qquad \text{or} \qquad V_y = (4x - 17)\text{ kN} \tag{E9}$$

$$M_z + [4(6-x)\text{ kN}]\left(\frac{6-x}{2}\right) - (7\text{ kN})(6-x) = 0 \qquad \text{or} \qquad M_z = (-2x^2 + 17x - 30)\text{ kN}\cdot\text{m} \tag{E10}$$

ANS. $V_y = (4x - 17)\text{ kN}$ $\quad M_z = (-2x^2 + 17x - 30)\text{ kN}\cdot\text{m}$

Check: Differentiating shear force and bending moment, we obtain

$$\frac{dV_y}{dx} = 4 = -p_y, \qquad \frac{dM_z}{dx} = -4x + 17 = -V_y \tag{E11}$$

Equation (E11) shows that Equations (6.17) and (6.18) are satisfied.

(b) Substituting $x = 2$ m into Equations (E3) and (E4) we obtain the values of V_y and M_z just before point B,

ANS. $V_y(2^-) = -10\text{ kN},$ $\quad M_z(2^-) = +10\text{ kN}\cdot\text{m}$

where the superscripts – refer to just before $x = 2$ m. Substituting $x = 2$ m into Equations (E6) and (E7) we obtain the values of V_y and M_z just after point B,

ANS. $V_y(2^+) = -5\text{ kN},$ $\quad M_z(2^+) = -2\text{ kN}\cdot\text{m}$

where the superscripts + refer to just after $x = 2$m.

COMMENTS

1. In Figures 6.40a and 6.40b the left part after the imaginary cut was taken and the distance from A was labeled x. In Figure 6.40c the right part of the imaginary cut was taken, and the distance from the right end was labeled $(6 - x)$. These free-body diagrams emphasize that x defines the location of the imaginary cut, irrespective of the part used in drawing the free-body diagram. Furthermore, the distance (coordinate) x is always measured from the same point in all free-body diagrams, which in this problem is point A.
2. We note that $V_y(2^+) - V_y(2^-) = 5$ kN, which is the magnitude of the applied external force at point B. Similarly, $M_z(2^+) - M_z(2^-) = -12$ kN·m, which is the magnitude of the applied external moment at point B. This emphasizes that the external point force causes a jump in internal shear force, and the external point moment causes a jump in the internal bending moment. We will make use of these observations in the next section in plotting the shear force—bending moment diagrams.
3. We can obtain V_y and M_z in each segment by integrating Equations (6.17) and (6.18). Observe that the shear force and bending moment jump by the value of applied force and moment, respectively causing additional difficulties in determining integration constants. Thus, the free-body approach is easier then the method of integration in this case.
4. For beam deflection, Section 7.4* introduces a method based on the integration approach, that eliminates drawing free-body diagrams for each segment to account for jumps in the loading. But that method requires an additional concept—*discontinuity functions* (also called *singularity functions*).

6.4 SHEAR AND MOMENT DIAGRAMS

Shear and moment diagrams are plots of internal shear force and internal bending moment as a function of x. By looking at these plots, we can immediately see the maximum values of the shear force and the bending moment, as well as the location of these maximum values. One way of making these plots is to determine the shear force and bending moment as a function of x, as in Section 6.3, and plot the results. However, for simple loadings there exists an easier alternative. We first discuss how the distributed forces are accounted, then how to account for the point forces and moments.

6.4.1 Distributed Force

The graphical technique described in this section is based on the interpretation of an integral as the area under a curve. The minus signs[7] in Equations (6.17) and (6.18) lead to positive areas being subtracted and negative areas being added. To overcome this problem of flip-flop of sign in the graphical procedure, we introduce $V = -V_y$. Let V_1 and V_2 be the values of V at x_1 and x_2, respectively. Let M_1 and M_2 be the of values of M_z at x_1 and x_2, respectively. Equations (6.17) and (6.18) can be written in terms of V as $dV/dx = p_y$ and $dM_z/dx = V$. Integration then yields

$$V_2 = V_1 + \int_{x_1}^{x_2} p_y\, dx \tag{6.19}$$

$$M_2 = M_1 + \int_{x_1}^{x_2} V\, dx \tag{6.20}$$

The key idea is to recognize that the values of the integrals in Equations (6.19) and (6.20) are the areas under the load curve p_y and the curve defining V, respectively. If we know V_1 and M_1, then by adding or subtracting the areas under the respective curves, we can find V_2 and M_2. We then move to point 2, where we now know the shear force and bending moment, and consider it as point 1 for the next segment of the beam. Moving in this bootstrap manner, we go across the beam accounting for the distributed forces.

Shear force curve

Recall that p_y is positive in the positive y direction. Thus in Figure 6.41*a* and *b*, $p_y = +w$, and from Equation (6.19) we obtain $V_2 = V_1 + w(x_2 - x_1)$. Similarly, in Figure 6.46*c* and *d*, $p_y = -w$, and from Equation (6.19) we obtain $V_2 = V_1 - w(x_2 - x_1)$. The term $w(x_2 - x_1)$ is the area of the rectangle and represents the magnitude of the integral in Equation (6.19).The line joining the values of V_1 and V_2 is a straight line because the integral of a constant function will result in a linear function.

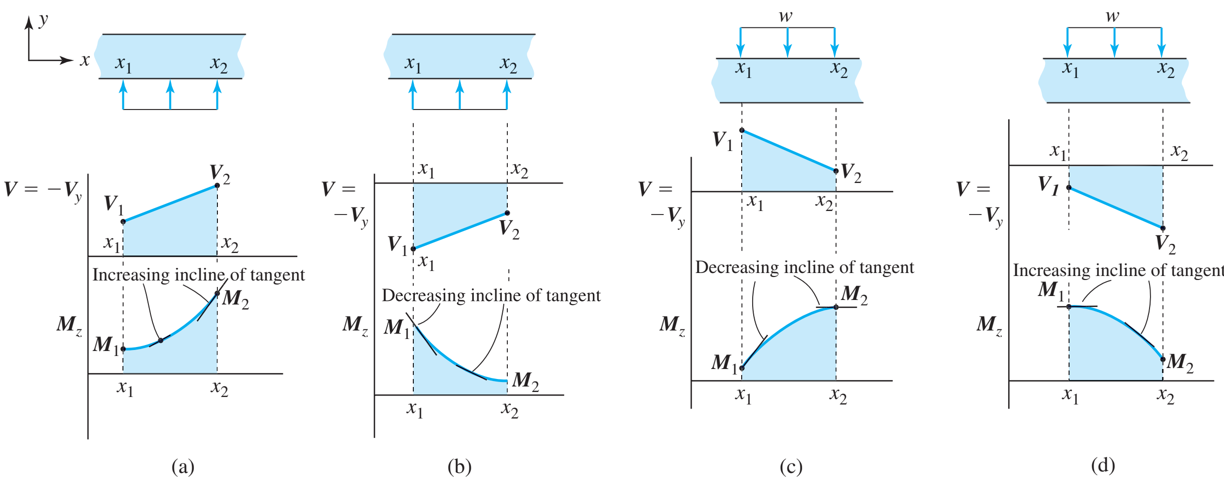


Figure 6.41 Shear and moment diagrams for uniformly distributed load.

Bending moment curve

The integral in Equation (6.20) represents the area under the curve defining V, that is, the areas of the trapezoids shown by the shaded regions in Figure 6.41. In Figure 6.41*a* and *c*, V is positive and we add the area to M_1 to get M_2. In Figure 6.41*b* and *d*, V is negative and we subtract the area from M_1 to get M_2. As V is linear between x_1 and x_2, the integral in Equation (6.20) will generate a quadratic function. But what would be the curvature of the moment curve, concave or convex? To answer this question, we note that the derivative of the moment curve—that is, the slope of the tangent—is equal to the value on the shear force diagram. To avoid some ambiguities associated with the sign[8] of a slope, we consider the inclination of the tangent to the moment curve, $|dM_z/dx| = |V|$. If the magnitude of V is increasing, the inclination of the tangent to the moment curve must increase, as shown in Figure 6.41*a* and *d*. If the magnitude of V is decreasing, the inclination of the tangent to the moment curve must decrease, as shown in Figure 6.41*b* and *c*.

[7]This is a consequence of trying to stay mathematically consistent while keeping the directions of shear force and shear stress the same. See footnote 6.
[8]We avoid statements such as "increasing negative slope," which could mean more negative or less negative. "Decreasing negative slope." is similarly ambiguous.

An alternative approach to getting the curvature of the moment curve is to note that if we substitute Equation (6.18) into Equation (6.17), we obtain $d^2\boldsymbol{M}_z/dx^2 = p_y$. If p_y is positive, then the curvature of the moment curve is positive, and hence the curve is concave, as shown in Figure 6.41*a* and *b*. If p_y is negative, then the curvature of the moment curve is negative, and the curve is convex, as shown in Figure 6.41*c* and *d*.

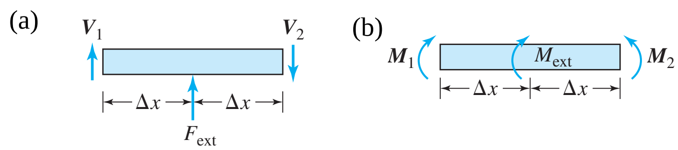
We call our conclusions the *curvature rule* for quadratic $\boldsymbol{M}_z$ curves:

> The curvature of the $\boldsymbol{M}_z$ curve must be such that the incline of the tangent to the $\boldsymbol{M}_z$ curve must increase (or decrease) as the magnitude of $\boldsymbol{V}$ increases (or decreases).
>
> or (6.21)
>
> The curvature of the moment curve is concave if p_y is positive, and convex if p_y is negative.

6.4.2 Point Force and Moments

It was noted in Comment 2 of Example 6.10 that the values of the internal shear force and the bending moment jump as one crosses an applied point force and moment, respectively. In Section 4.2.8 on axial force diagrams and in Section 5.2.6 on torque diagrams we used a template to give us the correct direction of the jump. We use the same idea here.

A template is a small segment (Δx tends to zero in Figure 6.42) of a beam on which the external moment M_{ext} and an external force F_{ext} are drawn. The directions of F_{ext} and M_{ext} are arbitrary. The ends at $+\Delta x$ and $-\Delta x$ from the applied external force and moment represent the imaginary cut just to the left and just to the right of the applied external forces and moments. On these cuts the internal shear force and the internal bending moment are drawn. Equilibrium equations are written for this $2\Delta x$ segment of the beam to obtain the template equations.



Template Equations $\quad \boldsymbol{V}_2 = \boldsymbol{V}_1 + F_{ext} \quad\quad \boldsymbol{M}_2 = \boldsymbol{M}_1 + M_{ext}$

Figure 6.42 Beam templates and equations for (*a*) Shear force (*b*) Moment.

Shear force template

Notice that the internal forces $\boldsymbol{V_1}$ and $\boldsymbol{V_2}$ are drawn opposite to the direction of positive internal shear forces, as per the definition $\boldsymbol{V} = -\boldsymbol{V}_y$, which is an additional artifact of the procedure to remember. To avoid this, we note that the sign of F_{ext} is the same as the direction in which $\boldsymbol{V_2}$ will move relative to $\boldsymbol{V_1}$. In the future we will not draw the shear force template but use the following observation:

- $\boldsymbol{V}$ will jump in the direction of the external point force.

Moment template

On the moment template, the internal moments are drawn according to our sign convention, discussed in Section 6.2.6. Unlike the observation about the jump in $\boldsymbol{V}$, there is no single observation that is valid for all coordinate systems. Thus the moment template must be drawn and the corresponding template equation used as follows.

If the external moment on the beam is in the direction of the assumed moment M_{ext} on the template, then the value of $\boldsymbol{M_2}$ is calculated according to the template equation. If the external moment on the beam is opposite to the direction of M_{ext} on the template, then $\boldsymbol{M_2}$ is calculated by changing the sign of M_{ext} in the template equation.

6.4.3 Construction of Shear and Moment Diagrams

Figure 6.43 is used to elaborate the procedure for constructing shear and moment diagrams as we outline it next.

Step 1 Determine the reaction forces and moments.

The free-body diagram for the entire beam is drawn, and the reaction forces and moments are calculated at the supports at A and B, as shown in Figure 6.43.

Step 2 Draw and label the vertical axes for $\boldsymbol{V}$ and $\boldsymbol{M_z}$ along with the units to be used.

We show $\boldsymbol{V} = -\boldsymbol{V_y}$ on the axis to remind ourselves that the positive and negative values read from the plots are for $\boldsymbol{V}$, whereas the formula that will be developed in Section 6.6 for the bending shear stress will be in terms of $\boldsymbol{V_y}$.

Step 3 Draw the beam with all forces and moments. At each change of loading draw a vertical line.

The vertical lines define the segments of the beam between two points x_1 and x_2 where the values of shear force and moment will be calculated. The vertical lines also represent points where $\boldsymbol{V}$ and $\boldsymbol{M_z}$ values may jump, such as at point C in cases 2 and 3 in Figure 6.43.

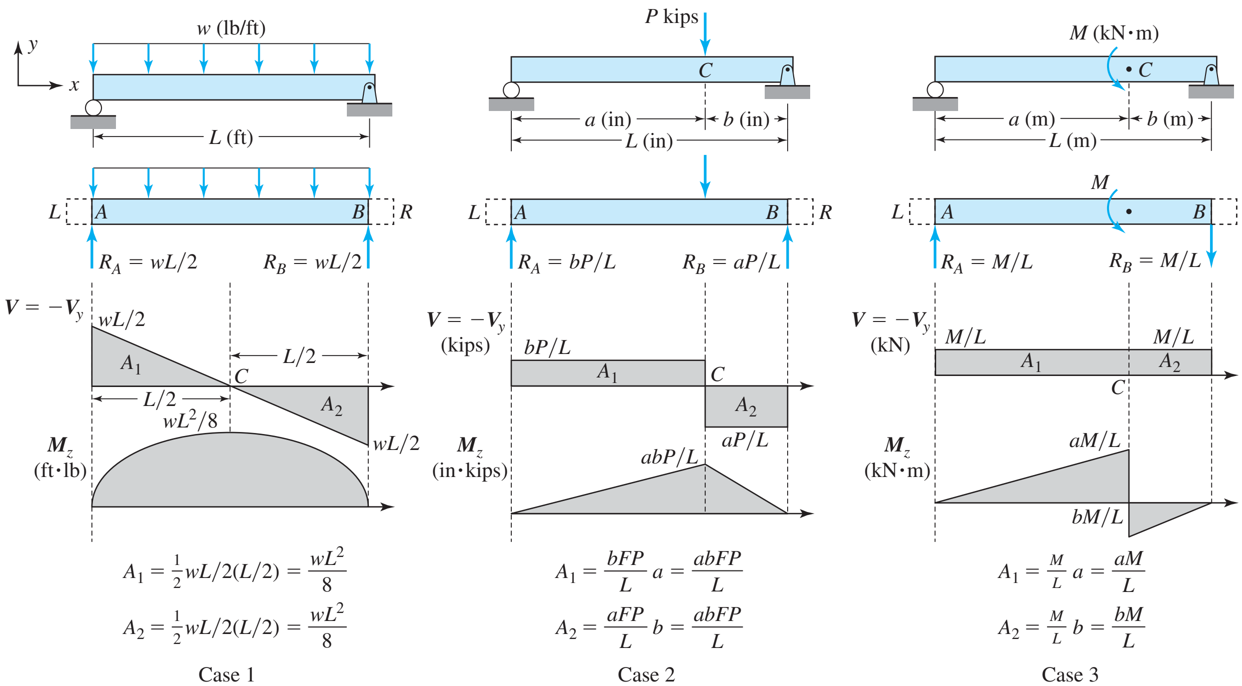


Figure 6.43 Construction of shear and moment diagrams.

Step 4 Consider imaginary extensions on the left and right ends of the beam. $\boldsymbol{V}$ and $\boldsymbol{M_z}$ are zero in these imaginary extensions.

In the imaginary left extension, LA at the beams shown in Figure 6.43, $\boldsymbol{V_1}$ and $\boldsymbol{M_1}$ are zero, and we can start our process at this segment. Point A (the start of the beam) can now be treated like any other point on the beam at which there is a point force and/or point moment. At the right imaginary extension BR, the values of the shear force and bending moment must return to zero, providing a check on our solution procedure.

Shear force diagram

Step 5 If there is a point force, then increase the value of $\boldsymbol{V}$ in the direction of the point force.

Just before point A in Figure 6.43, $\boldsymbol{V_1} = 0$ as we are in the imaginary extension. As we cross point A, the value of $\boldsymbol{V}$ jumps upward (positive) by the value of the reaction force R_A, which is in the upward direction.

At point C in case 2 we jump in the direction of P, which is pointed downward; that is, we subtract P from the value of $\boldsymbol{V_1}$. In other words, $\boldsymbol{V_2} = bP/L - P = (b - L)P/L = -aP/L$ just after point C, as shown.

The reaction force R_B is upward in cases 1 and 2, so we add the value of R_B to $\boldsymbol{V_1}$. In case 3 R_B is downward, so we subtract the value of R_B from $\boldsymbol{V_1}$. As expected in all cases, we return to a zero value for force $\boldsymbol{V}$ in the imaginary extension BR.

Step 6 Compute the area under the curve of the distributed load. Add the area to the value of $\boldsymbol{V}_1$ if p_y is positive, and subtract it if p_y is negative, to obtain the value of $\boldsymbol{V}_2$.

In case 1 the area under the distributed force is wL and p_y is negative. Therefore we subtract wL from the value of $\boldsymbol{V}$ just after A $(+wL/2)$ to get the value of $\boldsymbol{V}$ just before B $(-wL/2)$.

Step 7 Repeat Steps 5 and 6 until the imaginary extension at the right of the beam is reached. If the value of $\boldsymbol{V}$ is not zero in the imaginary extension, then check Steps 5 and 6 for each segment of the beam.

For the three simple cases considered in Figure 6.43, this step is not required.

Step 8 Draw additional vertical lines at any point where the value $\boldsymbol{V}$ is zero. Determine the location of these points by using geometry.

The points where $\boldsymbol{V}$ is zero represent the location of the maximum or minimum values of the bending moment because $d\boldsymbol{M}_z/dx = 0$ at these points. In case 1 $\boldsymbol{V} = 0$ at point C. The location can be found by using similar triangles.

Step 9 Calculate the areas under the $\boldsymbol{V}$ curve and between two adjacent vertical lines.

Areas A_1 and A_2 can be found and recorded as shown in Figure 6.48.

Moment diagram

Step 10 If there is a point moment, then use the moment template and the template equation to determine the direction of the jump.

In case 3 there is a point moment at point C. Comparing the direction of the moment at C to that in the template in Figure 6.42, we conclude that $M_{ext} = -M$. Just before C, $\boldsymbol{M_1} = aM/L$. As per the template equation, $\boldsymbol{M_2} = aM/L - M = (a - L)M/L = -bM/L$, which is the value just after C.

In all three cases there is no point moment at A, hence our starting value is zero. If there were a point moment at A, we would use the moment template and the template equation to determine the starting value as we move from the imaginary segment to just right of A.

Step 11 To move from the right of one vertical line to the left of the next vertical line, add the areas under the $\boldsymbol{V}$ curve if $\boldsymbol{V}$ is positive, and subtract the areas if $\boldsymbol{V}$ is negative. Draw the curve according to the curvature rule in Equation (6.21).

In all three cases the area A_1 is positive and we add the value of the area to the value of the moment at point A to obtain the moment just before C. In cases 1 and 2 the area A_2 is negative. Hence we subtract the value of A_2 from the moment value just after C to get a zero value just before B. In case 3 A_2 is positive and we add the value of A_2 to the moment value just after C.

In cases 2 and 3 the $\boldsymbol{V}$ curve is constant in each segment, and hence the $\boldsymbol{M_z}$ curve is linear in each segment. In case 1 the $\boldsymbol{V}$ curve is linear. Hence the $\boldsymbol{M_z}$ curve is quadratic and we need to determine the curvature of the curve. The inclination of the tangent to the $\boldsymbol{M_z}$ curve at A is nonzero, and it decreases to zero at C. Thus the inclination of the tangent decreases as the magnitude of $\boldsymbol{V}$ decreases. Similarly, as we move from C to B, the inclination of the tangent increases from zero to a nonzero value, consistent with the increasing magnitude of $\boldsymbol{V}$.

Alternatively, in case 1 $p_y = -w$, and hence the curvature of the moment curve is convex.

Step 12 Repeat Steps 10 and 11 until you reach the imaginary extension on the right of the beam. If the value of $\boldsymbol{M}_z$ is not zero in the imaginary extension, then check Steps 10 and 11 for each segment of the beam.

This procedure is applied and elaborated in Examples 6.11 and 6.12.

6.5 STRENGTH BEAM DESIGN

This section addresses two issues. The first relates to choosing a standard, commonly manufactured beam cross section that will be cheapest to use. The second issue relates to determining the maximum tensile or compressive bending normal stress.

6.5.1 Section Modulus

In the design of steel beams, the tensile and compressive strength are usually assumed to be equal. We calculate the magnitude of the maximum bending normal stress using Equation (6.12), which can be written as $\sigma_{max} = \boldsymbol{M}_{max} y_{max}/I_{zz}$, where $\boldsymbol{M}_{max}$ is the magnitude of the maximum internal bending moment, and y_{max} the distance of the point farthest from the neutral axis. The moment of inertia I_{zz} and y_{max} depend on the geometry of the cross section. Thus, Equation (6.12) requires two variables to determine the best geometric shape in a particular design. A variable called **section modulus** S, simplifies the equation for the maximum bending normal stress

$$S = \frac{I_{zz}}{y_{max}} \qquad \sigma_{max} = \frac{M_{max}}{S} \tag{6.22}$$

Section C.6 give the section modulus S for steel beams of standard shapes and Example 6.12 shows its use in design.

6.5.2 Maximum Tensile and Compressive Bending Normal Stresses

Section 3.1 observed that a brittle material usually ruptures when the maximum tensile normal stress exceeds the ultimate tensile stress of the material. Cracks in materials propagate due to tensile stress. Adhesively bonded material debonds from tensile normal stress called *peel stress*. Thus a structure designed for maximum normal stress may fail when the maximum tensile stress is less in magnitude than the maximum compressive stress. Similarly, failure may occur when the maximum compressive normal stress is less than the maximum tensile normal stress. This may happen because *buckling*, which is discussed in Chapter 11. Proper beam design must take into account failure due to tensile or compressive normal stresses.

To account for these stresses, it may be necessary to determine two stress values—the maximum tensile and compressive bending normal stress. These may differ when the top and the bottom of the beam are at different distances from the neutral axis of the cross section. Since both $\boldsymbol{M_z}$ and y affect the sign of the bending normal stress in Equation (6.12), stresses must be checked at four points:

- On the top and bottom surfaces on the cross-section location where $\boldsymbol{M_z}$ is a maximum positive value.
- On the top and bottom surfaces on the cross-section location where $\boldsymbol{M_z}$ is a maximum negative value.

Example 6.11 elaborates this issue.

EXAMPLE 6.11

Figure 6.44 shows a loaded beam and cross section. (a) Draw the shear force and bending moment diagrams for the beam, and determine the maximum shear force and bending moment. (b) Determine the maximum tensile and compressive bending normal stress in the beam.

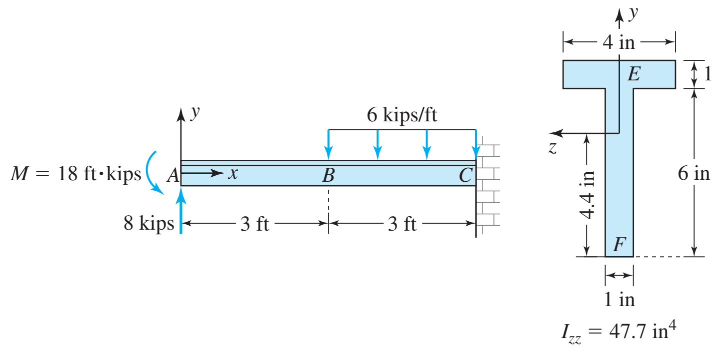


Figure 6.44 Beam and loading in Example 6.11.

PLAN

(a) We can determine the reaction force and moment at wall C and follow the procedure for drawing shear and moment diagrams described in Section 6.4.3. (b) We can find σ_{xx} from Equation (6.12) at points E and F at those cross sections where the $\boldsymbol{M_z}$ value is maximum positive and maximum negative. From these four values we can find the maximum tensile and compressive bending normal stresses.

SOLUTION

(a) We draw the shear force and bending moment diagram as per the procedure outlined in Section 6.4.3.

Step 1: From the free-body diagram shown in Figure 6.45 we can determine the value of the reaction force R_w by equilibrium of forces in the y direction. By equilibrium of moment about point C, we can determine the reaction moment M_w. The values of these reactions are

$$R_w = 10 \text{ kips} \tag{E1}$$

$$M_w = 3 \text{ ft·kips} \tag{E2}$$

Step 2: We draw and label the axes for $\boldsymbol{V}$ and $\boldsymbol{M_z}$ and record the units.

Step 3: The beam is shown in Figure 6.45 with all forces and moments acting on it. Vertical lines at points *A*, *B*, and *C* are drawn as shown.

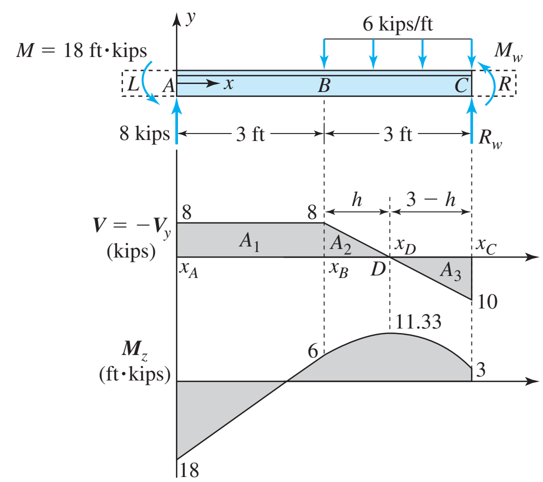


Figure 6.45 Shear and moment diagrams in Example 6.11.

Step 4: We draw imaginary extensions *LA* and *CR* to the beam.

Shear force diagram in Figure 6.45

Steps 5, 6, 7: In segment *LA* the shear force is zero, and hence $\boldsymbol{V_1} = 0$. The 8-kips force at *A* is upward, so we jump to a value of $\boldsymbol{V_2} = +8$ kips just to the right of point *A*. $p_y = 0$ in segment *AB*, and hence the value of $\boldsymbol{V}$ remains at 8 kips just before *B*. Since there is no point force at *B*, there is no jump in $\boldsymbol{V}$ at *B*.

In segment *BC* the area under the distributed load is 18 kips. As p_y is negative, we subtract the area from 8 kips to get a value of –10 kips just before *C*. We join it by a straight line as p_y is uniform in *BC*.

The reaction force R_w is upward, so we add the value to –10 kips to get a zero value just after *C*, confirming the correctness of our solution.

Step 8: At point *D*, where $\boldsymbol{V} = 0$, we draw another vertical line. To find the location of point *D*, we use the two similar triangles on either side of point *D* to get the value of *h* in Equation (E3).

$$\frac{8\text{ kips}}{h} = \frac{10\text{ kips}}{3\text{ ft} - h} \qquad \text{or} \qquad h = 1.333\text{ ft} \tag{E3}$$

Step 9: We calculate areas A_1, A_2, and A_3 as

$$A_1 = (8)(3) = 24 \qquad A_2 = \frac{1}{2}(8)(h) = 5.33 \qquad A_3 = \frac{1}{2}10(3 - h) = 8.33 \tag{E4}$$

Moment diagram in Figure 6.45

Steps 10, 11, 12: In segment *LA* the bending moment is zero, and hence $\boldsymbol{M_1} = 0$. Comparing the 18-ft·kips couple at point *A* with M_{ext} in the moment template in Figure 6.47, we obtain $M_{ext} = -18$ ft·kips. Hence from the template equation $\boldsymbol{M_2} = -18$ ft·kips just to the right of point *A*.

The area A_1 is positive, so we add its value to –18 ft·kips to obtain $\boldsymbol{M_2} = +6$ ft·kips just before *B*. As $\boldsymbol{V}$ was constant in *AB*, we join the moments at points *A* and *B* by a straight line, as shown.

The area A_2 is positive, so we add its value to +6 ft·kips to obtain $\boldsymbol{M_2} = +11.33$ ft·kips just before *D*. As $\boldsymbol{V}$ is linear between *B* and *D*, the integral will result in a quadratic function. The magnitude of the shear force is decreasing. Hence the incline of the tangent to the moment curve must decrease as we move from point *B* toward point *D*, resulting in the convex curve shown between *B* and *D*. Alternatively, since p_y is negative between *B* and *D*, the curve is convex.

The area A_3 is negative, so we subtract its value from 11.33 ft·kips to obtain +3 ft·kips just before *C*. As $\boldsymbol{V}$ is linear between *D* and *C*, the integral will result in a quadratic function. Since the magnitude of the shear force is increasing. Hence the incline of the tangent to the moment curve must increase as we move from point *D* toward *C*, resulting in the convex curve shown between *D* and *C*.

Comparing the moment M_w at *C* with M_{ext} in the template in Figure 6.42, we obtain $M_{ext} = -M_w = -3$ ft·kips. Hence from the template equation $\boldsymbol{M_2} = 0$ just to the right of point *C*. That is, in the imaginary segment *CR* the moment is zero as expected, confirming the correctness of our construction.

From Figure 6.45 we see that the maximum values of $\boldsymbol{V}$ and $\boldsymbol{M_z}$ are –10 kips and –18 ft·kips, respectively. Recollect that, $\boldsymbol{V} = -\boldsymbol{V_y}$. This gives us the maximum values of the shear force and the bending moment.

ANS. $(\boldsymbol{V_y})_{max} = 10\text{ kips} \qquad (\boldsymbol{M_z})_{max} = -18\text{ ft}\cdot\text{kips}$

(b) The maximum positive moment occurs at *D* ($\boldsymbol{M_D} = +11.33$ ft·kips = 136 in·kips) and the maximum negative moment occurs at *A* ($\boldsymbol{M_A} = -18$ ft·kips = –216 in.·kips). We can evaluate the bending normal stress at points *E* ($y_E = +2.6$ in) and *F* ($y_F = -4.4$ in.) on the cross sections at *A* and *D* using Equation (6.12):

- On cross section *A*, at point *E*, the bending normal stress is

$$\sigma_{AE} = -\frac{(-216 \text{ in.}\cdot\text{kips})(2.6 \text{ in.})}{47.7 \text{ in.}^4} = +11.8 \text{ ksi} \tag{E5}$$

- On cross section A, at point F, the bending normal stress is

$$\sigma_{AF} = -\frac{(-216 \text{ in.}\cdot\text{kips})(-4.4 \text{ in.})}{47.7 \text{ in.}^4} = -19.9 \text{ ksi} \tag{E6}$$

- On cross section D, at point E, the bending normal stress is

$$\sigma_{DE} = -\frac{(136 \text{ in.}\cdot\text{kips})(2.6 \text{ in.})}{47.7 \text{ in.}^4} = -7.4 \text{ ksi} \tag{E7}$$

- On cross section D, at point F, the bending normal stress is

$$\sigma_{DF} = -\frac{(136 \text{ in.}\cdot\text{kips})(-4.4 \text{ in.})}{47.7 \text{ in.}^4} = +12.6 \text{ ksi} \tag{E8}$$

From the results in Equations (E5), (E6), (E7), and (E8), it is clear that the maximum tensile bending normal stress occurs at point F on the cross section at D. The maximum compressive bending normal stress occurs instead at point F on the cross section at A.

ANS. $\sigma_{DF} = 12.6 \text{ ksi (T)}$ $\sigma_{AF} = 19.9 \text{ ksi (C)}$

COMMENTS

1. In practice we need not write each step. Equations (E3) through (E4) suffice for drawing the shear and moment diagrams.
2. In Figure 6.45 we see that at point D, where shear force is zero, there is a local maximum in the bending moment. But the maximum moment in the beam is at point A, where a point moment is applied.
3. If we were determining the *magnitude* of the maximum bending normal stress, then we need to evaluate stress one point only—where the moment is a maximum (cross section A) and where y is also a maximum (point F).
4. We could use the template shown in Figure 6.46 to determine the direction of the jump in the moment. It can be verified that the moment jumps at A and C will be as before. This shows that the direction of M_{ext} on the template is immaterial. Thus there is no need to memorize the template, which can be drawn before starting on the shear and moment diagrams.

Template Equation: $\boldsymbol{M}_2 = \boldsymbol{M}_1 - M_{ext}$

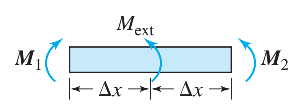


Figure 6.46 Alternative template and equation.

EXAMPLE 6.12

Consider the beam shown in Figure 6.38. Select the lightest W- or S-shaped beams from these given in Appendix E if the allowable bending normal stress is 53 MPa in tension or compression.

PLAN

We can draw the shear and moment diagrams using the procedure described in Section 6.4.3. From the moment diagram we can find the maximum moment. Using the allowable bending normal stress of 53 MPa and Equation (6.22), we can find the minimum sectional modulus. Using Section C.6, we can make a list of the beams for which the sectional modulus is just above the one we determined and choose the lightest beam we can use.

SOLUTION

Step 1: The reaction forces at points A and D were determined in Example 6.10.

Step 2: The beam with all forces and moments acting on it is shown in Figure 6.47. Vertical lines at points A, B, C, and D are shown as drawn.

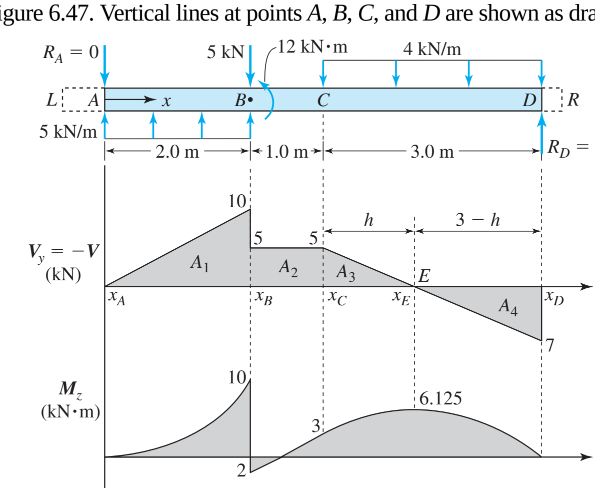


Figure 6.47 Shear and moment diagrams in Example 6.12.

Step 3: We draw imaginary extensions LA and DR to the beam.

Step 4: We label the axes for $\boldsymbol{V}$ and $\boldsymbol{M_z}$ and record the units.

Shear force diagram in Figure 6.47

Steps 5, 6, 7: In segment *LA*, $\boldsymbol{V_1} = 0$. Since R_A is zero, there is no jump at *A* and we start our diagram at zero.

In segment *AB*, $p_y = +5$ kN/m. Hence we add the area of $5 \times 2 = 10$ kN to obtain $\boldsymbol{V_2} = 10$ kN just before point *B* and draw a straight line between the values of $\boldsymbol{V}$ at *A* and *B*, as shown in Figure 6.47. At *B* the point force of 5 kN is downward. Thus we jump downward by 5 kN to obtain $\boldsymbol{V_2} = 5$ kN just after point *B*.

In segment *BC*, $p_y = 0$; hence the value of $\boldsymbol{V}$ does not change until point *C*.

In segment *CD*, $p_y = -4$ kN/m. Hence we subtract the area of $4 \times 3 = 12$ kN to obtain $\boldsymbol{V_2} = -7$ kN just before point *D* and draw a straight line between the values of $\boldsymbol{V}$ at *C* and *D*, as shown in Figure 6.47. The reaction force at *D* is upward, so we jump upward by 7 kN to obtain $\boldsymbol{V_2} = 0$ kN just after point *D*. That is, in the imaginary segment *DR* the shear force is zero as expected, confirming the correctness of our construction.

Step 8: At point *E*, where $\boldsymbol{V_y} = 0$, we draw another vertical line. To find the location of point *E*, we use the two similar triangles on either side of point *E* to obtain h:

$$\frac{5\text{ m}}{h} = \frac{7\text{ m}}{3\text{ m} - h} \qquad \text{or} \qquad h = 1.25\text{ m} \tag{E1}$$

Step 9: We calculate the areas A_1 through A_4

$$A_1 = \frac{1}{2}10 \times 2 = 10 \qquad A_2 = 5 \times 1 = 5 \qquad A_3 = \frac{1}{2}5h = 3.125 \qquad A_4 = \frac{1}{2}7(3-h) = 6.125 \tag{E2}$$

Bending moment diagram in Figure 6.47

Steps 10, 11, 12: In segment *LA* the bending moment is zero, and hence $\boldsymbol{M_1} = 0$. As there is no point moment at *A*, we start our moment diagram at zero.

As $\boldsymbol{V}$ is positive in segment *AB*, we add the area A_1 to obtain $\boldsymbol{M_2} = +10$ kN·m just before *B*. As $\boldsymbol{V}$ is linear in *AB*, the integral will result in a quadratic function between *A* and *B*. As the magnitude of the shear force is increasing, the incline of the tangent to the moment curve must increase as we move from point *A* toward point *B*, resulting in the concave curve shown between *A* and *B*. Alternatively, since p_y is positive in *AB*, the moment curve is concave.

Comparing the moment 12 kN·m at *B* with M_{ext} in the template in Figure 6.42, we obtain $M_{ext} = -12$ kN·m. Hence from the template equation $\boldsymbol{M_2} = 10 - 12 = -2$ kN·m just to the right of point *B*.

As $\boldsymbol{V}$ is positive in segment *BC*, we add the area A_2 to obtain the value of $\boldsymbol{M_2} = +3$ kN·m just before *C*. As $\boldsymbol{V}$ is constant between *B* and *C*, the integral will result in a linear function, so we draw a straight line between *B* and *C*.

As $\boldsymbol{V}$ is positive in segment *CE*, we add the area A_3 to obtain the value of $\boldsymbol{M_2} = +6.125$ kN·m just before *E*. As $\boldsymbol{V}$ is linear between *C* and *E*, the integral will result in a quadratic function. As the magnitude of the shear force is decreasing, the incline of the tangent to the moment curve must also decrease as we move from point *C* toward *E*, resulting in the convex curve between *C* and *E*, as shown. Alternatively, since p_y is negative in *CE*, the moment curve is convex.

As $\boldsymbol{V}$ is positive in *ED*, we add the area A_4 to obtain the value of $\boldsymbol{M_2} = 0$ just before *D*. As $\boldsymbol{V}$ is linear between *E* and *D*, the integral will result in a quadratic function. As the magnitude of the shear force is increasing, the incline of the tangent to the moment curve must also increase as we move from point *E* toward *D*, resulting in the convex curve between *E* and *D*, as shown. Alternatively, since p_y is negative in *ED*, the moment curve is convex.

As there is no point moment at *D*, there will be no jump in the moment at *D*. Hence we obtain a zero value for the moment in the imaginary segment *DR* as expected, confirming the correctness of our construction.

From the moment diagram in Figure 6.47 the maximum moment is $\boldsymbol{M_{max}} = 10$ kN·m. Noting that the allowable bending normal stress is 53 MPa, Equation (6.22) yields

$$\sigma_{max} = \frac{10(10^3)\text{ N}}{S} \le 53(10^6)\text{ N/m}^2 \qquad \text{or} \qquad S \ge 188.7(10^3)\text{ mm}^3 \tag{E3}$$

From Section C.6 we obtain the following list of W- and S-shaped beams that have a section modulus close to that given in Equation (E3):

W150 × 29.8	$S = 219 \times 10^3\text{ mm}^3$	and	S200 × 27.4	$S = 236 \times 10^3\text{ mm}^3$
W200 × 22.5	$S = 194.2 \times 10^3\text{ mm}^3$		S180 × 30	$S = 198.3 \times 10^3\text{ mm}^3$

The lightest beam is W200 × 22.5 as it has a mass of only 22.5 kg/m.

ANS. W200 × 22.5

COMMENTS

1. This example demonstrate the use of the section modulus in selecting the beam cross section from a set of standard shapes. But the section modulus can also be used with nonstandard shapes.
2. The alternative template shown in Figure 6.46 could have been used in this example. It would result in the same jumps as shown in Figure 6.47.
3. Again, only Equations (E1) through (E2) are needed to obtain the shear and moment diagrams. From here on, the shear and moment diagrams will be drawn without additional explanations.

QUICK TEST 6.1

Time: 20 minutes/Total: 20 points

Answer true or false and justify each answer in one sentence. Grade yourself with the answers given in Appendix E. Assume linear elastic, homogeneous material unless stated otherwise.

1. If you know the geometry of the cross section and the bending normal strain at one point on a cross section, then the bending normal strain can be found at any point on the cross section.

2. If you know the geometry of the cross section and the maximum bending normal stress on a cross section, then the bending normal stress at any point on the cross section can be found.

3. A rectangular beam with a 2-in. × 4-in. cross section should be used with the 2-in. side parallel to the bending (transverse) forces.

4. The best place to drill a hole in a beam is through the centroid.

5. In the formula $\sigma_{xx} = -\boldsymbol{M}_z y / I_{zz}$, y is measured from the bottom of the beam.

6. The formula $\sigma_{xx} = -\boldsymbol{M}_z y / I_{zz}$ can be used to find the normal stress on a cross section of a tapered beam.

7. The equations $\int_A \sigma_{xx}\, dA = 0$ and $\boldsymbol{M}_z = -\int_A y\sigma_{xx}\, dA$ cannot be used for nonlinear materials.

8. The equation $\boldsymbol{M}_z = -\int_A y\sigma_{xx}\, dA$ can be used for nonhomogeneous cross sections.

9. The internal shear force jumps by the value of the applied transverse force as one crosses it from left to right.

10. The internal bending moment jumps by the value of the applied concentrated moment as one crosses it from left to right.

PROBLEM SET 6.3

Equilibrium of shear force and bending moment

6.59 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.59; (b) show that your results satisfy Equations (6.17) and (6.18).

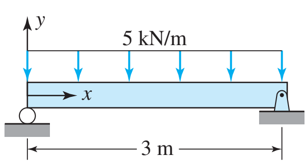


Figure P6.59

6.60 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.60; (b) show that your results satisfy Equations (6.17) and (6.18).

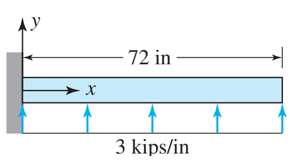


Figure P6.60

6.61 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.61; (b) show that your results satisfy Equations (6.17) and (6.18).

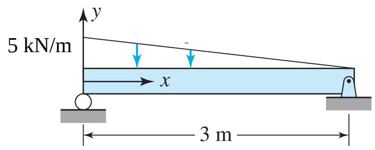


Figure P6.61

6.62 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.62; (b) show that your results satisfy Equations (6.17) and (6.18).

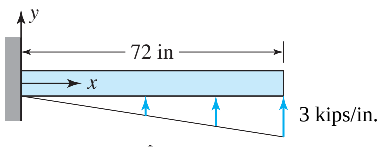


Figure P6.62

6.63 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.63; (b) show that your results satisfy Equations (6.17) and (6.18).

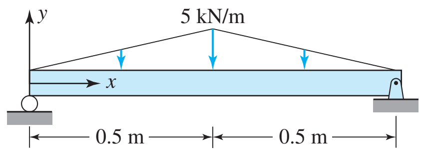


Figure P6.63

6.64 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.64; (b) show that your results satisfy Equations (6.17) and (6.18).

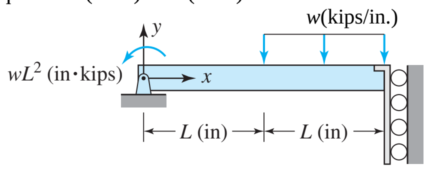


Figure P6.64

6.65 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.65; (b) show that your results satisfy Equations (6.17) and (6.18).

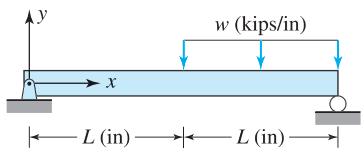


Figure P6.65

6.66 (a) Write the equations for shear force and bending moments as a function of x for the entire beam shown in Figure P6.66; (b) show that your results satisfy Equations (6.17) and (6.18).

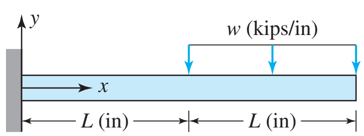


Figure P6.66

6.67 Consider the beam shown in Figure P6.67. (a) Write the shear force and moment equations as a function of x in segments AB and BC. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point B?

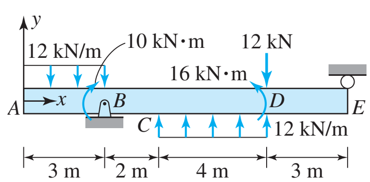


Figure P6.67

6.68 Consider the beam shown in Figure P6.67. (a) Write the shear force and moment equations as a function of *x* in segments *CD* and *DE*. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point *D*?

6.69 Consider the beam shown in Figure P6.69. (a) Write the shear force and moment equations as a function of *x* in segments *AB* and *BC*. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point *B*?

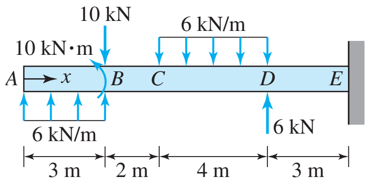


Figure P6.69

6.70 Consider the beam shown in Figure P6.69. (a) Write the shear force and moment equations as a function of *x* in segments *CD* and *DE*. (b) Show that your results satisfy Equations (6.17) and (6.18). (c) What are the shear force and bending moment values just before and just after point *D*?

6.71 During skiing, the weight of a person is often all on one ski. The ground reaction is modeled as a distributed force $p(x)$ and a concentrated fore P is modeled as shown in Figure P6.71. (a) Find shear force and bending moment as a function of x across the ski. (b) The ski is 50 mm wide and the thickness of the ski varies as shown. Determine the maximum bending normal stress. Use of spread sheet recommended.

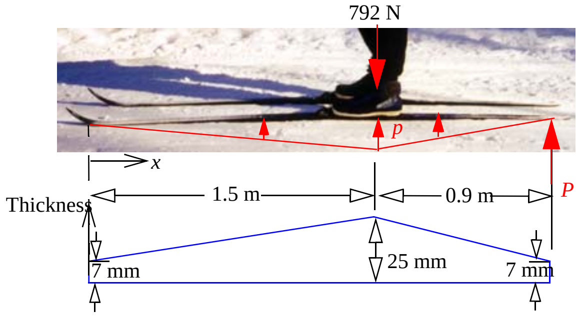


Figure P6.71

Shear and moment diagrams

6.72 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.72.

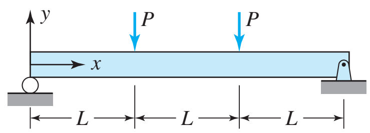


Figure P6.72

6.73 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.73.

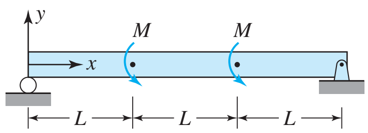


Figure P6.73

6.74 For the beam shown in Figure P6.59, draw the shear force bending moment diagram. Determine the maximum values of shear force and bending moment.

6.75 For the beam shown in Figure P6.60, draw the shear force bending moment diagram. Determine the maximum values of shear force and bending moment.

6.76 A man whose mass is 80 kg is sitting in the middle of a flat bottom boat, as shown in Figure 6.76. The weight of the boat per unit length between *A* and *B* is 130 N/m. To a first approximation assume the resisting water pressure acts between A and B and is uniform. Draw the shear force and bending moment diagram between *A* and *B*.

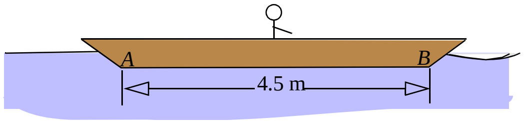


Figure P6.76

6.77 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.77. Determine the maximum values of shear force and bending moment.

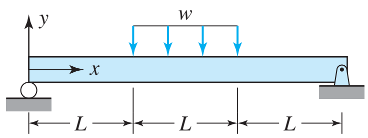


Figure P6.77

6.78 Draw the shear and moment diagrams for the beam and loading shown in Figure P6.78. Determine the maximum values of shear force and bending moment.

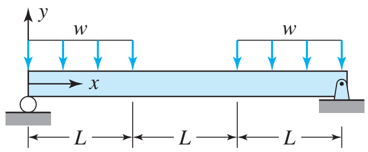


Figure P6.78

6.79 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.67.

6.80 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.69.

6.81 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.81.

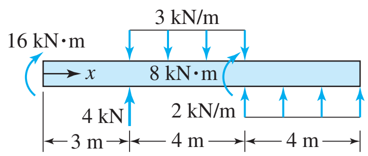


Figure P6.81

6.82 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.82.

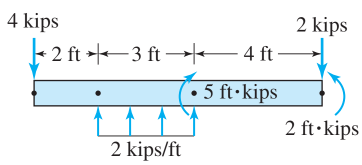


Figure P6.82

6.83 Determine the maximum values of shear force and bending moment. for the beam shown in Figure P6.83.

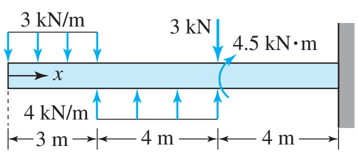


Figure P6.83

6.84 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.84.

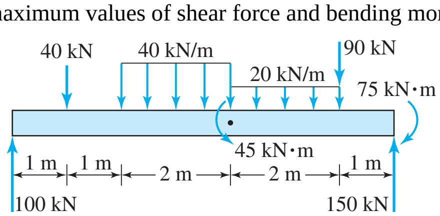


Figure P6.84

6.85 Determine the maximum value of the shear force and bending moment for the beam shown in Figure 6.85.

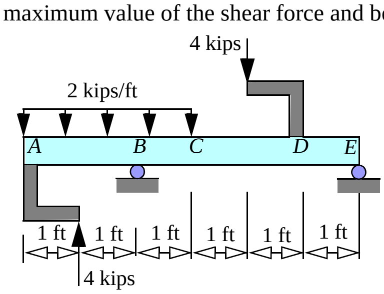


Figure P6.85

6.86 Determine the maximum values of shear force and bending moment for the beam shown in Figure P6.86.

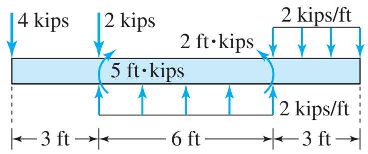


Figure P6.86

Maximum bending normal stress

6.87 A diver weighing 200 lb stands at the edge of the diving board, as shown in Figure 6.87. The diving board cross section is 16 in. x 1 in. Determine the maximum bending normal stress in the diving board.

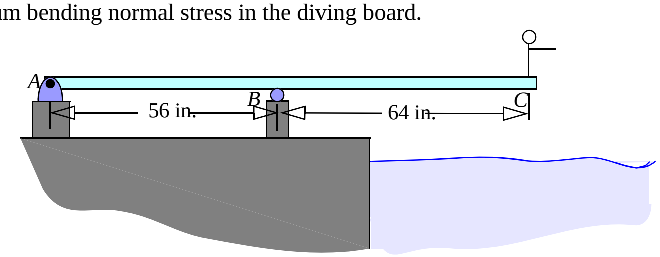


Figure P6.87

6.88 A diver weighing 200 lb stands at the edge of the diving board, as shown in Figure 6.87. The diving board cross section is 16 in. x 1 in. and has a weight of 60 lb. Model the weight of the diving board as a uniform distributed load of 0.5 lb/in. along the length. Determine the maximum bending normal stress in the diving board.

6.89 The diving board shown in Figure 6.87 has a cross section of 18 in. x 1 in. The allowable bending normal stress 10 ksi. What is the maximum force to the nearest pound that the board can sustain when the diver jumps on it before a dive. Neglect the weight of the diving board.

6.90 A father and his son are playing on a seesaw, as shown in Figure 6.90. The wooden plank of the see saw is 12 ft x 10 in. x 1.5 in. and is hinged in the middle. The weights of the father and son are 225 lb and 80 lb, respectively. The mass of the father m_F and mass of the son m_s times the acceleration a are the inertial forces acting on them at the time the plank is horizontal. Neglecting the weight of the plank, determine the maximum bending normal stress.

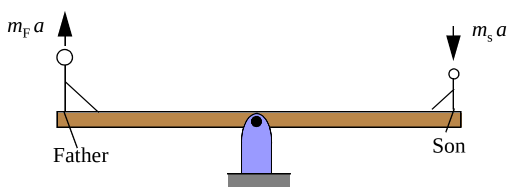


Figure P6.90

6.91 A mother and her daughter are on either side of the seesaw with the teenager son standing in the middle as shown in Figure 6.91. The wooden plank of the seesaw is 3.5 m x 250 mm x 40 mm and is hinged in the middle. The mass of the mother, son, and daughter are m_m= 70 kg, m_s= 80 kg, and m_d= 40 kg, respectively. At the time the plank is horizontal, inertial forces of mass times the acceleration a acts on the mother and daughter. Neglecting the weight of the plank, determine the maximum bending normal stress.

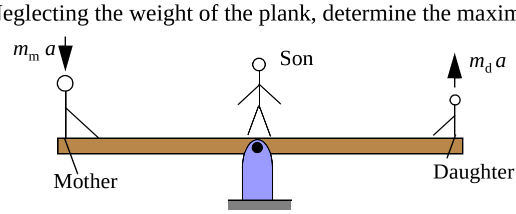


Figure P6.91

Design problems

6.92 A beam, its loading, and its cross section are as shown Figure P6.92. Determine the intensity w of the distributed load if the maximum bending normal stress is limited to 10 ksi (C) and 6 ksi (T). The second area moment of inertia is I_{zz} = 47.73 in.^{4}.

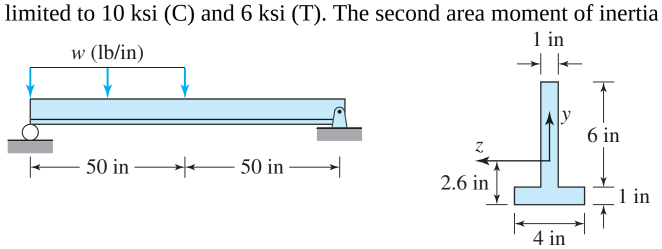


Figure P6.92

6.93 Two pieces of lumber are glued together to form the beam shown in Figure P6.93. Determine the intensity w of the distributed load if the maximum tensile bending normal stress in the glue is limited to 800 psi (T) and the maximum bending normal stress in wood is limited to 1200 psi.

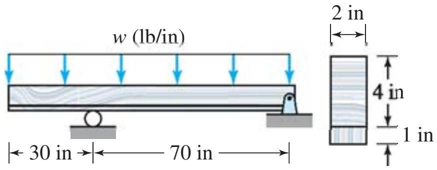


Figure P6.93

6.94 The beam shown in Figure P6.64 has a load w = 25 lb/in. and L = 72 in. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 21 ksi in tension and compression.

6.95 The beam shown in Figure P6.65 has a load w = 0.4 kips/in. and L = 48 in. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 16 ksi in tension and compression.

6.96 The beam shown in Figure P6.66 has a load w = 0.15 kips/in. and L = 48 in. Select the lightest W- or S-shaped beams from Section C.6 if the allowable bending normal stress is 21 ksi in tension and compression.

6.97 Consider the beam shown in Figure P6.67. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 180 MPa in tension and compression.

6.98 Consider the beam shown in Figure P6.69. Select the lightest W- or S-shaped beam from Section C.6 if the allowable bending normal stress is 225 MPa in tension and compression.

6.99 The wind pressure on a signpost is approximated as a uniform pressure, as shown Figure P6.99. A similar signpost is to be designed using a hollow square steel beam for the post. The outer dimension of the square is to be 12 in. If the allowable bending normal stress is 24 ksi and the pressure p = 33 lb/ft^2, determine the inner dimension of the lightest hollow beam to the nearest $\frac{1}{8}$ in.

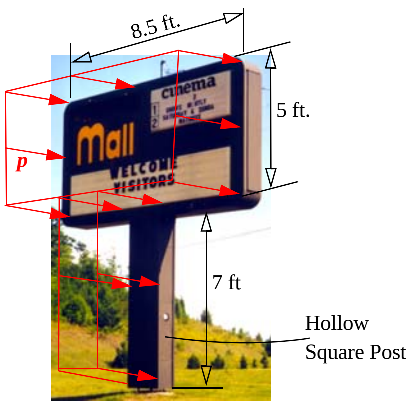


Figure P6.99

Stress concentration

6.100 The allowable bending normal stress in the stepped circular beam shown in Figure P6.100 is 200 MPa and P = 200 N. Determine the smallest fillet radius that can be used at section B. Use stress concentration graphs given in Section C.4.

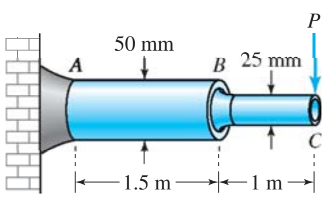


Figure P6.100

6.101 The allowable bending normal stress in the stepped circular beam shown in Figure P6.101 is 48 ksi. Determine the maximum intensity of the distributed load w assuming the fillet radius is: (a) 0.3 in.; (b) 0.5 in. Use stress concentration graphs given in Section C.4.

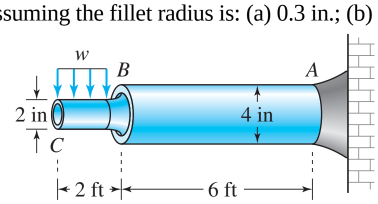


Figure P6.101

Fatigue

6.102 The fillet radius is 5 mm in the stepped aluminum circular beam shown in Figure P6.100. What should be the peak value of the cyclic load P to ensure a service life of one-half million cycles? Use the S–N curve shown in Figure 3.36.

6.103 The beam in Figure P6.101 is made from a steel alloy that has the S–N curve shown in Figure 3.36. The peak intensity of the cyclic distributed load is $w = 80$ lbs/in. and the fillet radius is 0.36 in. What is the predicted service life of the beam?

Stretch Yourself

6.104 A simply supported 3-m-long beam has a uniformly distributed load of 10 kN/m over the entire length of the beam. If the beam has the composite cross section shown in Figure P6.104, determine the maximum bending normal stress in each of the three materials. Use $E_{al} = 70$ GPa, $E_w = 10$ GPa, and $E_s = 200$ GPa. [Hint: Use Equations (6.14) and (6.16)].

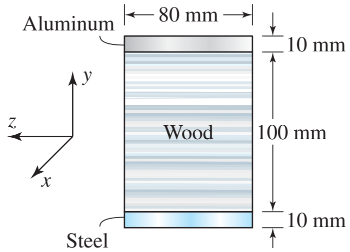


Figure P6.104

6.105 A steel ($E_{steel} = 200$ GPa) tube of outside diameter of 240 mm is attached to a brass ($E_{brass} = 100$ GPa) tube to form the cross section shown in Figure P6.105. Determine the maximum bending normal stress in steel and brass. [Hint: Use Equation (6.16)]

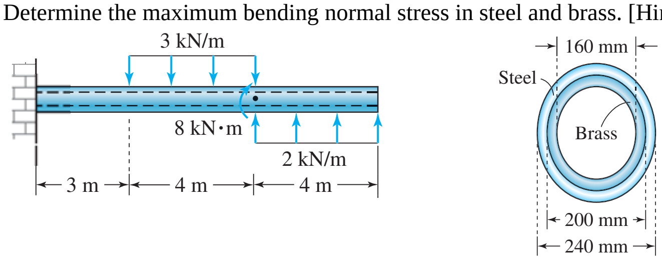


Figure P6.105

6.6 SHEAR STRESS IN THIN SYMMETRIC BEAMS

In Section 6.2.6 we observed that the maximum bending shear stress has to be nearly an order of magnitude less than the maximum bending normal stress for our theory to be valid. But shear stress plays an important role in bending, particularly when beams are constructed by joining a number of beams together to increase stiffness. In this section we develop a theory that can be used for calculating the bending shear stress.

Figure 6.48*a* and *b* shows the bending of four wooden strips that are *separate* and *glued together*, respectively. In Figure 6.48*a* each wooden strip slides relative to the other in the longitudinal direction. But in Figure 6.48*b* the relative sliding is prevented by the shear resistance of the glue—that is, the shear stress in the glue. One may thus hypothesize that in any beam there will be shear stresses on imaginary surfaces parallel to the axis of the beam.

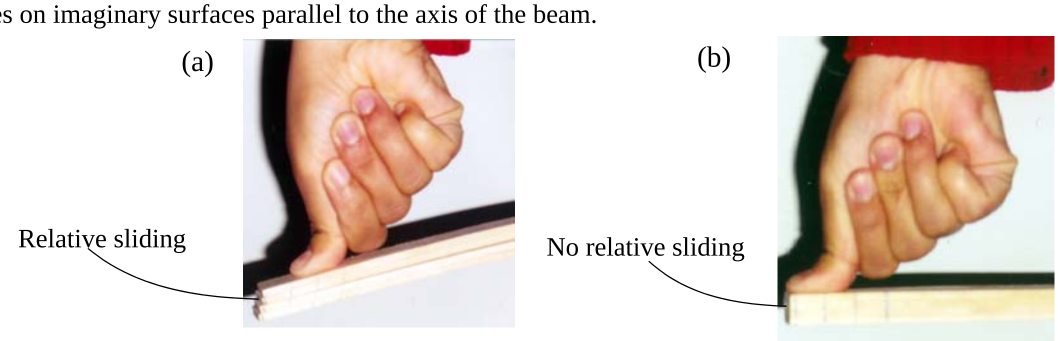


Figure 6.48 Effect of shear stress in bending. (a) Separate beams. (b) Glued beams.

Notice that the beam in Figure 6.48*a* has significantly more curvature (it bends more) than that in Figure 6.48*b*, even though the forces exerted in both cases are approximately the same. This phenomenon of increasing stiffness (see Problem

6.20) at the expense of introducing shear stress is exploited in the design of lightweight structures. In metal beams, the flanges are designed for carrying most of the normal stress in bending, and the webs are designed for carrying most of the shear stress (see Figure 6.28). In sandwich beams two stiff panels are separated by a soft core material. The stiff panels are designed to carry the normal stress and the soft core is designed to carry the shear stress.

6.6.1 Shear Stress Direction

Before developing formulas, it is worthwhile to understand the character of the shear stresses in bending and to determine their direction by inspection.

Consider the beam in Figure 6.49*a*. The beam is constructed by gluing five pieces of wood together. From the evidence of the photographs in Figure 6.48, we know that shear stress will exist at each glued surface to resist the relative sliding of the wood strips. If we take a small element Δx of strips 3 and 5, we obtain Figure 6.49*c* and *d*. On the glued surface between wooden strips 2 and 3 there will be a shear stress τ_{zx} as the outward normal of the surface is in the *z* direction and the internal shear force is in the *x* direction. On the glued surface between wooden strips 4 and 5 there will be a shear stress τ_{yx}, as the outward normal of the surface is in the *y* direction and the internal shear force is in the *x* direction.

Because of the bending load *P*, a normal stress distribution across the cross section will develop as shown in Figure 6.49*b*. From Equation (6.12), we know the bending normal stress σ_{xx} will vary along the length of the beam as the moment $\boldsymbol{M_z}$ varies. The bending normal stress distribution is such that there is no resultant axial force over the *entire* cross section. But if we only take a *part* of the cross section, as in Figure 6.49*b* and *c*, then there will be an axial force generated that varies along the length as shown in Figure 6.49*c* and *d*.

On the small element Δx, the equivalent shear force from the bending shear stresses must balance the change in the equivalent normal axial force, as shown Figure 6.54*e* and *f*. Thus in bending, the shear stress must balance the variations in the normal stress σ_{xx} along the length of the beam.[9]

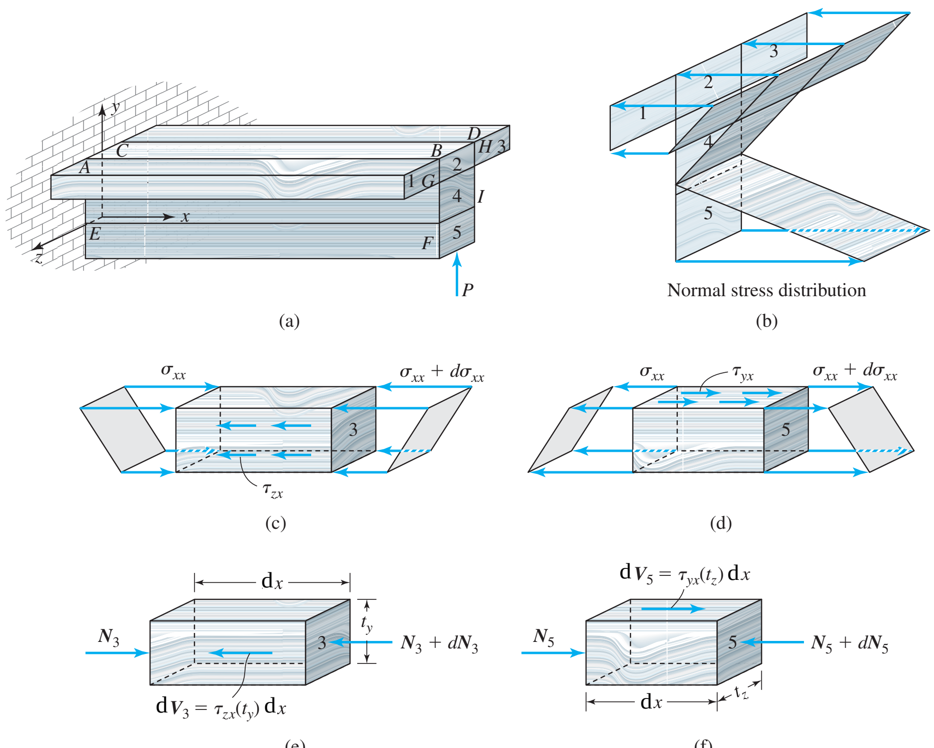


Figure 6.49 Shear stress on different surfaces in bending.

The preceding shows that shear stress develops on surfaces cut parallel to the axis of the beam. But from the symmetry of shear stresses $\tau_{xy} = \tau_{yx}$ and $\tau_{xz} = \tau_{zx}$. These stresses, τ_{xy} and τ_{xz}, are on the cross sections perpendicular to the axis of the beam.

[9]From the field of elasticity it is known that in the absence of body forces, the equilibrium at a point requires $\partial\sigma_{xx}/\partial x + \partial\tau_{yx}/\partial y + \partial\tau_{zx}/\partial z = 0$ (see Problem 1.105). Thus if σ_{xx} varies with *x*, then τ_{yx} (or τ_{xy}) must vary with *y*, and τ_{zx} (or τ_{xz}) must vary with *z*. See Problem 6.136 for additional details.

We know from Equation (6.10) that on the cross section of the beam the resultant of the shear stress τ_{xy} distribution is the shear force V_y. Thus the direction (sign) of τ_{xy} should be the same as that of V_y. But the shear force V_z that would be statically equivalent to τ_{xz} must be zero, as there is no external force in the z direction. This means that τ_{xz} must reverse sign (and direction) on the cross section if the net force from it is zero. We also know that the y axis is the axis of symmetry, and the loading is in the plane of symmetry. Therefore all stresses including τ_{xz} must be symmetric about the y axis. In other words, the shear stress τ_{xz} will reverse its direction as one crosses the y axis on the cross section. This sometimes implies that the shear stress τ_{xz} will be zero at the y axis.

Consider now a circular cross section that is glued together from nine wooden strips, as shown in Figure 6.50*a*. Once more shear stresses will develop along each glued surface, to resist relative sliding between two adjoining wooden strips, and the shear stress value must balance the change in axial force due to the variation in σ_{xx}. The outward normal of the surface will be in a different direction for each glued surface on which we consider the shear stress. If we define a tangential coordinate s that is in the direction of the tangent to the center line of the cross section, then the outward normal to the glued surface will be in the s direction and the shear stress will be τ_{sx}. Once more by the symmetry of shear stresses, $\tau_{xs} = \tau_{sx}$. At a point if the s direction and the y direction are the same, then τ_{xs} will equal $\pm\tau_{xy}$. If the s direction and the z direction are the same at a point, then τ_{xs} will equal $\pm\tau_{xz}$.

It should be noted that in Figure 6.49*e* and *f* and in Figure 6.50*b* the shear force that balances the change in the axial force $\boldsymbol{N}$ is shown on only one surface. The surface on the other end of the free-body diagram is always assumed to be a free surface. That is, the shear stress is zero on these other surfaces. The origin of the s coordinate is chosen to be one of the free surfaces and will be used in the next section in developing shear stress formulas. In a beam cross section the top and bottom and the side surfaces are always assumed to be surfaces on which shear stress is zero.

In Figure 6.49*e* and *f* and in Figure 6.50*b* we notice that the shear force expression contains the product of the shear stress and the thickness t of the cross section at that point. This product is the **shear flow** q.

$$q = \tau_{xs} t \tag{6.23}$$

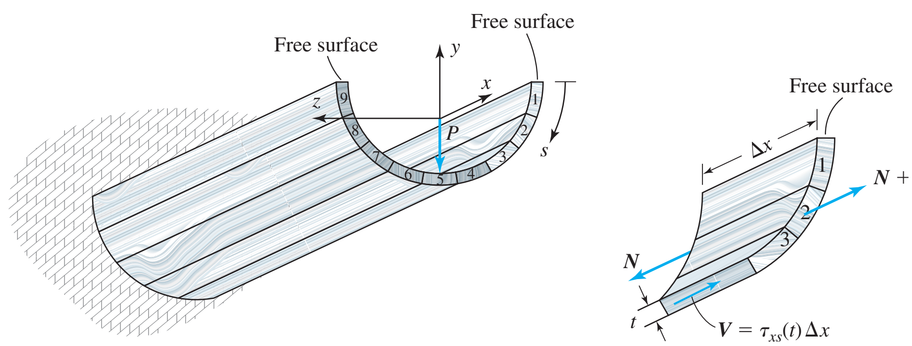


Figure 6.50 Shear stress in circular cross section.

The units of the shear flow are force per unit length. The terminology is from fluid flow in channels, but it is used extensively to discuss shear stresses in thin cross sections, probably because of the image of an actual flow helps in discussing shear stress directions, as elaborated further in the next section and Example 6.13.

6.6.2 Shear Flow Direction by Inspection

The shear flow and shear stress along the center line of the cross section are drawn in a direction that satisfies the following rules:

1. The resultant force in the y direction is in the same direction as V_y.
2. The resultant force in the z direction is zero.
3. It is symmetric about the y axis. This requires that shear flow change direction as one crosses the y axis on the center line. Sometimes this will imply that shear stress is zero at the point(s) where the center line intersects the y axis.

EXAMPLE 6.13

Assuming a positive shear force $\boldsymbol{V}_y$, sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure 6.51.

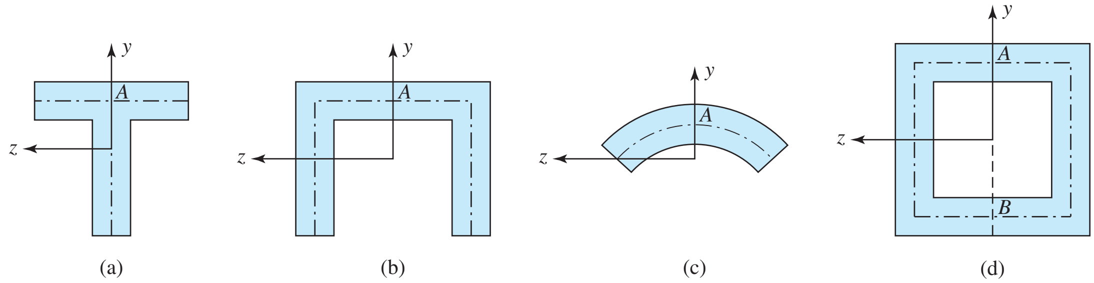


Figure 6.51 Cross sections in Example 6.13.

PLAN

With the outward normal of the cross section in the positive *x* direction, the positive shear force $\boldsymbol{V}_y$ will be in the positive *y* direction according to the sign convention in Section 6.2.6. We can determine the direction of the flow in each cross section to satisfy the rules described at the end of Section 6.6.2.

SOLUTION

(a) On the cross section shown in Figure 6.52*a* the shear flow (shear stress) from *C* to *A* will be in the positive *y* direction, since $\boldsymbol{V}_y$ on the cross section is in the positive *y* direction. At point *A* in the flange the flow will break in two and go in opposite directions, as shown in Figure 6.52*a*. The resultant force due to shear flow from *A* to *D* will cancel the force due to shear flow from *A* to *E*, satisfying the condition of zero resultant force in the *z* direction and the condition of symmetric flow about the *y* axis.

(b) On the cross section shown in Figure 6.52*b*, the shear flow from *C* to *E* and from *D* to *F* will be in the positive *y* direction. This satisfies the condition of symmetry about the *y* axis and is consistent with direction of $\boldsymbol{V}_y$. In the flange the two flows will approach point *A* from opposite directions. The resultant force due to shear flow from *A* to *E* will cancel the force due to shear flow from *A* to *F*, satisfying the condition of zero resultant force in the *z* direction and the condition of symmetric flow about the *y* axis.

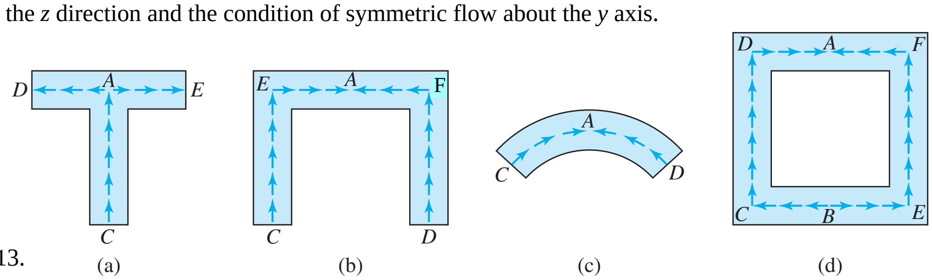


Figure 6.52 Shear flow in Example 6.13.

(c) On the cross section shown in Figure 6.52*c* the shear flows from points *C* and *D* will approach point *A* in opposite directions. This ensures the condition of symmetry, and the condition of zero force in the *z* direction is met.

(d) The shear flow from *C* to *D* and the shear flow from *E* to *F* have to be in the positive *y* direction to satisfy the condition of symmetry about the *y* axis and to have the same direction as $\boldsymbol{V}_y$. At points *A* and *B* the shear flows must change direction to ensure symmetric shear flows about the *y* axis. The force from the shear flows in *BC* and *DA* will cancel the force from the shear flows in *BE* and *FA*, ensuring the condition of a zero force in the *z* direction.

COMMENTS

1. The shear flow (shear stress) is zero at the following points because these points are on the free surface: points *C*, *D*, and *E* in Figure 6.52*a*; points *C* and *D* in Figure 6.52*b* and *c*.
2. At point *A* in Figure 6.52*b*, *c*, and *d* the shear flow will be zero, but it will not be zero in Figure 6.52*a* as we can appreciate by analogy to fluid flow. In Figure 6.52*a* the shear flows at point *A* in branches *AD* and *AE* add up to the value of shear flow at point *A* in branch *CA*. With no other branch at point *A* in Figure 6.52*b*, *c*, and *d* the values of the shear flow are equal and opposite, which is possible only if the value of shear flow is zero.
3. The term *flow* invokes an image that helps in visualizing the direction of shear stress.
4. By examining the direction of the stress components in the Cartesian system, we can determine whether a stress component is positive or negative τ_{xy} or τ_{xz}, as shown in Figure 6.53. Note that τ_{xy} is positive in all cases, a consequence of positive shear force $\boldsymbol{V}_y$. But τ_{xz} can be positive or negative, depending on the location of the point.

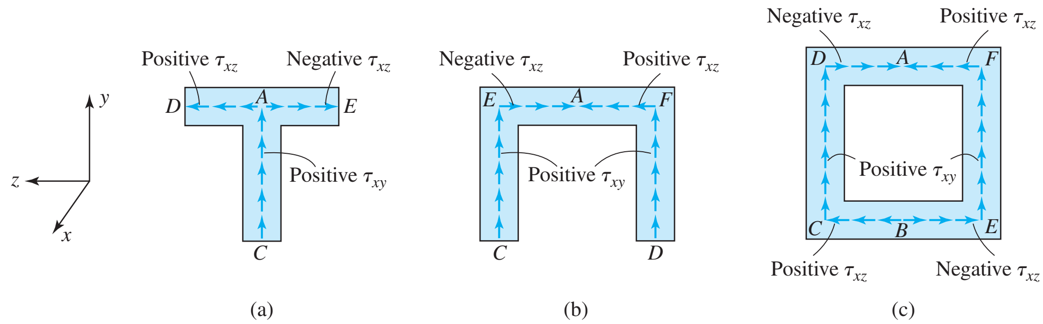


Figure 6.53 Directions and signs of stress components in Example 6.13.

6.6.3 Bending Shear Stress Formula

The previous section and Example 6.13 highlight that the bending shear stress is τ_{xy} in the web and τ_{xz} in the flange, whereas for symmetric curvilinear cross sections it depends on the location of the point. To develop a single formula applicable to all situations, we define a tangential coordinate s in the direction of the tangent to the center line of the cross section, starting from a free surface. In this section we derive the formula for bending shear stress τ_{xs}.

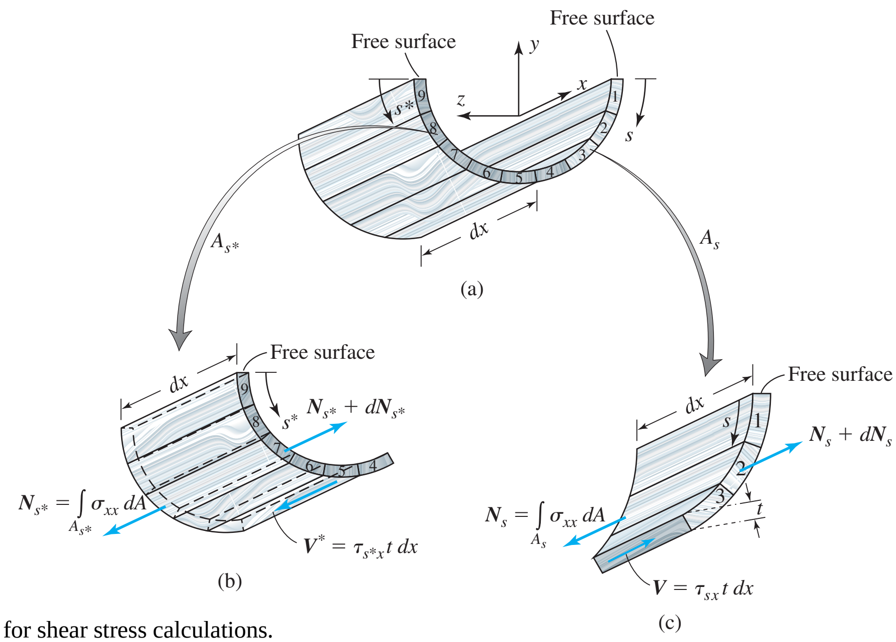


Figure 6.54 Differential element of beam for shear stress calculations.

Consider a differential element of a wooden beam with circular cross section, as shown in Figure 6.54*a*. Consider the shear stress acting on the surface between wooden pieces 3 and 4. We can consider two possible free-body diagrams, shown in Figure 6.54*b* and *c*. The axial force $\mathbf{N}_s$ (or $\mathbf{N}_{s*}$) acting on the part of cross section A_s (or A_{s*}) varies because of the variation of the bending stress σ_{xx} along the length of the beam. If the shear stress does not change across the thickness, the shear force $\mathbf{V}$ (or $\mathbf{V^*}$) is equal to the product of the shear stress multiplied by the area $t\,dx$, as shown. The assumption of constant shear stress in the thickness direction is a good approximation if the thickness is small.

Assumption 9: The beam is thin perpendicular to the center line of the cross section.

By equilibrium of forces in Figure 6.54*c*, we obtain $\mathbf{N}_s + d\mathbf{N}_s - \mathbf{N}_s + \tau_{sx} t\, dx = 0$ or

$$\tau_{sx} t = -\frac{d\mathbf{N}_s}{dx} = -\frac{d}{dx}\int_{A_s} \sigma_{xx}\, dA \qquad (6.24)$$

Substituting Equation (6.12) into Equation (6.24) and noting that the moment $\boldsymbol{M_z}$ and the area moment of inertia I_{zz} do not vary over the cross section, we obtain

$$\tau_{sx}t = \frac{d}{dx}\left(\frac{\boldsymbol{M}_z}{I_{zz}}\int_{A_s} y\, dA\right) = \frac{d}{dx}\left(\frac{\boldsymbol{M}_z Q_z}{I_{zz}}\right) \tag{6.25}$$

where Q_z is referred to as the first moment of the area A_s and is defined as

$$\boxed{Q_z = \int_{A_s} y\, dA} \tag{6.26}$$

Assumption 10: The beam is not tapered.

Assumption 10 implies that I_{zz} and Q_z are not a function of x, and these quantities can be taken outside the derivative sign. We obtain

$$\tau_{sx}t = (Q_z/I_{zz})d\boldsymbol{M}_z/dx$$

Substituting Equation (6.18), we obtain the formula for bending shear stress:

$$\boxed{\tau_{sx} = \tau_{xs} = -\left(\frac{\boldsymbol{V}_y Q_z}{I_{zz}t}\right)} \tag{6.27}$$

In Equation (6.27) the shear force $\boldsymbol{V_y}$ can be found either by equilibrium or by drawing the shear force diagram. Also, t is the thickness at the point where the shear stress is being found, and I_{zz} is known from the geometry of the cross section. The direction of s, identification of the area A_s, and the calculation of Q_z are the critical new elements. We record the following observations before discussing in detail the calculation of Q_z.

- Area A_s is the area between the free surface and the point where the shear stress is being evaluated.
- s is the direction from the free surface in the area A_s used in the calculation of Q_z.

6.6.4 Calculating Q_z

Figure 6.55 shows the area A_s between the top free surface and the point at which the shear stress is being found (line s–s). From Equation (6.26) we note that Q_z is the first moment of the area A_s about the z axis. The integral in Equation (6.26) is the numerator in the definition of the centroid of the area A_s. Analogous to the moment due to a force, the first moment of an area can be found by placing the area A_s at its centroid and finding the moment about the neutral axis. That is, Q_z is the product of area A_s and the distance of the centroid of the area A_s from the neutral axis, as shown in Figure 6.55. Alternatively, Q_z can be found by using the bottom surface as the free surface, shown as Q_{z*} in Figure 6.55.

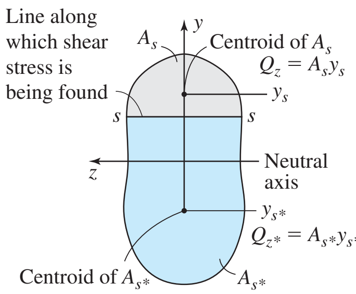


Figure 6.55 Calculation of Q_z.

At the top surface, which is a free surface, the value of Q_z is zero, as the area A_s is zero. When we reach the bottom surface after starting from the top, the value of Q_z is once more zero because $A_s = A$, and from Equation (6.9), $\int_A y\, dA = 0$. If Q_z starts with a zero value at the top and ends with a zero value at the bottom, then it must reach a maximum value somewhere on the cross section.

To see where Q_z reaches a maximum value, consider the change in Q_z as the line s–s moves downward in Figure 6.55. Toward the neutral axis, the moment of the area Q_z increases as we add the moments from the additional areas. When the line s–s crosses the neutral axis, then the new additional area below the axis produces a negative moment because the centroid of this area is in the negative y direction. In other words, Q_z increases up to the neutral axis and then starts decreasing. Thus Q_z is maximum at the neutral axis. From Equation (6.27), it follows that bending shear stress is maximum at the neutral axis of a cross section. In summary

- Q_z is zero at the top and bottom surface.
- Q_z is maximum at the neutral axis.
- The maximum bending shear stress on a cross section is at the neutral axis.
- The maximum bending shear stress in the beam will be at the neutral axis on a cross section where V_y is maximum.

We can write $A = A_s + A_{s*}$ in Equation (6.9) and write the integral as $\int_{A_s} y\,dA + \int_{A_{s*}} y\,dA = 0$ to obtain $Q_z + Q_{z*} = 0$. This implies that Q_z and Q_{z*} will have the same magnitude but opposite signs. Thus, if we used Q_z or Q_{z*} in Equation (6.27), we would get the same magnitude of the shear stress, but which would give the correct sign (or direction)?[10] The answer is that both will give the correct sign, provided the s direction in Equation (6.27) is measured from the free surface used in the calculation of Q_z.

We can find the magnitude and the direction of the bending shear stress in two ways:

1. Use Equation (6.27) to find the magnitude of the shear stress. Use the rules described in Section 6.6.2 to determine the direction of the shear stress.
2. Alternatively, follow the sign convention described in Section 6.2.6 to determine the shear force V_y. The shear stress is found from Equation (6.27), and the direction of the shear stress is determined using the subscripts, as elaborated in Section 1.3.

6.6.5 Shear Flow Formula

The formula for shear flow can be obtained by substituting Equation (6.27) into Equation (6.23) to get

$$\boxed{q = -\frac{V_y Q_z}{I_{zz}}} \tag{6.28}$$

Equation (6.28) can be used in two ways. It can be used for finding the magnitude of the shear flow at a point, and the direction of shear flow can then be found by inspection following the rules described in Section 6.6.2. Alternatively, the sign convention for the shear force V_y is followed and the shear flow is determined from Equation (6.28). A positive value of shear flow implies that the flow is in the positive s direction, where s is measured from the free surface used in the calculation of Q_z.

One application of Equation (6.28) is the determination of the spacing between mechanical fasteners holding strips of beams together. Nails or screws are examples of mechanical fasteners used in wooden beams. Bolts or rivets are examples of mechanical fasteners used in metal beams. Figure 6.56 shows two strips of beams held together by a row of mechanical fasteners. Suppose the fasteners are spaced at intervals Δs, and each fastener can support a shear force V_F. Then the row of fasteners can support an average shear force per unit length of $V_F/\Delta s$, which can be approximated as the shear flow in the beam, or, $q \approx V_F / \Delta s$.

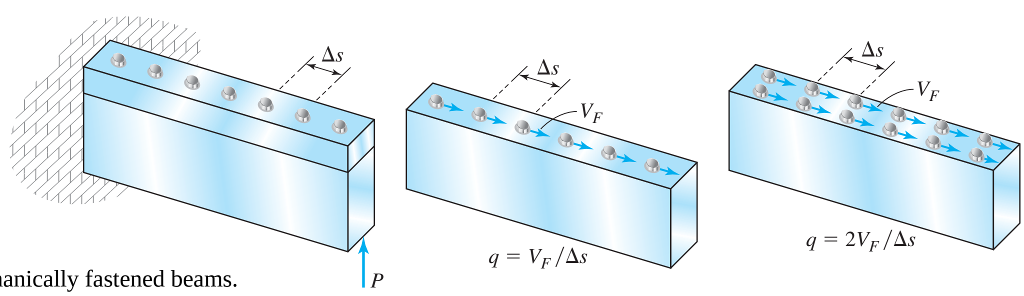


Figure 6.56 Spacing in mechanically fastened beams.

[10]In many textbooks the bending shear stress formula gives only the correct magnitude. The correct sign (or direction) has to be found by inspection. In this book inspection as well as subscripts in the formulas will be used in determining the direction of shear stress.

Thus once we know the shear flow from Equation (6.28), we can find the spacing in a row of fasteners as $\Delta s \approx V_F / q$. If there is more than one row of fasteners holding two pieces of wood together, then each row of fasteners can carry an average shear flow of $V_F/\Delta s$. Thus the total shear flow carried by two rows is $2V_F/\Delta s$, which is then approximated by the shear flow in the beam, $q = 2V_F / \Delta s$. Thus once we know the shear flow from Equation (6.28), we can use it to determine the spacing between the fasteners, once the shear force that the fasteners can support is known. Alternatively, if the spacing is known, then we can find the shear force carried by each fastener. Example 6.18 further elaborates on this discussion.

6.6.6 Bending Stresses and Strains

In symmetric bending about the z axis, the significant stress components in Cartesian coordinates are σ_{xx} and τ_{xy} in the web and σ_{xx} and τ_{xz} in the flange. We can find σ_{xx} from Equation (6.12), but from Equation (6.27) we get τ_{xs}. How do we get τ_{xy} or τ_{xz} from τ_{xs}? There are two alternatives.

1. Follow the sign convention for the shear force to determine V_y. Using Equation (6.27), get $\tau_{xs.}$ Note that the positive s direction is from the free surface to the point where the shear stress is found. Draw the stress cube using the argument of subscripts as described in Section 1.3. Now look at the shear stress in the Cartesian coordinates and determine the direction and sign of the stress component (τ_{xy} or τ_{xz}).
2. Alternatively, use Equation (6.27) to find the magnitude of τ_{xs}, and determine the direction of the shear stress by inspection, as described in Section 6.6.1. Draw the stress cube. Now look at the shear stress in the Cartesian coordinates and determine the direction and sign of the stress component (τ_{xy} or τ_{xz}).

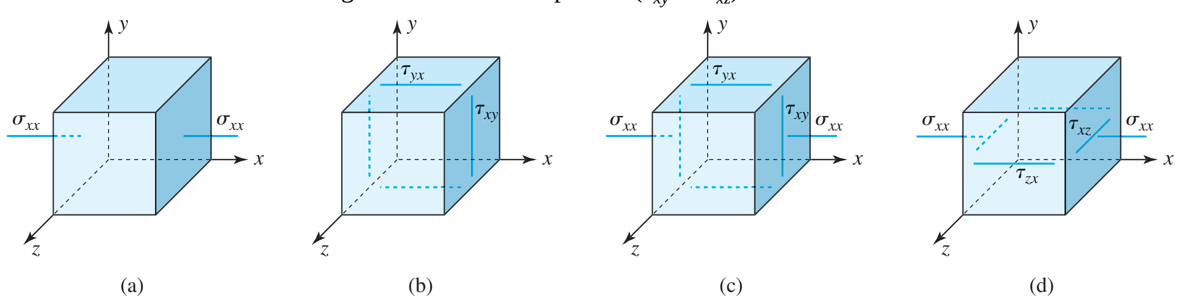


Figure 6.57 Stress elements in symmetric bending of beams: (*a*) top or bottom; (*b*) neutral axis; (*c*) any point on web. (*d*) any point on flange.

In beam bending problems there are four possible stress elements, as shown in Figure 6.57. At the top and bottom surfaces of the beam the bending shear stress τ_{xy} is zero, and the bending normal stress σ_{xx} is maximum at a cross section. The state of stress at the top and bottom is shown on in Figure 6.57*a*. No arrows are shown in the figures, as the normal stress could be tensile or compressive. At the neutral axis σ_{xx} is zero and τ_{xy} is maximum in a cross section, as shown by the stress element in Figure 6.57*b*. At any point on the web σ_{xx} and τ_{xy} are nonzero, whereas at any point in the flange σ_{xx} and τ_{xz} are nonzero, as shown in Figure 6.57*c* and *d*.

From the generalized Hooke's law given by Equations (3.12*a*) through (3.12*f*), we obtain the strains

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \tag{6.29}$$

The normal strains in the y and z directions are due to the Poisson effect.

Consolidate your knowledge

1. With the book closed, derive Equation (6.27).

EXAMPLE 6.14

A positive shear force $\boldsymbol{V}$ acts on the thin rectangular cross section shown in Figure 6.58. Determine the shear stress τ_{xs} due to bending about the z axis as a function of s and sketch it.

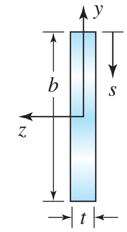


Figure 6.58 Cross section in Example 6.14.

PLAN

We can find Q_z by taking the first moment of the area between the top surface and the surface located at an arbitrary point s. By substituting Q_z as a function of s in Equation (6.27), we can obtain τ_{xs} as a function of s.

SOLUTION

We can draw the area A_s between the top surface and some arbitrary location s in Figure 6.59a and determine the first moment about the z axis to find Q_z.

$$Q_z = st\left(\frac{b}{2} - \frac{s}{2}\right) = \frac{st(b-s)}{2} \tag{E1}$$

Substituting Equation (E1) and the area moment of inertia $I_{zz} = tb^3/12$ into Equation (6.27), we obtain

$$\tau_{xs} = -\frac{\boldsymbol{V}st(b-s)/2}{(tb^3/12)t} \tag{E2}$$

ANS. $\tau_{xs} = \dfrac{-6\boldsymbol{V}s(b-s)}{b^3 t}$

Suppose we take the positive x direction normal to this page. Since we obtain a negative sign for the shear stress, the direction of the shear stress has to be in the negative s direction, as shown in Figure 6.59*b*.

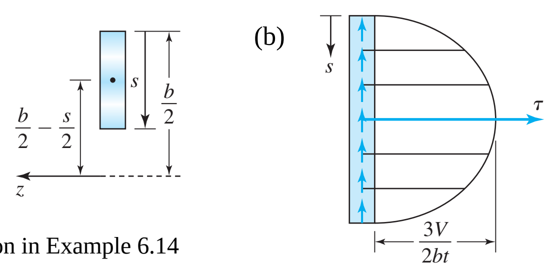


Figure 6.59 (a) Calculation of Q_z in Example 6.14. (b) Shear stress distribution in Example 6.14

COMMENTS

1. Figure 6.59*b* shows that the shear stress is zero at the top ($s = 0$) and the bottom ($s = b$) and is maximum at the neutral axis, as expected. The maximum bending shear stress at a cross section can be written $\tau_{max} = 1.5\boldsymbol{V}/A$, where A is the cross-sectional area.
2. The shear force is in the positive y direction. Hence the shear stress on the cross section should be in the positive direction, as shown in Figure 6.59*b*.
3. Note that the s direction is in the negative y direction. Hence $\tau_{xy} = -\tau_{xs}$, which is confirmed by the direction of shear stress in Figure 6.59*b*.
4. Substituting $\tau_{xy} = -\tau_{xs} = 6\boldsymbol{V}s(b-s)/b^3 t$ into Equation (6.13) and noting that $dA = t\,ds$, we obtain by integration

$$\boldsymbol{V}_y = \int_0^b \frac{6\boldsymbol{V}s(b-s)}{b^3 t} t\,ds = \frac{6\boldsymbol{V}}{b^3}\left(\frac{bs^2}{2} - \frac{s^3}{3}\right)\Bigg|_0^b = \boldsymbol{V}$$

which once more confirms our results.

EXAMPLE 6.15

A positive shear force $V_y = 30$ kN acts on the thin cross sections shown in Figure 6.60 (not drawn to scale). Determine the shear stress at points *B*, *C*, *D*, and *E*. Report the answers as τ_{xy} or τ_{xz}.

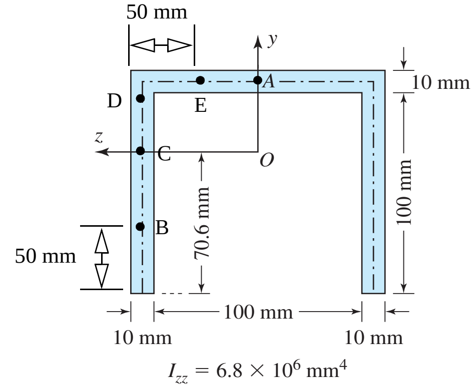


Figure 6.60 Cross sections in Example 6.15.

PLAN

V_y, I_{zz} and t in Equation (6.27) are known. Hence the shear stress will be determined if Q_z is determined at the given points. We make an imaginary cut perpendicular to the center line, and we draw the area A_s between the bottom and the imaginary cut and calculate Q_z.

SOLUTION

Figure 6.61 shows the areas A_s that can be used for finding Q_z at points *B*, *C*, *D*, and *E*. The distance from the centroid of the areas A_s to the *z* axis can be found and multiplied by the area A_s to obtain Q_z at each point:

$$(Q_z)_B = (0.05 \text{ m})(0.01 \text{ m})[-(0.0706 \text{ m} - 0.025 \text{ m})] = -22.8(10^{-6}) \text{ m}^3 \tag{E1}$$

$$(Q_z)_C = (0.0706 \text{ m})(0.01 \text{ m})[-0.0706/2 \text{ m}] = -24.92(10^{-6}) \text{ m}^3 \tag{E2}$$

$$(Q_z)_D = (0.10 \text{ m})(0.01 \text{ m})[-(0.0706 \text{ m} - 0.05 \text{ m})] = -20.6(10^{-6}) \text{ m}^3 \tag{E3}$$

$$(Q_z)_E = (0.10 \text{ m})(0.01 \text{ m})[-(0.0706 \text{ m} - 0.05 \text{ m})] + (0.05 \text{ m})(0.01 \text{ m})(0.105 \text{ m} - 0.0706 \text{ m}) \quad \text{or} \quad (Q_z)_E = -3.4(10^{-6}) \text{ m}^3 \tag{E4}$$

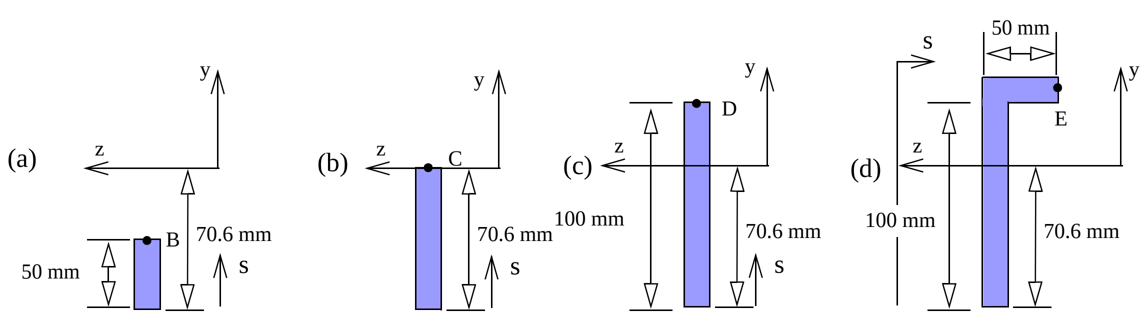


Figure 6.61 Area As for calculations of Q_z in Example 6.15.

The shear stress at the points can be found from Equation (6.27):

$$(\tau_{xs})_B = -\left[\frac{V_y(Q_z)_B}{I_{zz}t_B}\right] = -\frac{[30(10^3) \text{ N}][-22.8(10^{-6}) \text{ m}^3]}{[6.8(10^{-6}) \text{ m}^4](0.01 \text{ m})} = 10.5(10^6) \text{ N/m}^2 \tag{E5}$$

$$(\tau_{xs})_C = -\left[\frac{V_y(Q_z)_C}{I_{zz}t_C}\right] = -\frac{[30(10^3) \text{ N}][-24.92(10^{-6}) \text{ m}^3]}{[6.8(10^{-6}) \text{ m}^4](0.01 \text{ m})} = 10.99(10^6) \text{ N/m}^2 \tag{E6}$$

$$(\tau_{xs})_D = -\left[\frac{V_y(Q_z)_D}{I_{zz}t_D}\right] = -\frac{[30(10^3) \text{ N}][-20.6(10^{-6}) \text{ m}^3]}{[6.8(10^{-6}) \text{ m}^4](0.01 \text{ m})} = 9.09(10^6) \text{ N/m}^2 \tag{E7}$$

$$(\tau_{xs})_E = -\left[\frac{V_y(Q_z)_E}{I_{zz}t_E}\right] = -\frac{[30(10^3) \text{ N}][-3.4(10^{-6}) \text{ m}^3]}{[6.8(10^{-6}) \text{ m}^4](0.01 \text{ m})} = 1.5(10^6) \text{ N/m}^2 \tag{E8}$$

In Figure 6.61, the *s* direction is in the positive *y* direction at points *B*, *C*, and *D* and negative *z* direction at point *E*. Thus, the stress results are

ANS. $(\tau_{xy})_B = 10.5$ MPa $\quad (\tau_{xy})_C = 11.0$ MPa $\quad (\tau_{xy})_D = 9.1$ MPa $\quad (\tau_{xy})_B = -1.5$ MPa

COMMENTS

1. The signs of the bending shear stress components in this example are consistent with those in Figure 6.53b where they were determined by inspection.
2. In Equation (E4) we added the first moment of the area of the horizontal piece in Figure 6.61d to the $(Q_z)_D$ calculated in Equation (E3). We could also have written the integral over the entire area A_s as a sum of integrals over its parts.
3. The maximum bending shear stress will occur at point *C*—that is, at the neutral axis.
4. The bending shear stress at *A* will be zero, because the shear flow will go in opposite direction at the axis of symmetry.

EXAMPLE 6.16

A positive shear force V_y = 30 N acts on the thin cross sections shown in Figure 6.62 (not drawn to scale). Determine the shear flow along the center lines and sketch it.

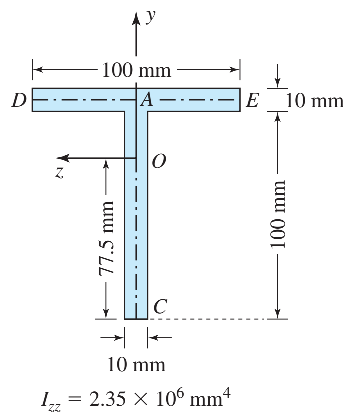


Figure 6.62 Cross sections in Example 6.16.

PLAN

V_y and I_{zz} are known in Equation (6.28). Hence the shear flow along the center line will be determined if Q_z is determined along the centerline. Noting that the cross section is symmetric about the *y* axis, the shear flow needs to be found only on one side of the *y* axis.

SOLUTION

(a) Figure 6.63 shows the areas A_s that can be used for finding the shear flows in *DA* and *CA* of the cross section in Figure 6.62*a*. The parameters s_1 and s_2 are defined from the free surface to the point where the shear flow is to be found. The distance from the centroid of the areas A_s to the *z* axis can be found and Q_z calculated as

$$Q_1 = s_1(0.01\ \text{m})(0.105\ \text{m} - 0.0775\ \text{m}) = 0.275s_1(10^{-3})\ \text{m}^3 \tag{E1}$$

$$Q_2 = (s_2(0.01\ \text{m}))[-(0.0775\ \text{m} - s_2/2)] = -(0.775s_2 - 5s_2^2)(10^{-3})\ \text{m}^3 \tag{E2}$$

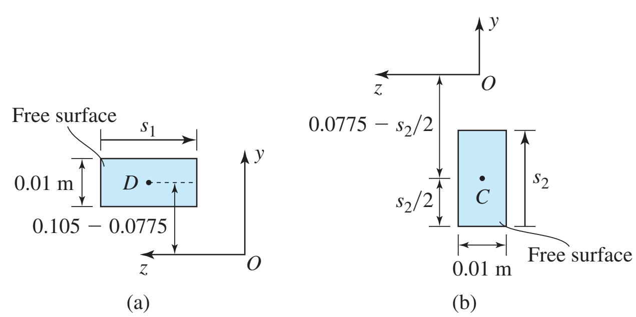


Figure 6.63 Calculation of Q_z in part (a) of Example 6.16.

Substituting V_y, I_{zz}, and Equations (E1) and into Equation (6.28), we find the shear flow in *DA* and *CA* of the cross section in Figure 6.62*a*:

$$q_1 = -\frac{(30\ \text{N})[0.275s_1(10^{-3})\ m^3]}{2.35(10^{-6})\ \text{m}^4} = -3.51s_1\ \text{kN/m} \tag{E3}$$

$$q_2 = -\frac{(30\ \text{N})[-(0.775s_2 - 5s_2^2)(10^{-3})\ \text{m}^3]}{2.35(10^{-6})\ \text{m}^4} = (9.89s_2 - 63.83s_2^2)\ \text{kN/m} \tag{E4}$$

ANS. $q_1 = -3.51s_1\ \text{kN/m} \qquad q_2 = (9.89s_2 - 63.83s_2^2)\ \text{kN/m}$

The shear flow q_1 is negative, implying that the direction of the flow is opposite to the direction of s_1. The values of q_1 can be calculated and plotted as shown in Figure 6.64*a*. By symmetry the flow in *AE* can also be plotted. The values of q_2 are positive between *C* and *A*, implying the flow is in the direction of s_2. The values of q_2 can be calculated from and plotted as shown in Figure 6.64*a*.

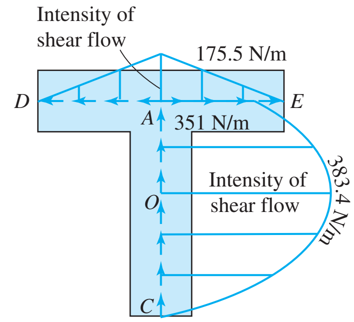


Figure 6.64 Shear flows on cross sections in Example 6.16.

COMMENTS

1. In Example 6.13 the direction of flow was determined by inspection, whereas in this example it was determined using formulas. A comparison of Figures 6.64 and 6.52*a* shows the same results. Thus, inspection can be used to check our results. Alternatively, we could calculate the magnitude of the shear flow (or stress) from formulas and then determine the direction of the shear flow by inspection.
2. In Figure 6.64 the flow value at point *A* in *CA* is 351 N/m, which is the sum of the flows in *AD* and *AE*. Thus the behavior of shear flow is similar to that of fluid flow in a channel.
3. Figure 6.64 shows that the shear flow in the flanges varies linearly. The shear flow in the web varies quadratically, and its maximum value is at the neutral axis.

EXAMPLE 6.17

A beam is loaded as shown in Figure 6.65. The cross section of the beam is shown on the right and has an area moment of inertia $I_{zz} = 40.83\ \text{in.}^4$ (a) Determine the maximum bending normal and shear stresses. (b) Determine the bending normal and shear stresses at point *D* on a section just to the right of support *A*. Point *D* is just below the flange. (c) Show the results of parts (a) and (b) on stress cubes.

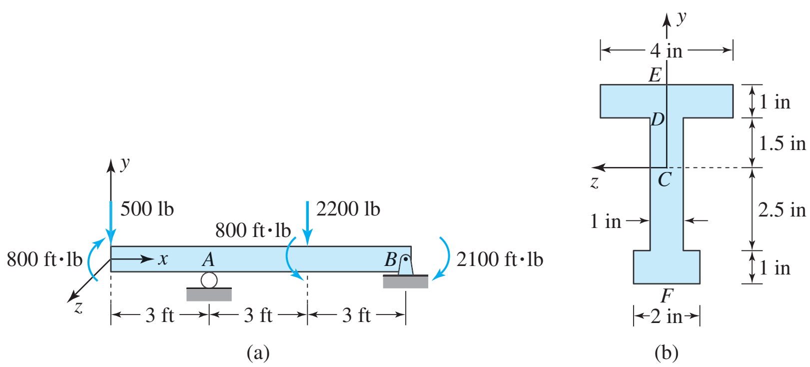


Figure 6.65 Beam and loading in Example 6.17.

PLAN

We can draw the shear force and bending moment diagrams and determine the maximum bending moment $\boldsymbol{M}_{max}$, the maximum shear force $(\boldsymbol{V}_y)_{max}$, and the value of the bending moment $\boldsymbol{M}_A$ and the shear force $(\boldsymbol{V}_y)_A$ just to the right of support *A*. Using Equations (6.12) and (6.27), we can determine the required stresses and show the results on a stress cube.

SOLUTION

By considering the free-body diagram of the entire beam, we can find the reaction forces at *A* and *B* and draw the shear force and bending moment diagrams in Figure 6.66. The areas under the shear force curve are

$$A_1 = 500 \times 3 = 1500 \qquad A_2 = 1000 \times 3 = 3000 \qquad A_3 = 1200 \times 3 = 3600 \tag{E1}$$

From Figure 6.66 we can find the maximum shear force and moment, as well as the values of shear force and moment just to the right of support *A*:

$$(\boldsymbol{V}_y)_{\max} = 1200\ \text{lbs} \qquad \boldsymbol{M}_{max} = 2300\ \text{ft}\cdot\text{lbs} \qquad (\boldsymbol{V}_y)_A = -1000\ \text{lbs} \qquad \boldsymbol{M}_A = -700\ \text{ft}\cdot\text{lbs} \tag{E2}$$

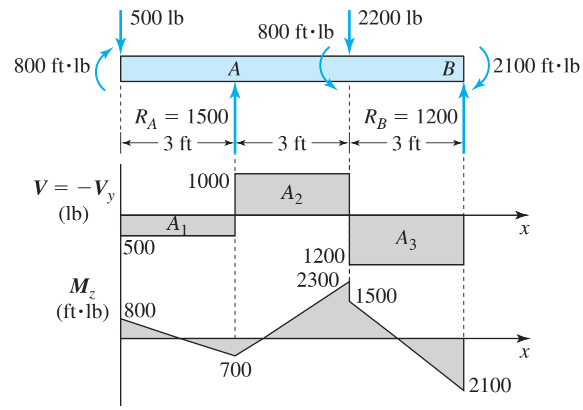


Figure 6.66 Shear force and bending moment diagrams in Example 6.17.

(a) Point F is the point farthest away from the neutral axis. Hence the maximum bending normal stress will occur at point F. Substituting $y_F = -3.5$ and $\boldsymbol{M}_{max}$ into Equation (6.12), we obtain

$$\sigma_{max} = -\frac{[(2300)(12)\ \text{in.}\cdot\text{lbs}](-3.5\ \text{in.})}{40.83\ \text{in.}^4} = 2365.7\ \text{lbs/in.}^2 \tag{E3}$$

ANS. $\sigma_{max} = 2366\ \text{psi (T)}$

The maximum bending shear stress will occur at the neutral axis in the section where V_y is maximum. We can draw the area A_s between the top surface and the neutral axis (NA) as shown in Figure 6.67 and determine the first moment about the z axis to find Q_z, in Equation (E4)

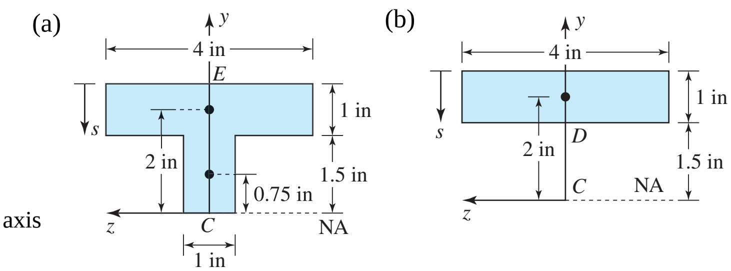


Figure 6.67 Calculation of Q_z in Example 6.17 (a) at neutral axis (b) at D.

$$Q_{NA} = (4\ \text{in.})(1\ \text{in.})(2\ \text{in.}) + (1.5\ \text{in.})(1\ \text{in.})(0.75\ \text{in.}) = 9.125\ \text{in.}^3 \tag{E4}$$

Substituting $\boldsymbol{V}_{max}$ and Equation (E4) into Equation (6.27), we obtain

$$(\tau_{xs})_{max} = -\frac{(1200\ \text{lbs})(9.125\ \text{in.}^3)}{(40.83\ \text{in.}^4)(1\ \text{in.})} = -268.2\ \text{lbs/in.}^2 \tag{E5}$$

ANS. $(\tau_{xs})_{max} = -268\ \text{psi}$

(b) Substituting $y_D = 1.5$ in.and $\boldsymbol{M}_A$ into Equation (6.12), we obtain the value of the normal stress at point D on a section just right of A as

$$(\sigma_{xx})_D = -\frac{[(-700)(12)\ \text{in.}\cdot\text{lbs}](1.5\ \text{in.})}{(40.83\ \text{in.}^4)} = 308.6\ \text{lbs/in.}^2 \tag{E6}$$

ANS. $(\sigma_{xx})_D = 309\ \text{psi (T)}$

We can draw the area A_s between the free surface at the top and point D as shown in Figure 6.67b and find Q_z at D.

$$Q_D = (4\ \text{in.})(1\ \text{in.})(2\ \text{in.}) = 8\ \text{in.}^3 \tag{E7}$$

Substituting $(\boldsymbol{V}_y)_A$ and Equation (E7) into Equation (6.27), we obtain the value of the shear stress at point D on a section just right of A as

$$(\tau_{xs})_D = -\frac{(-1000\ \text{lbs})(8\ \text{in.}^3)}{(40.83\ \text{in.}^4)(1\ \text{in.})} = 196\ \text{lbs/in.}^2 \tag{E8}$$

ANS. $(\tau_{xs})_D = 196\ \text{psi}$

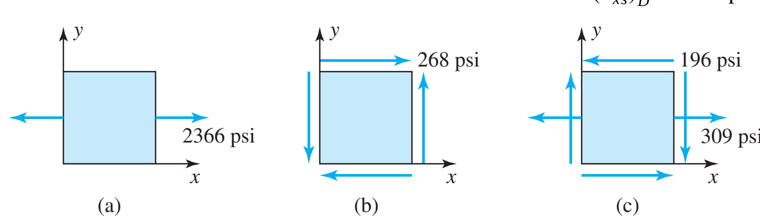


Figure 6.68 Stress elements in Example 6.17. (*a*) Maximum bending normal stress. (*b*) Maximum bending shear stress. (*c*) Bending and normal shear stresses at point D just to the right of A.

(c) In Figures 6.67*a* and *b* the coordinate *s* is in the opposite direction to *y* at points *D* and the neutral axis. Hence at both these points $\tau_{xy} = -\tau_{xs}$, and from Equations (E5) and (E8) we obtain

ANS. $(\tau_{xy})_{max} = 268 \text{ psi}$ $(\tau_{xy})_D = -196 \text{ psi}$

We can show these results along with the normal stress values on the stress elements in Figure 6.68.

COMMENTS

1. The maximum value of $\boldsymbol{V}$ is –1200 lbs but $\boldsymbol{V} = -\boldsymbol{V}_y$. Hence the maximum value of $\boldsymbol{V}_y$ is a positive value, as given in Equation (E2).
2. $\boldsymbol{V}_y$ is positive in Equation (E2), thus we expect $(\tau_{xy})_{max}$ to be positive. Just after support *A* the shear force $\boldsymbol{V}_y$ is negative, thus we expect that $(\tau_{xy})_D$ will be negative, as shown in Figure 6.68.
3. Note that the maximum bending shear stress in the beam given by Equation (E5) is nearly an order of magnitude smaller than the maximum bending normal stress given by Equation (E3). This is consistent with the requirement for validity of our beam theory, as was remarked in Section 6.2.6. If in some problem the maximum bending shear stress were nearly the same as the maximum bending normal stress, then that would indicate that the assumptions of beam theory are not valid and the theory needs to be modified to account for shear stress.

EXAMPLE 6.18

A wooden cantilever box beam is to be constructed by nailing four pieces of lumber in one of the two ways shown in Figure 6.69. The allowable bending normal and shear stresses in the wood are 750 psi and 150 psi, respectively. The maximum force that the nails can support is 100 lb. Determine the maximum value of load *P* to the nearest pound, the spacing of the nails to the nearest half inch, and the preferred nailing method.

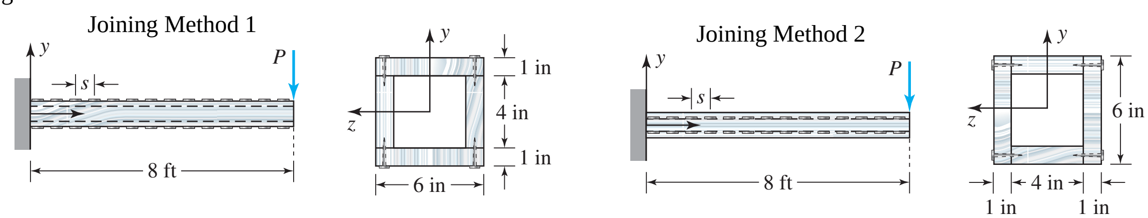


Figure 6.69 Wooden beams in Example 6.18.

PLAN

The maximum bending normal and shear stresses for both beams can be found in terms of *P*. These maximum values can be compared to the allowable stress values, and the limiting value on force *P* can be found. The shear flow at the junction of the wood pieces can be found using Equation (6.28). The spacing of the nails for each joining method can be found by dividing the allowable force in the nail by the shear flow. The method that gives the greater spacing between the nails is better as fewer nails will be needed.

SOLUTION

We can draw the shear force and bending moment diagrams for the beams as shown in Figure 6.70a and calculate the maximum shear force and moment

$$(\boldsymbol{V}_y)_{max} = -P \text{ lb} \qquad \boldsymbol{M}_{max} = -8P \text{ ft} \cdot \text{lb} = -96P \text{ in} \cdot \text{lb} \tag{E1}$$

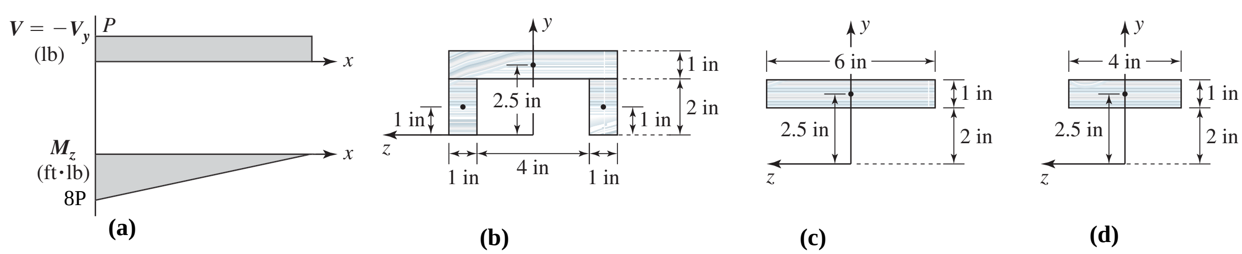


Figure 6.70 (a) Shear and moment diagrams. (b) Q_z at neutral axis. (c) Q_z at nails in joining method 1. (d) Q_z at nails in joining method 2

The area moment of inertia about the z axis can be found as

$$I_{zz} = \frac{1}{12}(6 \text{ in.})(6 \text{ in.})^3 - \frac{1}{12}(4 \text{ in.})(4 \text{ in.})^3 = 86.67 \text{ in.}^4 \tag{E2}$$

Substituting Equations (E1) and (E2) and $y_{max} = \pm 3$ in, we can find the magnitude of the maximum bending normal stress from Equation (6.12) in terms of *P*. Using the allowable normal stress as 750 psi, we can obtain one limiting value on *P*,

$$\sigma_{max} = \left|\frac{M_{max} y_{max}}{I_{zz}}\right| = \frac{(96P)(3 \text{ in.})}{86.67 \text{ in.}^4} \le 750 \text{ lbs/in.}^2 \qquad \text{or} \qquad P \le 225.7 \text{ lbs} \tag{E3}$$

Figure 6.70*b* shows the area A_s for the calculation of Q_z at the neutral axis:

$$Q_{NA} = (6\text{ in.})(1\text{ in.})(2.5\text{ in.}) + 2(2\text{ in.})(1\text{ in.})(1\text{ in.}) = 19\text{ in.}^3 \tag{E4}$$

We also note that at the neutral axis, the thickness perpendicular to the center line is $t = 1$ in. + 1 in. = 2 in. Substituting Equations (E1), (E2), and (E4) and $t = 2$ in. into Equation (6.27), we can obtain the magnitude of the bending shear stress in terms of *P*. Using the allowable shear stress as 150 psi, we can obtain another limiting value on *P*.

$$\tau_{max} = \left|\frac{P(19\text{ in.}^3)}{(86.67\text{ in.}^4)(2\text{ in.})}\right| \le 150\text{ lbs/in.}^2 \qquad \text{or} \qquad P \le 1368\text{ lbs} \tag{E5}$$

If the maximum value of P is determined from Equation (E3), then Equation (E5) will be satisfied. Rounding downward we determine the maximum value of force P to the nearest pound.

ANS. $P_{max} = 225$ lbs

To find the shear flow on the surface joined by the nails, we make imaginary cuts through the nails and draw the area A_s, as shown in Figure 6.70*c* and *d*. We can then find Q_z for each joining method:

$$Q_1 = (6\text{ in.})(1\text{ in.})(2.5\text{ in.}) = 15\text{ in.}^3 \qquad Q_2 = (4\text{ in.})(1\text{ in.})(2.5\text{ in.}) = 10\text{ in.}^3 \tag{E6}$$

From Equations (E1) and (E3) we obtain the shear force as $\boldsymbol{V}_y = -225$ lb. Substituting this value along with Equations (E2) and (E6) into Equation (6.28), we obtain the magnitude of the shear flow for each joining method,

$$q_1 = \left|\frac{\boldsymbol{V}_y Q_1}{I_{zz}}\right| = \left|\frac{(225\text{ lbs})(15\text{ in.}^3)}{86.67\text{ in.}^4}\right| = 39.94\text{ lbs/in.} \tag{E7}$$

$$q_2 = \left|\frac{\boldsymbol{V}_y Q_2}{I_{zz}}\right| = \left|\frac{(225\text{ lbs})(10\text{ in.}^3)}{86.67\text{ in.}^4}\right| = 26.96\text{ lbs/in.} \tag{E8}$$

This shear flow is to be carried by two rows of nails for each of the joining methods. Thus each row resists half of the flow. Using this fact, we can find the spacing between the nails,

$$\frac{100\text{ lbs}}{\Delta s_1} = \frac{q_1}{2} \qquad \text{or} \qquad \Delta s_1 = \frac{2(100\text{ lbs})}{39.94\text{ lbs/in.}} = 5.1\text{ in.} \tag{E9}$$

$$\frac{100}{\Delta s_2} = \frac{q_2}{2} \qquad \text{or} \qquad \Delta s_2 = \frac{2(100\text{ lbs})}{26.96\text{ lbs/in.}} = 7.7\text{ in.} \tag{E10}$$

As $\Delta s_2 > \Delta s_1$, fewer nails will be used in joining method 2. Rounding downward to the nearest half inch, we obtain the nail spacing.

ANS. Use joining method 2 with a nail spacing of 7.5 in.

COMMENTS

1. In this particular example only, the magnitudes of the stresses were important; the sign did not play any role. This will not always be the case, particularly in later chapters when we consider combined loading and stresses on different planes.
2. From visualizing the imaginary cut surface of the nails, we observe that the shear stress component in the nails is τ_{yx} in joining method 1 and τ_{zx} in joining method 2.
3. In Section 6.6.1 we observed that the shear stresses in bending balance the changes in axial force due to σ_{xx}. The shear stresses in the nails balance σ_{xx}, which acts on a greater area in joining method 1 (6 in. wide) than in joining method 2 (4 in. wide). This is reflected in the higher value of Q_z, which led to a higher value of shear flow for joining method 1 than for joining method 2, as shown by Equations (E7) and (E8).
4. The observations in comment 3 are valid as long as σ_{xx} is the same for both joining methods at any location. If I_{zz} and y_{max} were different, then it is possible to arrive at a different answer. See Problem 6.126

PROBLEM SET 6.4

Bending normal and shear stresses

6.106 For a positive shear force $\boldsymbol{V}_y$, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.106. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

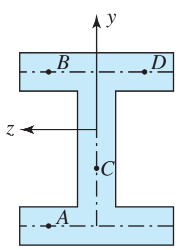


Figure P6.106

6.107 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.107. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative

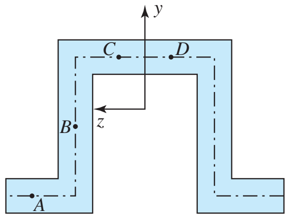


Figure P6.107

6.108 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.108. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

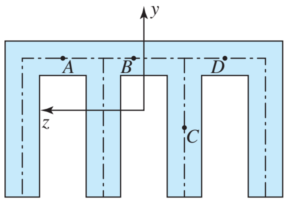


Figure P6.108

6.109 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.109. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

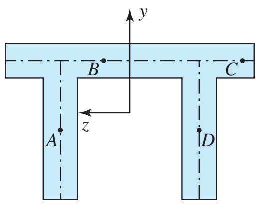


Figure P6.109

6.110 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.109. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

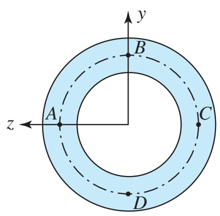


Figure P6.110

6.111 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.111. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

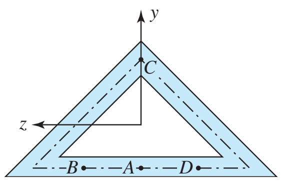


Figure P6.111

6.112 For a positive shear force V_y, (a) sketch the direction of the shear flow along the center line on the thin cross sections shown in Figure P6.112. (b) At points *A*, *B*, *C*, and *D*, determine whether the stress component is τ_{xy} or τ_{xz} and whether it is positive or negative.

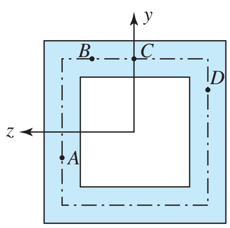


Figure P6.112

6.113 A cross section (not drawn to scale) of a beam that bends about the z axis is shown in Figure 6.113. The shear force acting at the cross section is 5 kips. Determine the bending shear stress at points *A*, *B*, *C*, and *D*. Report your answers as positive or negative τ_{xy} or τ_{xz}. Point *B* is just below the flange.

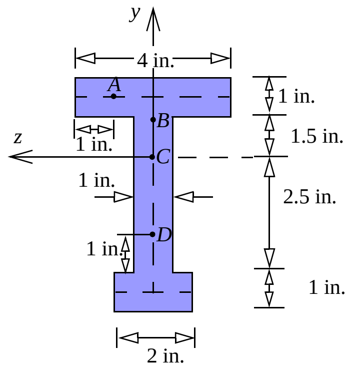


Figure P6.113

6.114 A cross section (not drawn to scale) of a beam that bends about the z axis is shown in Figure 6.71. The shear force acting at the cross section is -10 kN. Determine the bending shear stress at points *A*, *B*, *C*, and *D* Report your answers as positive or negative τ_{xy} or τ_{xz}. Point *B* is just below the flange.

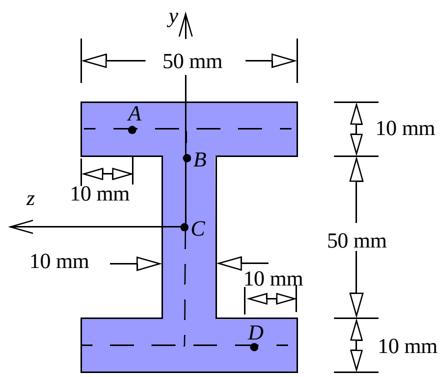


Figure 6.71

6.115 A cross section of a beam that bends about the z axis is shown in Figure 6.115. The internal bending moment and shear force acting at the cross section are $\boldsymbol{M}_z = 50$ in.-kips and $\boldsymbol{V}_y = 10$ kips, respectively. Determine the bending normal and shear stress at points *A*, *B*, and *C* and show it on stress cubes.Point *B* is just below the flange.

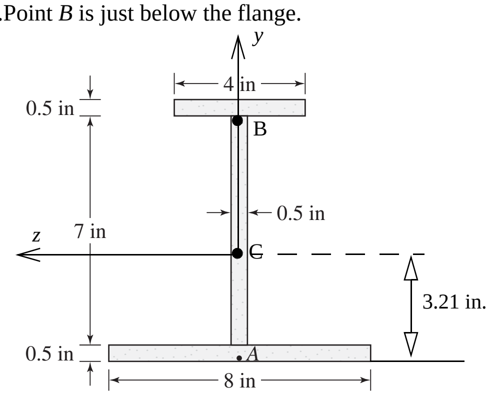


Figure P6.115

6.116 Determine the magnitude of the maximum bending normal stress and bending shear stress in the beam shown in Figure P6.116.

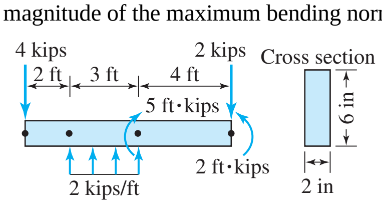


Figure P6.116

6.117 For the beam, loading, and cross section shown in Figure P6.117, determine (a) the magnitude of the maximum bending normal stress, and shear stress; (b) the bending normal stress and the bending shear stress at point *A*. Point *A* is just below the flange on the cross section just right of the 4 kN force. Show your result on a stress cube. The area moment of inertia for the beam was calculated to be $I_{zz} = 3.6 \times 10^6$ mm^4.

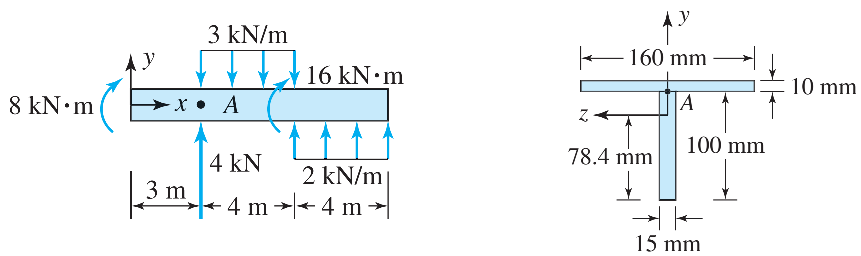


Figure P6.117

6.118 For the beam, loading, and cross section shown in Figure P6.118, determine (a) the magnitude of the maximum bending normal stress and shear stress; (b) the bending normal stress and the bending shear stress at point A. Point A is on the cross section 2 m from the right end. Show your result on a stress cube. The area moment of inertia for the beam was calculated to be $I_{zz} = 453\ (10^6)\ \text{mm}^4$.

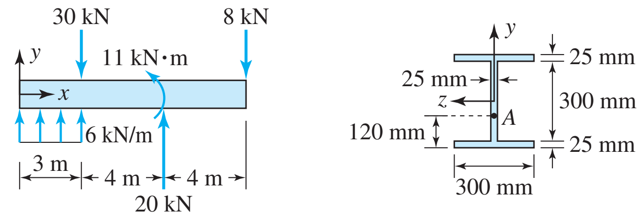


Figure P6.118

6.119 Determine the maximum bending normal and shear stress in the beam shown in Figure 6.119*a*. The beam cross section is shown in Figure 6.119*b*.

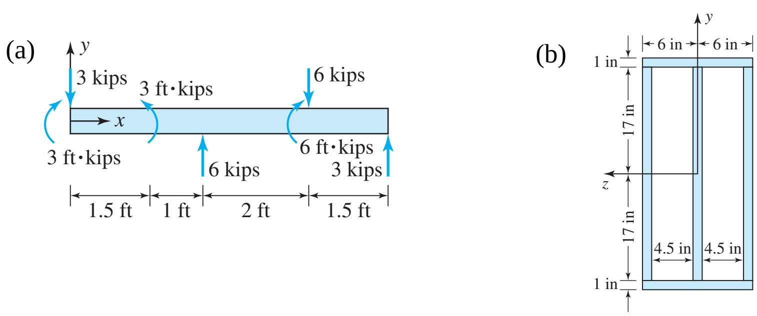


Figure P6.119

6.120 Two pieces of lumber are nailed together as shown in Figure P6.120. The nails are uniformly spaced 10 in apart along the length. Determine the average shear force in each nail in segments *AB* and *BC*.

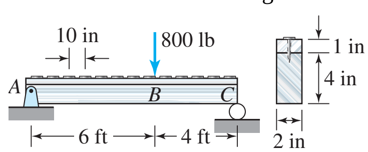


Figure P6.120

6.121 A cantilever beam is constructed by nailing three pieces of lumber, as shown in Figure P6.121. The nails are uniformly spaced at intervals of 75 mm. Determine the average shear force in each nail.

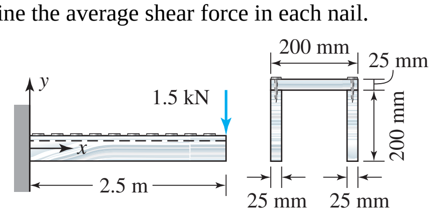


Figure P6.121

6.122 A cantilever beam is constructed by nailing three pieces of lumber, as shown in Figure P6.122. The nails are uniformly spaced at intervals of 75 mm. (a) Determine the shear force in each nail. (b) Which is the better nailing method, the one shown in Problem 6.99 or the one in this problem?

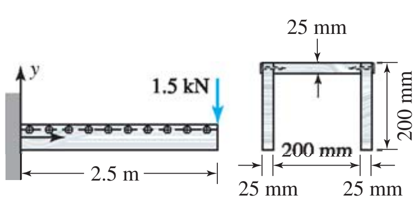


Figure P6.122

Design problems

6.123 The planks in a park bench are made from recycled plastic and are bolted to concrete supports, as shown in Figure P6.123. For the purpose of design the front plank is modeled as a simply supported beam that carries all the weight of two individuals. Assume that each person has a mass 100 kg and the weight acts at one-third the length of the plank, as shown. The allowable bending normal stress for the recycled plastic is 10 MPa and allowable bending shear stress is 2 MPa. The width d of the planks that can be manufactured is in increments of 2 cm, from 12 to 20 cm. To design the lightest bench, determine the corresponding values of the thickness t to the closest centimeter for the various values of d.

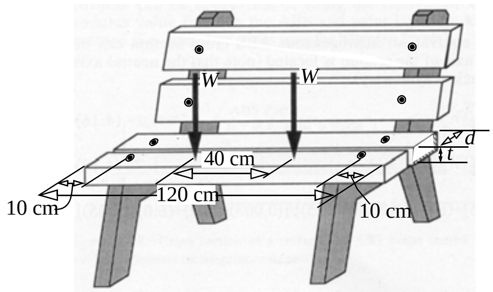


Figure P6.123

6.124 Two pieces of wood are glued together to form a beam, as shown in Figure P6.124. The allowable bending normal and shear stresses in wood are 3 ksi and 1 ksi, respectively. The allowable bending normal and shear stresses in the glue are 600 psi (T) and 250 psi, respectively. Determine the maximum moment M_{ext} that can be applied to the beam.

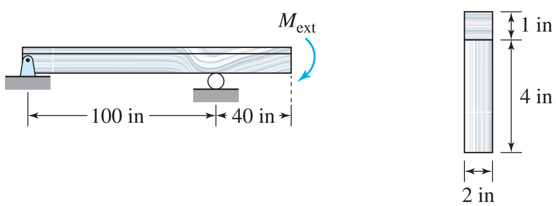


Figure P6.124

6.125 A wooden cantilever beam is to be constructed by nailing two pieces of lumber together, as shown in Figure P6.125. The allowable bending normal and shear stresses in the wood are 7 MPa and 1.5 MPa, respectively. The maximum force that the nail can support is 300 N. Determine the maximum value of load P to the nearest Newton and the spacing of the nails to the nearest centimeter.

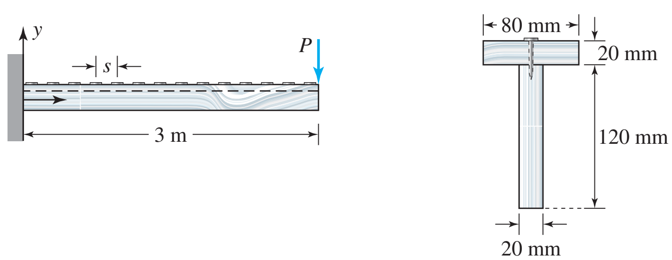


Figure P6.125

6.126 A wooden cantilever box beam is to be constructed by nailing four 1-in.× 6-in. pieces of lumber in one of the two ways shown in Figure P6.126. The allowable bending normal and shear stresses in the wood are 750 psi and 150 psi, respectively. The maximum force that a nail can support is 100 lb. Determine the maximum value of load P to the nearest pound, the spacing of the nails to the nearest half inch, and the preferred nailing method

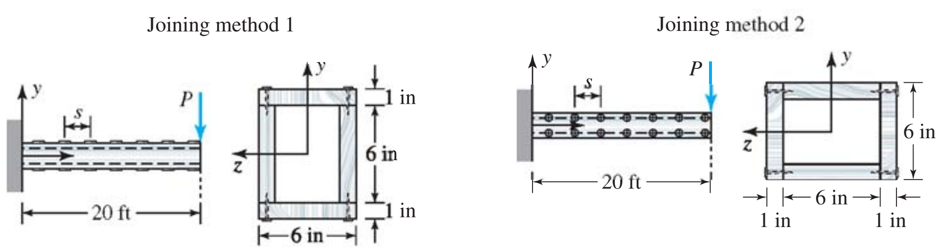


Figure P6.126

Historical problems

6.127 Leonardo da Vinci conducted experiments on simply supported beams and drew the following conclusion: "If a beam 2 braccia long (L) supports 100 libbre (W), a beam 1 braccia long ($L/2$) will support 200 libbre ($2W$). As many times as the shorter length is contained in the longer (L/α), so many times more weight (αW) will it support than the longer one." Prove this statement to be true by considering the two simply supported beams in Figure P6.127 and showing that $W_2 = \alpha W$ for the same allowable bending normal stress.

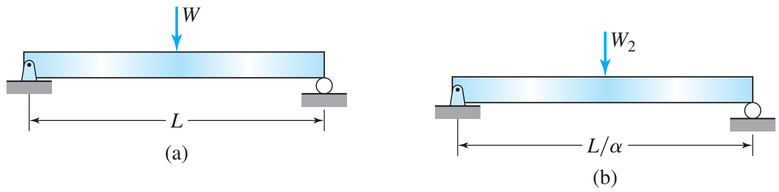


Figure P6.127

6.128 Galileo believed that the cantilever beam shown in Figure P6.128*a* would break at point *B*, which he considered to be a fulcrum point of a lever, with *AB* and *BC* as the two arms. He believed that the material resistance (stress) was uniform across the cross section. Show that the stress value σ that Galileo obtained from Figure P6.128*b* is three times smaller than the bending normal stress predicted by Equation (6.12).

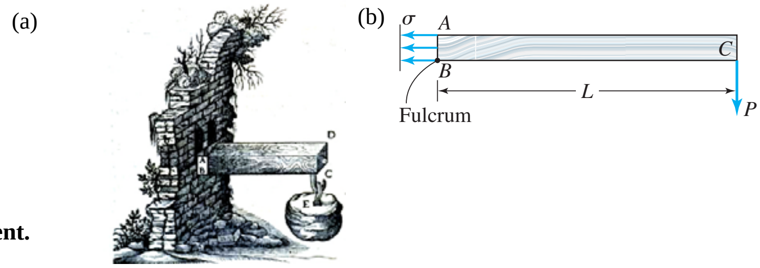


Figure P6.128 Galileo's beam experiment.

6.129 Galileo concluded that the bending moment due to the beam's weight increases as the square of the length at the built-in end of a cantilever beam. Show that Galileo's statement is correct by deriving the bending moment at the built-in end in the cantilever beam in terms of specific weight γ, cross-sectional area *A*, and beam length *L*.

6.130 In the simply supported beam shown in Figure P6.130. Galileo determined that the bending moment is maximum at the applied load and its value is proportional to the product *ab*. He then concluded that to break the beam with the smallest load *P*, the load should be placed in the middle. Prove Galileo's conclusions by drawing the shear force and bending moment diagrams and finding the value of the maximum bending moment in terms of *P*, *a*, and *b*. Then show that this value is largest when $a = b$.

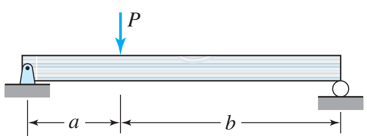


Figure P6.130

6.131 Mariotte, in an attempt to correct Galileo's strength prediction, hypothesized that the stress varied in proportion to the distance from the fulcrum, point *B* in Figure P6.128. That is, it varied linearly from point *B*. Show that the maximum bending stress value obtained by Mariotte is twice that predicted by Equation (6.12).

Stretch Yourself

6.132 A beam is acted upon by a distributed load $p(x)$. Let $\boldsymbol{M_A}$ and $\boldsymbol{V_A}$ represent the internal bending moment and the shear force at *A*. Show that the internal moment at *B* is given by

$$\boldsymbol{M}_B = \boldsymbol{M}_A - \boldsymbol{V}_A(x_B - x_A) + \int_{x_A}^{x_B} (x_B - x)p(x)\,dx \tag{6.30}$$

6.133 The displacement in the *x* direction in a beam cross section is given by $u = u_0(x) - y(dv/dx)(x)$. Assuming small strains and linear, elastic, isotropic, homogeneous material with no inelastic strains, show that

$$\boldsymbol{N} = EA\frac{du_0}{dx} - EAy_c\frac{d^2\mathrm{v}}{dx^2} \qquad \boldsymbol{M}_z = -EAy_c\frac{du_0}{dx} + EI_{zz}\frac{d^2\mathrm{v}}{dx^2}$$

where y_c is the *y* coordinate of the centroid of the cross section measured from some arbitrary origin, *A* is the cross-sectional area, I_{zz} is the area moment of inertia about the *z* axis, and $\boldsymbol{N}$ and $\boldsymbol{M_z}$ are the internal axial force and the internal bending moment. Note that if *y* is measured from the centroid of the cross section, that is, if $y_c = 0$, then the axial and bending problems decouple. In such a case show that $\sigma_{xx} = \boldsymbol{N}/A - \boldsymbol{M_z}y/I_{zz}$.

6.134 Show that the bending normal stresses in a homogeneous, linearly elastic, isotropic symmetric beam subject to a temperature change $\Delta T(x, y)$ is given by

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} + \frac{M_T y}{I_{zz}} - E\alpha\Delta T(x, y) \tag{6.31}$$

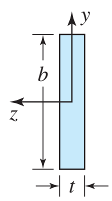
where $M_T = E\alpha\int_A y\Delta T(x, y)\ dA$, α is the coefficient of thermal expansion, and E is the modulus of elasticity.

6.135 In unsymmetrical bending of beams, under the assumption of plane sections remaining plane and perpendicular to the beam axis, the displacement u in the x direction can be shown to be $u = -y\, dv/dx - z\, dw/dx$, where y and z are measured from the centroid of the cross section, and v and w are the deflections of the beam in the y and z directions, respectively. Assume small strain, a linear, elastic, isotropic, homogeneous material, and no inelastic strain. Using Equations (1.8b) and (1.8c), show that

$$M_y = EI_{yz}\frac{d^2v}{dx^2} + EI_{yy}\frac{d^2w}{dx^2} \qquad M_z = EI_{zz}\frac{d^2v}{dx^2} + EI_{yz}\frac{d^2w}{dx^2} \qquad \sigma_{xx} = -\left(\frac{I_{yy}M_z - I_{yz}M_y}{I_{yy}I_{zz} - I_{yz}^2}\right)y - \left(\frac{I_{zz}M_y - I_{yz}M_z}{I_{yy}I_{zz} - I_{yz}^2}\right)z \tag{6.32}$$

Note that if either y or z is a plane of symmetry, then $I_{yz} = 0$. From Equation (6.32), this implies that the moment about the z axis causes deformation in the y direction only and the moment about the y axis causes deformation in the z direction only. In other words, the bending problems about the y and z axes are decoupled.

6.136 The equation $\partial\sigma_{xx}/\partial x + \partial\tau_{yx}/\partial y = 0$ was derived in Problem 1.105. Into this equation, substitute Equations (6.12) and (6.18) and integrate with y for beam cross section in Figure P6.136 and obtain the equation below.



$$\tau_{yx} = \frac{6V_y(b^2/4 - y^2)}{b^3 t}$$

Figure P6.136

Computer problems

6.137 A cantilever, hollow circular aluminum beam of 5-ft length is to support a load of 1200 lb. The inner radius of the beam is 1 in. If the maximum bending normal stress is to be limited to 10 ksi, determine the minimum outer radius of the beam to nearest $\frac{1}{16}$ in.

6.138 Table P6.138 shows the values of the distributed loads at several points along the axis of the rectangular beam shown in Figure P6.138. Determine the maximum bending normal and shear stresses in the beam.

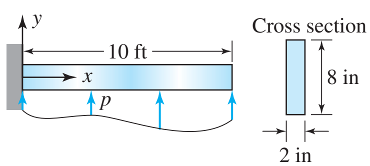


Figure P6.138

Table P6.138 Data for Problem 6.138

x (ft)	*p*(*x*) (lb/ft)
0	275
1	348
2	398
3	426
4	432
5	416

Table P6.138 Data for Problem 6.138

x (ft)	*p*(*x*) (lb/ft)
6	377
7	316
8	233
9	128
10	0

6.139 Let the distributed load $p(x)$ in Problem 6.138 be represented by $p(x) = a + bx + cx^2$. Using the data in Table P6.138, determine the constants a, b, and c by the least-squares method. Then find the maximum bending moment and the maximum shear force by analytical integration and determine the maximum bending normal and shear stresses.

6.7* CONCEPT CONNECTOR

Historically, an understanding of the strength of materials began with the study of beams. It did not, however, follow a simple course. Instead, much early work addressed mistakes, regarding the location of the neutral axis and the stress distribution across the cross section. The predicted values for the fracture loads on a beam did not correlate well with experiment. To make the pioneers' struggle in the dark more difficult, near fracture the stress–strain relationship is nonlinear, which alters the stress distribution and the location of the neutral axis.

6.7.1 History: Stresses in Beam Bending

The earliest known work on beams was by Leonardo da Vinci (1452–1519). In addition to his statements on simply supported beams, which are described in Problem 6.127, he correctly concluded that in a cantilevered, untapered beam the cross section farthest from the built-in end deflects the most. But it was Galileo's work that had the greatest early influence.

Galileo Galilei (1564–1642) (Figure 6.72) was born in Pisa. In 1581 he enrolled at the University of Pisa to study medicine, but the work of Euclid, Archimedes, and Leonardo attracted him to mathematics and mechanics. In 1589 he became professor of mathematics at the university, where he conducted his famous experiments on falling bodies, and the field of dynamics was born. He concluded that a heavier object takes the same time as a lighter object to fall through the same height, in complete disagreement with the popular Aristotelian mechanics. He paid the price for his views, for the proponents of Aristotelian mechanics made his stay at the university untenable, and he left in early 1592.

Figure 6.72 Galileo Galilei.

Fortunately, by the end of 1592 he was appointed professor of mathematics at the University of Padua. During this period he discovered his interest in astronomy. Based on sketchy reports, he built himself a telescope. On seeing moons in orbit around Jupiter in 1610, he found evidence for the Copernican theory, which held that the Earth is not the center of the universe. In 1616 Copernicus was condemned by the Church, and the Inquisition warned Galileo to leave theology to the Church. In 1632, however, he published his views, under the mistaken belief that the new pope, Maffeo Barberini, who was Galileo's admirer, would be more tolerant. Galileo was now condemned by the Inquisition and put under house arrest for the last eight years of his life. During this period he wrote *Two New Sciences*, in which he describes his work in mechanics, including the mechanics of materials. We have seen his contribution toward a concept of stress in Section 1.6. Here we discuss briefly his contributions on the bending of beams.

Figure P6.128 shows Galileo's illustration of the bending test. We described two of his insights in Problems 6.129 and 6.130. Three other conclusions of his, too, have influenced the design of beams ever since. First, a beam whose width is greater than its thickness offers greater resistance standing on its edge than lying flat, because the area moment of inertia is then greater. Second, the resisting moment—and thus the strength of the beam—increases as the cube of the radius for circular beams. Thus, the section modulus increases as the cube of the radius. Finally, the cross-sectional dimensions must increase at a greater rate than the length for constant strength cantilever beam bending due to its own weight. However, as we saw in Problem 6.128, Galileo incorrectly predicted the load-carrying capacity of beams, because he misjudged the stress distribution and the location of the neutral axis.

Credit for an important correction to the stress distribution goes to Edme Mariotte (1620–1684), who also discovered the eye's blind spot. Mariotte became interested in the strength of beams while trying to design pipes for supplying water to the palace of Versailles. His experiments with wooden and glass beams convinced him that Galileo's load predictions were greatly exaggerated. His own theory incorporated linear elasticity, and he concluded that the stress distribution is linear, with a zero stress value at the bottom of the beam. Mariotte's predicted values did not correlate well with experiment either, however. To explain why, he argued that beams loaded over a long time would have failure loads closer to his predicted values. While true, this is not the correct explanation for the discrepancy.

As we saw in Problem 6.131, the cause lay instead in an incorrect assumption about the location of the neutral axis. This incorrect location hindered many pioneers, including Claude-Louis Navier (1785-1836), who also helped develop the formulas for fluid flow, and the mathematician Jacob Bernoulli (1654-1705). (We saw some of Navier's contribution in Section 1.6 and will discuss Bernoulli's in Chapter 7 on beam deflection.) As a result, engineers used Galileo's theory in designing beams of brittle material such as stone, but Mariotte's theory for wooden beams.

Antoine Parent (1666–1716) was the first to show that Mariotte's stress formula does not apply to beams with circular cross section. Born in Paris, Parent studied law on the insistence of his parents, but he never practiced it, because he wanted to do mathematics. He also proved that, for a linear stress distribution across a rectangular cross section, the zero stress point is at the center, provided the material behavior is elastic. Unfortunately Parent published his work in a journal that he himself edited and published, not in the journal by the French Academy, and it was not widely read. More than half a century later, Charles Augustin Coulomb, whose contributions we saw in Section 5.5, independently deduced the correct location of the neutral axis. Coulomb showed that the stress distribution is such that the net axial force is zero (Equation (6.2)), independent of the material.

Jean Claude Saint-Venant (see Chapter 5) rigorously examined kinematic Assumptions 1 through 3. He demonstrated that these are met exactly only for zero shear force: the beam must be subject to couples only, with no transverse force. However, the shear stresses in beams had still not received much attention.

As mentioned in Section 1.6, the concept of shear stress was developed in 1781, by Coulomb, who believed that shear was only important in short beams. Louis Vicat's experiment in 1833 with short beams gave ample evidence of the importance of shear. Vicat (1786-1861), a French engineer, had earlier invented artificial cement. D. J. Jourawaski (1821-1891), a Russian railroad engineer, was working in 1844 on building a railroad between St. Petersburg and Moscow. A 180-ft-long bridge had to be built over the river Werebia, and Jourawaski had to use thick wooden beams. These thick beams were failing along the length of the fibers, which were in the longitudinal direction. Jourawaski realized the importance of shear in long beams and developed the theory that we studied in Section 6.6.

In sum, starting with Galileo, it took nearly 250 years to understand the nature of stresses in beam bending. Other historical developments related to beam theory will appear in Section 7.6.

6.8 CHAPTER CONNECTOR

In this chapter we established formulas for calculating normal and shear stresses in beams under symmetric bending. We saw that the calculation of bending stresses requires the internal bending moment and the shear force at a section. We considered only statically determinate beams. For these, the internal shear force and bending moment diagrams can be found by making an imaginary cut and drawing an appropriate free-body diagram. Alternatively, we can draw a shear force–bending moment diagram. The free-body diagram is preferred if stresses are to be found at a specified cross section. However, shear force–moment diagrams are the better choice if maximum bending normal or shear stress is to be found in the beam.

The shear force–bending moment diagrams can be drawn by using the graphical interpretation of the integral, as the area under a curve. Alternatively, internal shear force and bending moments can be found as a function of the x coordinate along the beam and plotted. Finding the bending moment as a function of x is important in the next chapter, where we integrate the moment–curvature relationship. Once we know how to find the deflection in a beam, we can solve problems of statically indeterminate beams.

We also saw that, to understand the character of bending stresses, we can draw the bending normal and shear stresses on a stress element. In many cases, the correct direction of the stresses can be obtained by inspection. Alternatively, we can follow the sign convention for drawing the internal shear force and bending moment on free-body diagrams, determine the direction using the subscripts in the formula. It is important to understand *both* methods for determining the direction of stresses. Shear–moment diagrams yield the shear force and the bending moment, following our sign convention. Drawing the bending stresses on a stress element is also important in stress or strain transformation, as described later.

In Chapter 8, on stress transformation, we will consider problems in which we first find bending stresses, using the stress formulas in this chapter. We then find stresses on inclined planes, including planes with maximum normal and shear stress. In Chapter 9, on strain transformation, we will find the bending strains and then consider strains in different coordinate systems, including coordinate systems in which the normal and shear strains are maximum. In Section 10.1 we will consider the combined loading problems of axial, torsion, and bending and the design of simple structures that may be determinate or indeterminate.

POINTS AND FORMULAS TO REMEMBER

- Our Theory is limited to (1) slender beams; (2) regions away from the neighborhood of stress concentration; (3) gradual variation in cross section and external loads; (4) loads acting in the plane of symmetry in the cross section; and (5) no change in direction of loading during bending.

- $$M_z = -\int_A y\sigma_{xx}\, dA \qquad (6.1) \qquad u = -y\frac{dv}{dx}, \quad v = v(x) \qquad (6.5)$$

- $$\text{small strain,} \qquad \varepsilon_{xx} = -\frac{y}{R} = -y\frac{d^2 v}{dx^2} \qquad (6.6a, b)$$

- where $\boldsymbol{M}_z$ is the internal bending moment that is drawn on the free-body diagram to put a point with positive y coordinate in compression; u and v are the displacements in the x and y directions, respectively; σ_{xx} and ε_{xx} are the bending (flexure) normal stress and strain; y is the coordinate measured from the neutral axis to the point where normal stress and normal strain are defined, and d^2v/dx^2 is the curvature of the beam.
- The normal bending strain ε_{xx} is a linear function of y.
- The normal bending strain ε_{xx} will be maximum at either the top or the bottom of the beam.
- Equations (6.1), (6.6a), and (6.6b) are independent of the material model.
- The following formulas are valid for material that is linear, elastic, and isotropic, with no inelastic strains.
- For homogeneous cross section:

- $$M_z = EI_{zz}\frac{d^2 v}{dx^2} \qquad (6.11) \qquad \sigma_{xx} = -\frac{M_z y}{I_{zz}} \qquad (6.12)$$

- where y is measured from the centroid of the cross section, and I_{zz} is the second area moment about the z axis passing through the centroid.
- EI_{zz} is the bending rigidity of a beam cross section.
- Normal stress σ_{xx} in bending varies linearly with y on a homogeneous cross section.
- Normal stress σ_{xx} is zero at the centroid ($y = 0$) and maximum at the point farthest from the centroid for a homogeneous cross section.
- The shear force $\boldsymbol{V}_y$ will jump by the value of the applied external force as one crosses it from left to right.
- $\boldsymbol{M}_z$ will jump by the value of the applied external moment as one crosses it from left to right.

- $$V_y = \int_A \tau_{xy}\, dA \qquad (6.13) \qquad \tau_{xs} = -\frac{V_y Q_z}{I_{zz} t} \qquad (6.27)$$

- where Q_z is the first moment of the area A_s about the z axis passing through the centroid, t is the thickness perpendicular to the centerline, A_s is the area between the free surface and the line at which the shear stress is being found, and the coordinate s is measured from the free surface used in computing Q_z.
- The direction of shear flow on a cross section must be such that (1) the resultant force in the y direction is in the same direction as $\boldsymbol{V}_y$; (2) the resultant force in the z direction is zero; and (3) it is symmetric about the y axis.
- Q_z is zero at the top and bottom surfaces and is maximum at the neutral axis.
- Shear stress is maximum at the neutral axis of a cross section in symmetric bending of beams.
- The bending strains are
- $$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad (6.29)$$

CHAPTER SEVEN

DEFLECTION OF SYMMETRIC BEAMS

Learning Objective

1. Learn to formulate and solve the boundary-value problem for the deflection of a beam at any point.

Greg Louganis, the American often considered the greatest diver of all time, has won four Olympic gold medals, one silver medal, and five world championship gold medals. He won both the springboard and platform diving competitions in the 1984 and 1988 Olympic games. In his incredible execution, Louganis and all divers (Figure 7.1*a*) makes use of the behavior of the diving board. The flexibility of the springboard, for example, depends on its thin aluminum design, with the roller support adjusted to give just the right unsupported length. In contrast, a bridge (Figure 7.1*b*) must be stiff enough so that it does not vibrate too much as the traffic goes over it. The stiffness in a bridge is obtained by using steel girders with a high area moment of inertias and by adjusting the distance between the supports. In each case, to account for the right amount of flexibility or stiffness in beam design, we need to determine the beam deflection, which is the topic of this chapter

(a) (b)

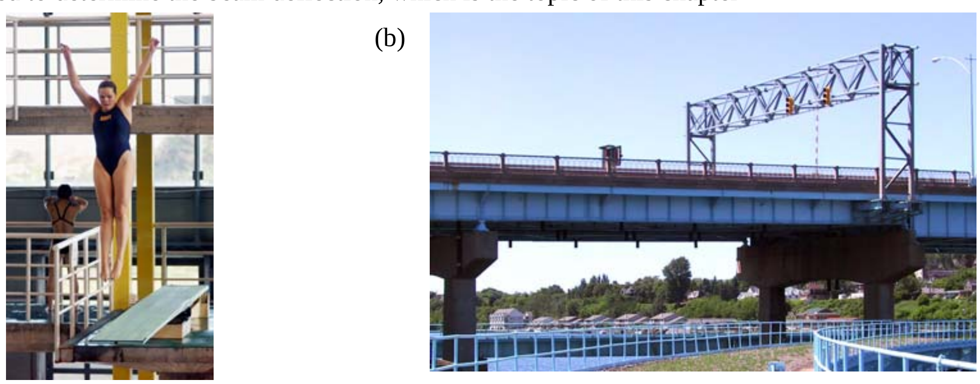


Figure 7.1 Examples of beam: (a) flexibility of diving board; and (b) stiffness of steel girders.

We can obtain the deflection of a beam by integrating either a second-order or a fourth-order differential equation. The differential equation, together with all the conditions necessary to solve for the integration constants, is called a **boundary-value problem**. The solution of the boundary-value problem gives the deflection at any location x along the length of the beam.

7.1 SECOND-ORDER BOUNDARY-VALUE PROBLEM

Chapter 6 considered the symmetric bending of beams. We found that if we can find the deflection in the y direction of one point on the cross section, then we know the deflection of all points on the cross section. In other words, the deflection at a cross section is independent of the y and z coordinates. However, the deflection can be a function of x, as shown in Figure 7.2. The deflected curve represented by v(x) is called the **elastic curve.**

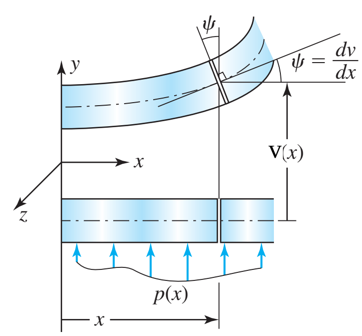


Figure 7.2 Beam deflection.

The deflection function v(x) can be found by integrating Equation (6.11) twice, provided we can find the internal moment as a function of x, as we did in Section 6.3. Equation (6.11), this second-order differential equation is rewritten for convenience:

$$\boldsymbol{M}_z = EI_{zz}\frac{d^2\mathrm{v}}{dx^2} \tag{7.1}$$

The two integration constants generated from Equation (7.12.a) are determined from boundary conditions, as discussed next, in Section 7.1.1.

As one moves across the beam, the applied load may change, resulting in different functions of x that represent the internal moment $\boldsymbol{M}_z$. In such cases there are as many differential equations as there are functions representing the moment $\boldsymbol{M}_z$. Each additional differential equation generates additional integration constants. These additional integration constants are determined from continuity (compatibility) equations, obtained by considering the point where the functional representation of the moment changes character. The continuity conditions will be discussed in Section 7.1.2. The mathematical statement listing all the differential equations and all the conditions necessary for solving for v(x) is called the **boundary-value problem** for the beam deflection.

7.1.1 Boundary Conditions

The integration of Equation (7.1) will result in v and $d\mathrm{v}/dx$. Thus, we are seeking conditions on v or $d\mathrm{v}/dx$. Figure 7.3 shows three types of support and the associated boundary conditions.

Note that for a second-order differential equation we need two boundary conditions. If on one end there is only one boundary condition, as in Figure 7.3*b* or *c*, then the remaining boundary condition must come from another location. Doubts about a boundary condition at a support can often be resolved by drawing an approximate deformed shape of the beam.

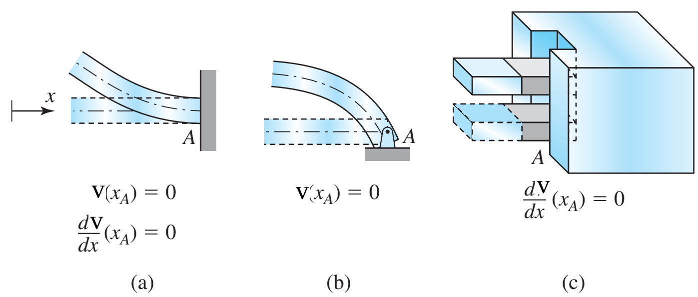


Figure 7.3 Boundary conditions for second-order differential equations. (*a*) Built-in end. (*b*) Simple support. (*c*) Smooth slot.

7.1.2 Continuity Conditions

Suppose that because of change in the applied loading, the internal moment $\boldsymbol{M}_z$ in a beam is represented by one function to the left of x_j and another function to the right of x_j. Then there are two second-order differential equations, and integration will produce two different displacement functions, one for each side of x_j, together, these will contain a total of four integration constants. Two of these four integration constants can be determined from the boundary conditions, as discussed in Section 7.1.1. The remaining two constants will have to be determined from conditions at x_j. Figure 7.4 shows that a discontinuous displacement at x_j implies a broken beam, and a discontinuous slope at x_j implies that a beam is kinked at x_j.

Assuming that the beam neither breaks nor kinks, then the displacement functions must satisfy the following conditions:

$$\mathrm{v}_1(x_j) = \mathrm{v}_2(x_j) \tag{7.2.a}$$

$$\frac{d\mathrm{v}_1}{dx}(x_j) = \frac{d\mathrm{v}_2}{dx}(x_j) \tag{7.2.b}$$

where v_1 and v_2 are the displacement functions to the left and right of x_j. The conditions given by Equations (7.2) are the **continuity conditions**, also known as compatibility conditions or matching conditions.

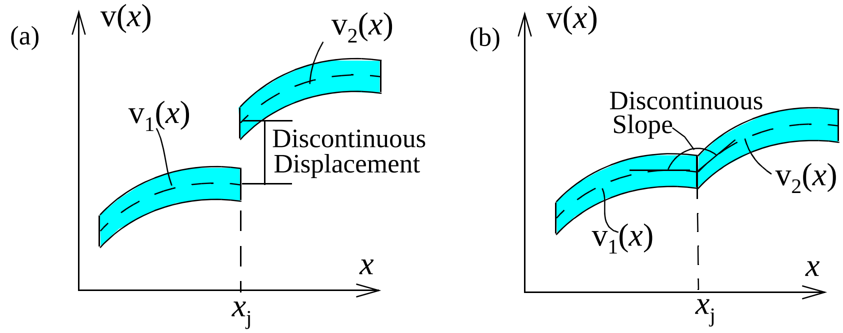


Figure 7.4 (*a*) Broken beam. (*b*) Kinked beam.

- Example 7.1 demonstrates the formulation and solution of a boundary-value problem with one second-order differential equation and the associated boundary conditions.
- Example 7.2 demonstrates the formulation and solution of a boundary-value problem with two second-order differential equations, the associated boundary conditions, and the continuity conditions.
- Example 7.3 demonstrates the formulation only of a boundary-value problem with multiple second-order differential equations, the associated boundary conditions, and the continuity conditions.
- Example 7.4 demonstrates the formulation and solution of a boundary-value problem with variable area moment of inertia, that is, I_{zz} is a function of x.

EXAMPLE 7.1

A beam has a linearly varying distributed load, as shown in Figure 7.5. Determine: (a) The equation of the elastic curve in terms of E, I, w, L, and x. (b) The maximum intensity of the distributed load if the maximum deflection is to be limited to 20 mm. Use $E = 200$ GPa, $I = 600\ (10^6)\ \text{mm}^4$, and $L = 8$ m.

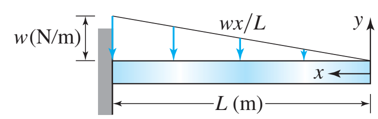


Figure 7.5 Beam and loading in Example 7.1.

PLAN

(a) We can make an imaginary cut at an arbitrary location x and draw the free-body diagram. Using equilibrium equations, the moment $\boldsymbol{M}_z$ can be written as a function of x. By integrating Equation (7.1) and using the boundary conditions that deflection and slope at $x = L$ are zero, we can find v(x). (b) The maximum deflection for this problem will occur at the free end and can be found by substituting $x = 0$ in the v(x) expression. By requiring that $|v_{max}| \le 0.02\ \text{m}$, we can find w_{max}.

SOLUTION

(a) Figure 7.6 shows the free-body diagram of the right part after making an imaginary cut at some location x. Internal moment and shear forces are drawn according to the sign convention discussed in Section 6.2.6. The distributed force is replaced by an equivalent force, and the internal moment is found by equilibrium of moment about point O.

$$\boldsymbol{M}_z = -\frac{1}{2}\frac{wx^2}{L}\left(\frac{x}{3}\right) = -\frac{1}{6}\frac{wx^3}{L} \tag{E1}$$

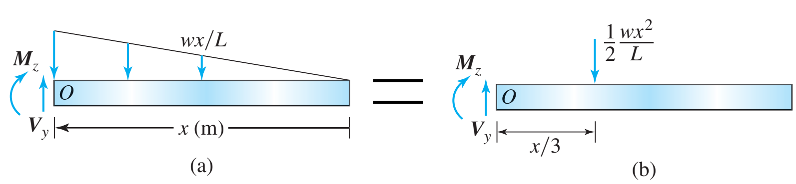


Figure 7.6 Free-body diagram in Example 7.1. (a) Imaginary cut on original beam. (b) Statically equivalent diagram.

We substitute Equation (E1) into Equation (7.1) and note the zero slope and deflection at the built-in end. The boundary-value problem can then be stated as follows:

- **Differential equation:**

$$EI_{zz}\frac{d^2v}{dx^2} = -\frac{1}{6}\frac{wx^3}{L} \tag{E2}$$

- **Boundary conditions:**

$$v(L) = 0 \tag{E3}$$

$$\frac{dv}{dx}(L) = 0 \tag{E4}$$

Equation (E2) can be integrated to obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{1}{24}\frac{wx^4}{L} + c_1 \tag{E5}$$

Substituting $x = L$ in Equation (E5) and using Equation (E4) gives the constant c_1:

$$-\frac{1}{24}\frac{wL^4}{L} + c_1 = 0 \qquad \text{or} \qquad c_1 = \frac{wL^3}{24} \tag{E6}$$

Substituting Equation (E6) into Equation (E5) we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{1}{24}\frac{wx^4}{L} + \frac{wL^3}{24} \tag{E7}$$

Equation (E7) can be integrated to obtain

$$EI_{zz}v = -\frac{1}{120}\frac{wx^5}{L} + \frac{wL^3}{24}x + c_2 \tag{E8}$$

Substituting $x = L$ in Equation (E8) and using Equation (E3) gives the constant c_2:

$$-\frac{1}{120}\frac{wL^5}{L} + \frac{wL^3}{24}L + c_2 = 0 \qquad \text{or} \qquad c_2 = -\frac{wL^4}{30} \tag{E9}$$

The deflection expression can be obtained by substituting Equation (E9) into Equation (E8) and simplifying.

ANS. $v(x) = -\dfrac{w}{120EI_{zz}L}(x^5 - 5L^4x + 4L^5)$

Dimension check: We note that all terms in the parentheses have the dimension of length to the power of five, that is, $O(L^5)$. Thus the answer is dimensionally homogeneous. But we can also check whether the left-hand side and any one term of the right-hand side has the same dimension,

$$w \to O\left(\frac{F}{L}\right) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{wx^5}{EI_{zz}L} \to O\left(\frac{(F/L)L^5}{(F/L^2)O(L^4)L}\right) \to O(L) \to \text{checks}$$

(b) By inspection it can be seen that the maximum deflection for this problem will occur at the free end. Substituting $x = 0$ in the deflection expression, we obtain $v_{max} = -wL^4/30EI_{zz}$. The minus sign indicates that the deflection is in the negative y direction, as expected. Substituting the given values of the variables and requiring that the magnitude of the deflection be less than 0.02 m, we obtain

$$|v_{max}| = \frac{w_{max}L^4}{30EI_{zz}} = \frac{w_{max}(8\text{ m})^4}{30[200(10^9\text{ N/m}^2)][600(10^{-6})\text{ m}^4]} \le 0.02\text{ m} \qquad \text{or} \qquad w_{max} \le 17.58(10^3)\text{ N/m} \tag{E10}$$

ANS. $w_{max} = 17.5$ kN/m

COMMENTS

1. From calculus we know that the maximum of a function occurs at the point where the slope of the function is zero. But the slope at $x = L$, where the deflection is maximum, is not zero. This is because $v(x)$ is a **monotonic function**— that is, a continuously increasing (or decreasing) function. For monotonic functions the maximum (or minimum) always occurs at the end of the interval. We intuitively recognized the function's monotonic character when we stated that the maximum deflection occurs at the free end.
2. If the dimension check showed that some term did not have the proper dimension, then we would backtrack, check each equation for dimensional homogeneity, and identify the error.

EXAMPLE 7.2

For the beam and loading shown in Figure 7.7, determine: (a) the equation of the elastic curve in terms of *E*, *I*, *L*, *P*, and *x*; (b) the maximum deflection in the beam.

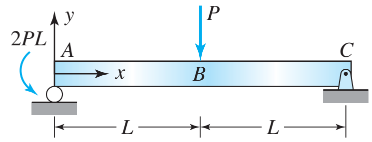


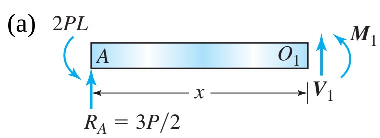
Figure 7.7 Beam and loading in Example 7.2.

PLAN

(a) The internal moment due to the load *P* at *B* will be represented by different functions in *AB* and *BC*, which can be found by making imaginary cuts and drawing free-body diagrams. We can write the two differential equations using Equation (7.1), the two boundary conditions of zero deflection at *A* and *C*, and the two continuity conditions at *B*. The boundary-value problem can be solved to obtain the elastic curve. (b) In each section we can set the slope to zero and find the roots of the equation that will give the location of zero slope. We can substitute the location values in the elastic curve equation derived in part (a) to determine the maximum deflection in the beam.

SOLUTION

(a) The free-body diagram of the entire beam can be drawn, and the reaction at *A* found as $R_A = 3P/2$ upward, and the reaction at *C* found as $R_C = P/2$ downward. Figure 7.8 shows the free body diagrams after imaginary cuts have been made and then internal shear force and bending moment drawn according to our sign convention.



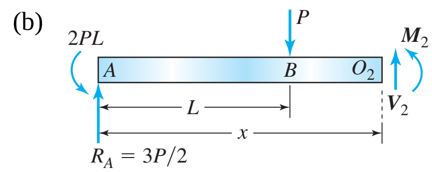


Figure 7.8 Free body diagrams in Example 7.2 after imaginary cut in (a) *AB* (b) *BC*.

By equilibrium of moments in Figure 7.8*a* and *b* we obtain the internal moments

$$M_1 + 2PL - R_A x = 0 \quad \text{or} \quad M_1 = \frac{3}{2}Px - 2PL \tag{E1}$$

$$M_2 + 2PL - R_A x + P(x - L) = 0 \quad \text{or} \quad M_2 = \frac{3}{2}Px - 2PL - P(x - L) \tag{E2}$$

Check: The internal moment must be continuous at *B*, since there is no external point moment at *B*. Substituting $x = L$ in Equations (E1) and (E2), we find $M_1 = M_2$ at $x = L$.

The boundary-value problem can be stated using Equation (7.1), (E1), and (E2), the zero deflection at points *A* and *C*, and the continuity conditions at *B* as follows:

- **Differential equations:**

$$EI_{zz}\frac{d^2 v_1}{dx^2} = \frac{3}{2}Px - 2PL, \qquad 0 \le x < L \tag{E3}$$

$$EI_{zz}\frac{d^2 v_2}{dx^2} = \frac{3}{2}Px - 2PL - P(x - L), \qquad L \le x < 2L \tag{E4}$$

- **Boundary conditions:**

$$v_1(0) = 0 \tag{E5}$$

$$v_2(2L) = 0 \tag{E6}$$

- **Continuity conditions:**

$$v_1(L) = v_2(L) \tag{E7}$$

$$\frac{dv_1}{dx}(L) = \frac{dv_2}{dx}(L) \tag{E8}$$

Integrating Equations (E3) and (E4) we obtain

$$EI_{zz}\frac{dv_1}{dx} = \frac{3}{4}Px^2 - 2PLx + c_1 \tag{E9}$$

$$EI_{zz}\frac{dv_2}{dx} = \frac{3}{4}Px^2 - 2PLx - \frac{P}{2}(x - L)^2 + c_2 \tag{E10}$$

Substituting $x = L$ in Equations (E9) and (E10) and using Equation (E8), we obtain

$$\frac{3}{4}PL^2 - 2PL^2 + c_1 = \frac{3}{4}PL^2 - 2PL^2 - 0 + c_2 \qquad \text{or} \qquad c_1 = c_2 \tag{E11}$$

Substituting Equation (E11) into Equation (E10) and integrating Equations (E9) and (E10), we obtain

$$EI_{zz}v_1 = \frac{1}{4}Px^3 - PLx^2 + c_1x + c_3 \tag{E12}$$

$$EI_{zz}v_2 = \frac{1}{4}Px^3 - PLx^2 - \frac{P}{6}(x-L)^3 + c_1x + c_4 \tag{E13}$$

Substituting $x = L$ in Equations (E12) and (E13) and using Equation (E7), we obtain

$$\frac{1}{4}PL^3 - PL^3 + c_1L + c_3 = \frac{1}{4}PL^3 - PL^3 - 0 + c_1L + c_4 \qquad \text{or} \qquad c_3 = c_4 \tag{E14}$$

Substituting $x = 0$ in Equation (E12) and using Equation (E5), we obtain

$$c_3 = 0 \tag{E15}$$

From Equation (E14),

$$c_4 = 0 \tag{E16}$$

Substituting $x = 2L$ and Equation (E16) into Equation (E13) and using Equation (E6), we obtain

$$\frac{1}{4}P(2L)^3 - PL(2L)^2 - \frac{P}{6}(L)^3 + c_1(2L) = 0 \qquad \text{or} \qquad c_1 = \frac{13}{12}PL^2 \tag{E17}$$

Substituting Equations (E15), (E16), and (E17) into Equations (E12) and (E13) and simplifying, we obtain the answer:

$$\textbf{ANS. } v_1(x) = \frac{P}{12EI_{zz}}(3x^3 - 12Lx^2 + 13L^2x) \qquad 0 \le x < L \tag{E18}$$

$$\textbf{ANS. } v_2(x) = \frac{P}{12EI_{zz}}[3x^3 - 12Lx^2 + 13L^2x - 2(x-L)^3] \qquad L \le x < 2L \tag{E19}$$

Dimension check: All terms in parentheses are dimensionally homogeneous, as all have the dimensions of length cubed. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension:

$$P \to O(F) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{Px^3}{EI_{zz}} \to O\left(\frac{FL^3}{(F/L^2)L^4}\right) \to O(L) \to \text{checks}$$

(b) Let dv_1/dx be zero at $x = x_1$. Differentiating Equation (E18), we obtain

$$\frac{P}{12EI_{zz}}(9x_1^2 - 24Lx_1 + 13L^2) = 0 \qquad \text{or} \qquad 9x_1^2 - 24Lx_1 + 13L^2 = 0 \tag{E20}$$

The roots of the quadratic equation are $x_1 = 1.91L$ and $x_1 = 0.756L$. . The admissible root is $x_1 = 0.756L$. , since Equation (E18), and hence Equation (E20), are valid only in the range from 0 to L. Substituting this root into Equation (E18), we obtain

$$v_1(0.756L) = \frac{P}{12EI_{zz}}(3 \times 0.756L^3 - 12L \times 0.756L^2 + 13L^2 \times 0.756L) = \frac{0.355PL^3}{EI_{zz}} \tag{E21}$$

To find the maximum deflection in BC, assume dv_2/dx to be zero at $x = x_2$. Differentiating Equation (E19) we obtain

$$\frac{P}{12EI_{zz}}[9x_2^2 - 24Lx_2 + 13L^2 - 6(x_2 - L)^2] = 0 \qquad \text{or} \qquad 3x_2^2 - 12Lx_2 + 7L^2 = 0 \tag{E22}$$

The roots of the quadratic equations in Equation (E22) are $x_2 = 0.709L$ and $x_2 = 3.29L$. Both roots are outside the range of L to $2L$ and hence are inadmissible. Thus in this problem the slope is zero only at $0.756L$, and the maximum deflection is given by Equation (E21).

$$\textbf{ANS.} \qquad v_{max} = \frac{0.355PL^3}{EI_{zz}}$$

COMMENT

1. When we made the imaginary cut in BC, we took the left part for drawing the free-body diagram. Had we taken the right part, we would have obtained the moment expression $\boldsymbol{M}_2 = (Px/2) - PL$, which is the simplified form of Equation (E21). We can start with this moment expression and obtain our results from integration and the conditions as shown. The values of the integration constants will be different, and there will be slightly more algebra, but the final result will be the same. The form of the moment expression used in the example made use of the observation that the continuity conditions are at $x = L$ and the terms in powers of $(x - L)$ will be zero. This form results in less algebra and simplified relations for the constants, as given by Equations (E11) and (E14).

EXAMPLE 7.3

Write the boundary-value problem for solving the deflection at any point x of the beam shown in Figure 7.9. Do not integrate or solve.

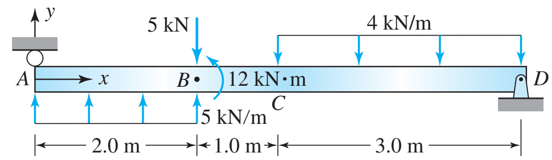


Figure 7.9 Beam and loading in Example 7.3.

PLAN

The moment expressions in each interval were found in Example 6.10. The differential equations can be written by substituting these moment expressions into Equation (7.1). We can also write the zero-deflection conditions at points A and D and the continuity conditions at points B and C to complete the boundary-value problem statement.

SOLUTION

From the free body diagram of the entire beam the reactions at A and D in Example 6.10 were found to be $R_A = 0$ and $R_D = 7$ kN. Figure 7.10 shows the free body diagrams used in Example 6.10 to obtain the internal moments

$$\boldsymbol{M}_1 = \left(\frac{5}{2}x^2\right) \text{ kN}\cdot\text{m} \qquad 0 \le x < 2 \text{ m} \tag{E1}$$

$$\boldsymbol{M}_2 = (5x - 12) \text{ kN}\cdot\text{m} \qquad 2 \text{ m} < x < 3 \text{ m} \tag{E2}$$

$$\boldsymbol{M}_3 = (-2x^2 + 17x - 30) \text{ kN}\cdot\text{m} \qquad 3 \text{ m} < x \le 6 \text{ m} \tag{E3}$$

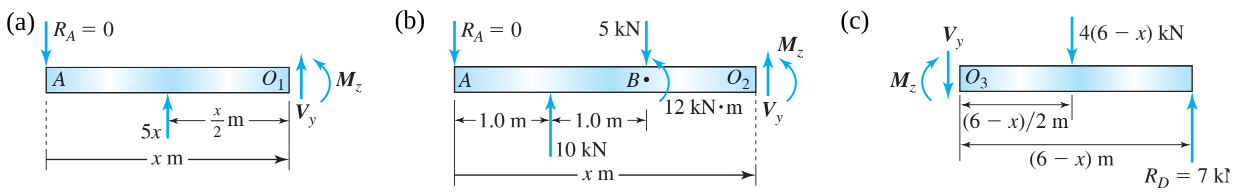


Figure 7.10 Free body diagrams in Example 7.3 after imaginary cut in (a) AB (b) BC (c) CD.

The boundary value problem can be written as described below.

- **Differential equations:**

$$EI_{zz}\frac{d^2v_1}{dx^2} = \left(\frac{5}{2}x^2\right) \text{ kN}\cdot\text{m} \qquad 0 \le x < 2 \text{ m} \tag{E4}$$

$$EI_{zz}\frac{d^2v_2}{dx^2} = (5x - 12) \text{ kN}\cdot\text{m} \qquad 2 \text{ m} < x < 3 \text{ m} \tag{E5}$$

$$EI_{zz}\frac{d^2v_3}{dx^2} = (-2x^2 + 17x - 30) \text{ kN}\cdot\text{m} \qquad 3 \text{ m} < x \le 6 \text{ m} \tag{E6}$$

- **Boundary conditions:**

$$v_1(0) = 0 \tag{E7}$$

$$v_3(6) = 0 \tag{E8}$$

- **Continuity conditions:**

$$v_1(2) = v_2(2) \tag{E9}$$

$$\frac{dv_1}{dx}(2) = \frac{dv_2}{dx}(2) \tag{E10}$$

$$v_2(3) = v_3(3) \tag{E11}$$

$$\frac{dv_2}{dx}(3) = \frac{dv_3}{dx}(3) \tag{E12}$$

COMMENTS

1. Equations (E4), (E5), and (E6) are three differential equations of order 2. Integrating these three differential equations would result in six integration constants. We have two boundary conditions and four continuity conditions. A properly formulated boundary-value problems will always have *exactly* the right number of conditions needed to solve a problem.
2. In Example 7.3 there were two differential equations and the resulting algebra was tedious. This example has three differential equations, which will make the algebra even more tedious. Fortunately there is a method, discussed in Section 7.4*, which reduces the algebra. This *discontinuity method* introduces functions that will let us write all three differential equations as a single equation and implicitly satisfy the continuity conditions during integration.

EXAMPLE 7.4

A cantilever beam with variable width $b(x)$ is shown in Figure 7.11. Determine the maximum deflection in terms of P, b_L, t, L, and E.

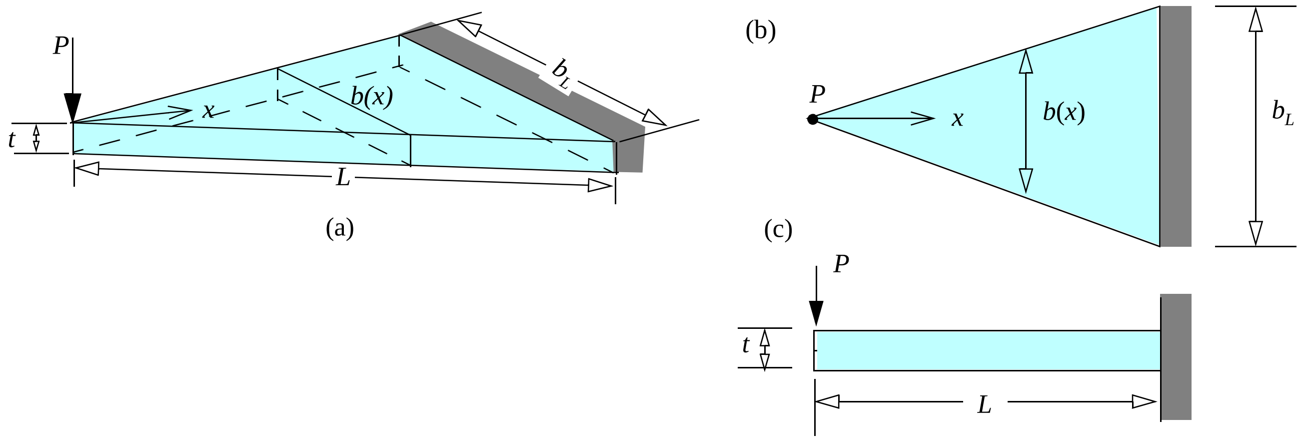


Figure 7.11 (a) Geometry of variable-width beam in Example 7.4. (b) Top view. (c) Front view.

PLAN

The area moment of inertia and the bending moment can also be found of x and substituted into Equation (7.1) to obtain the differential equation. The zero deflection and slope at $x = L$ are the boundary conditions necessary to solve the boundary-value problem for the elastic curve. The maximum deflection will be at $x = 0$ and can be found from the equation of the elastic curve.

SOLUTION

Noting that $b(x)$ is a linear function of x that passes through the origin and has a value of b_L at $x = L$, we obtain $b(x) = b_L x/L$ and the area moment of inertia as

$$I_{zz} = \frac{b(x)t^3}{12} = \left(\frac{b_L t^3}{12L}\right)x \tag{E1}$$

Figure 7.12 shows the free body diagram after an imaginary cut can be made at some location x. By equilibrium of moment at about O, we obtain the internal moment,

$$\boldsymbol{M}_z = -Px \tag{E2}$$

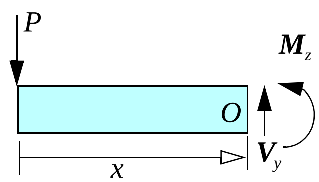


Figure 7.12 Free-body diagram in Example 7.4.

The boundary-value problem can be written as follows:

- **Differential equation:**

$$\frac{d^2 v}{dx^2} = \frac{M_z}{EI_{zz}} = \frac{-Px}{E(b_L t^3/12L)x} = -\frac{12PL}{Eb_L t^3} \tag{E3}$$

- **Boundary conditions:**

$$v(L) = 0 \tag{E4}$$

$$\frac{dv}{dx}(L) = 0 \tag{E5}$$

Integrating Equation (E3) we obtain

$$\frac{dv}{dx} = -\frac{12PL}{Eb_L t^3}x + c_1 \tag{E6}$$

Substituting Equation (E6) into Equation (E5), we obtain

$$0 = -\frac{12PL}{Eb_L t^3}L + c_1 \qquad \text{or} \qquad c_1 = \frac{12PL^2}{Eb_L t^3} \tag{E7}$$

Substituting Equation (E7) into Equation (E6) and integrating, we obtain

$$v = -\frac{12PL}{Eb_L t^3}\left(\frac{x^2}{2}\right) + \frac{12PL^2}{Eb_L t^3}x + c_2 \tag{E8}$$

Substituting Equation (E8) into Equation (E4), we obtain

$$0 = -\frac{12PL}{Eb_L t^3}\left(\frac{L^2}{2}\right) + \frac{12PL^2}{Eb_L t^3}L + c_2 \qquad \text{or} \qquad c_2 = -\frac{6PL^3}{Eb_L t^3} \tag{E9}$$

The maximum deflection will occur at the free end. Substituting $x = 0$ into Equation (E8) and using Equation (E9) we obtain the maximum deflection.

$$\textbf{ANS.} \qquad v_{max} = -\frac{6PL^3}{Eb_L t^3} \tag{E10}$$

COMMENTS

1. The beam taper must be gradual given the limitation on the theory described in Section 6.2.
2. We can calculate the maximum bending normal stress in any section by substituting $y = t/2$ and Equations (E1) and (E2) into Equation (6.12), to obtain

$$\sigma_{max} = \left| -Px\frac{t/2}{(b_L t^3/12L)x} \right| = \frac{6PL}{b_L t^2} \tag{E11}$$

3. Equation (E11) shows that the maximum bending normal stress is a constant throughout the beam. Such *constant-strength beams* are used in many designs where reduction in weight is a serious consideration. One such design is elaborated in comment 3.
4. In a *leaf spring* (see page 334), each leaf is considered an independent beam that bends about its own neutral axis because there is no restriction to sliding (see Problem 6.20). The variable-width beam is designed for constant strength, and b_L is found using Equation (E11). The width b_L is then divided into n parts, as shown in Figure 7.13*a*. Except for the main leaf *A*, all other leaf dimensions are found by taking the one-half leaf width on either side of the main leaf. In the assembled spring, the distance in each leaf from the applied load P is the same as in the original variable-width beam shown in Figure 7.11. Hence each leaf has the same allowable strength at all points. If $\bar{b}$ is the width of each leaf and $\bar{L}$ is the total length of the spring, so that $L = \bar{L}/2$, Equations (E10) and (E11) can be rewritten as

$$\delta = \frac{3P\bar{L}^3}{4nE\bar{b}t^3} \qquad \sigma_{max} = \frac{3P\bar{L}}{n\bar{b}t^2} \tag{7.3}$$

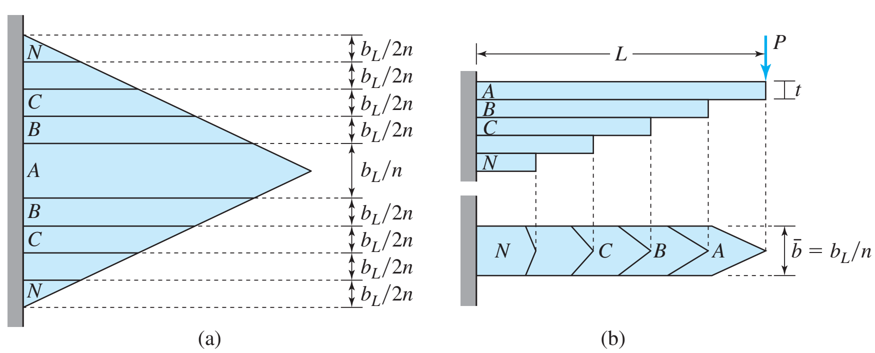


Figure 7.13 Explanation of leaf spring

5. The results can be used in design as in Example 3.6.

MoM In Action: Leaf Springs

When it comes to leaf springs, necessity was the mother of invention. Humans realized very early the mechanical advantage of a spring force from bending. For example, most early civilizations had longbows. However, thongs and ropes lose much of their elasticity in dampness and rain. Metals do not, and in 200 B.C.E. Philo of Byzantium proposed using bronze springiness as a source of power. By the early sixteenth century spring-powered clocks attained an accuracy of one minute a day—far better than the weight-driven clocks seen in the towers of Renaissance Italy. The discovery revolutionized navigation, enabling world exploration and European colonial power.

Around the same time, overland travel drove a different kind of spring development. Wagons and carriages felt every bump in the road, and the solution was the first suspension system: leather straps attached to four posts of a chassis suspended the carriage body and isolated it from the chassis. For all its advantages, however, the system did not prevent forward and backward sway, and the high center of gravity left the carriage susceptible to rollover. The problems were significantly reduced by the introduction of *cart springs* (Figure 7.14*a*) or what we call *leaf springs* today. Edouard Phillips (1821–1889) developed the theory of leaf springs (see Example 7.4) while studying the spring suspension in freight trains. It was one of the first applications of the mechanics of materials to engineering design problems.

(a)

(b)

Figure 7.14 Leaf springs in (a) cart; (b) conventional vehicles.

We still use the term *suspension system* today, although cars, trucks, and railways are all supported on springs rather than suspended. To increase bending rigidity, leaf springs have a curve (Figure 7.14*b*). When the curve is elliptical, the springs are referred to as *semi-elliptical* springs. The Ford Model T had a non-elliptical curve, but with the Corvette leaf spring design reached its zenith. Unlike the traditional longitudinal mounting of one spring per wheel, the spring in a Corvette was mounted transversely. This eliminated one leaf spring and significantly reduced the tendency to rollover. A double wishbone design allowed for independent articulation of each wheel. The one-piece fiberglass material practically eliminated fatigue failure, reducing the weight by two thirds compared to steel springs.

With the growth of front wheel drive in the 1970s, automobiles turned instead to coil springs, which require less space and provide each wheel with independent suspension. However, leaf springs continue to be used in trucks and railways, to distribute their heavy loads over larger spans.

Both coil and leaf spring systems are part of a *passive suspension* system, which involve a trade-off between comfort, control, handling, and safety. Those factors are driving newer design systems called *active suspension*, in which the amount of spring force is externally controlled. It took 400 years for leaf springs to reach their zenith, but need has no zenith, and necessity is still the mother of invention in suspension design.

PROBLEM SET 7.1

Second-order boundary-value problems

7.1 For the beam shown in Figure P7.1, determine in terms of *P*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

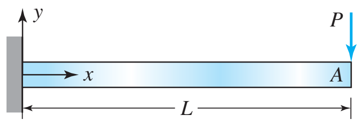


Figure P7.1

7.2 For the beam shown in Figure P7.2, determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

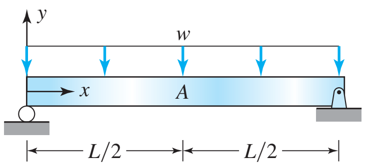


Figure P7.2

7.3 For the beam shown in Figure P7.3, determine in terms of *P*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

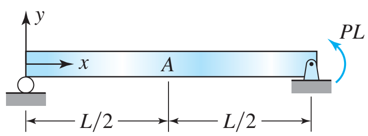


Figure P7.3

7.4 For the beam shown in Figure P7.4, determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

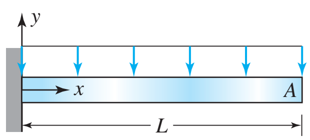


Figure P7.4

7.5 For the beam shown in Figure P7.5, determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

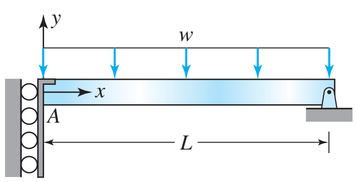


Figure P7.5

7.6 For the beam shown in Figure P7.6, determine in terms of *P*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection of the beam at point *A*.

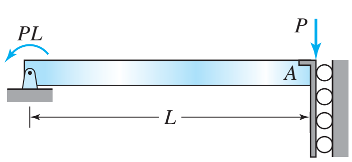


Figure P7.6

7.7 The cantilever beam in Figure P7.7 is acted upon by a distributed bending moment *m* per unit length. Determine (a) the elastic curve in terms of *m*, *E*, *I*, *L*, and *x*; (b) the deflection at $x = L$.

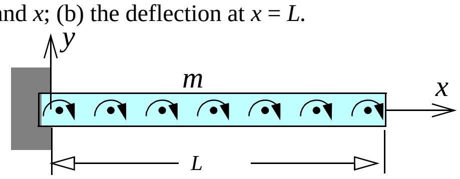


Figure P7.7

7.8 For the beam shown in Figure P7.8 determine the deflection at point *A* in terms of *w, L, E,* and *I*.

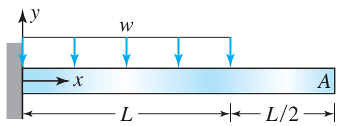


Figure P7.8

7.9 For the beam shown in Figure P7.9 determine the deflection at point *A* in terms of *P, L, E,* and *I*.

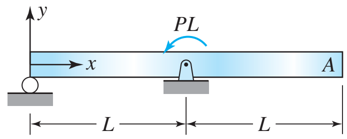


Figure P7.9

7.10 For the beam and loading shown in Figure P7.10, determine the deflection at point *A* in terms of *P, L, E,* and *I*.

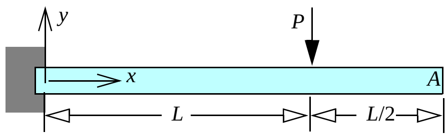


Figure P7.10

7.11 For the beam shown and loading in Figure P7.11, determine the deflection at point *A* in terms of *w, L, E,* and *I*.

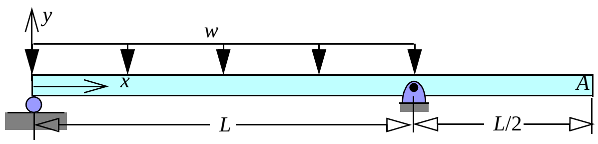


Figure P7.11

7.12 For the beam and loading shown in Figure P7.11, determine the deflection at point *A* in terms of *w, L, E,* and *I*

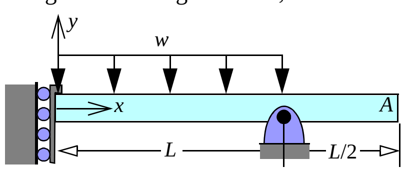


Figure P7.12

7.13 In Table P7.13, v_1 and v_2 represents the deflection in segment *AB* and *BC*. For the beam shown in Figure P7.2, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

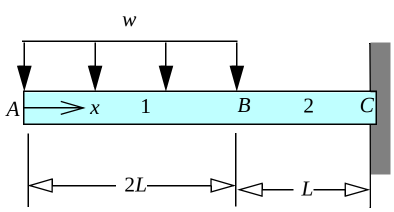


Figure P7.13

TABLE P7.13 **Potential Boundary and Continuity Conditions**

(a) $v_1(0) = 0$	(e) $v_2(2L) = 0$	(i) $v_1(L) = v_2(L)$
(b) $v_1(L) = 0$	(f) $v_2(3L) = 0$	(j) $v_1(2L) = v_2(2L)$
(c) $v_2(L) = 0$	(g) $\frac{dv_1}{dx}(0) = 0$	(k) $\frac{dv_1}{dx}(L) = \frac{dv_2}{dx}(L)$
(d) $v_1(2L) = 0$	(h) $\frac{dv_2}{dx}(3L) = 0$	(l) $\frac{dv_1}{dx}(2L) = \frac{dv_2}{dx}(2L)$

7.14 In Table P7.13, v_1 and v_2 represents the deflection in segment AB and BC. For the beam shown in Figure P7.14, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

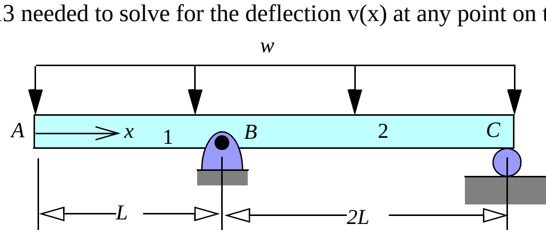


Figure P7.14

7.15 In Table P7.13, v_1 and v_2 represents the deflection in segment *AB* and *BC*. For the beam shown in Figure P7.15, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

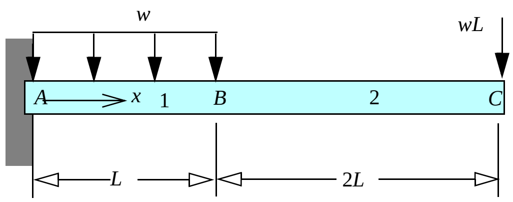


Figure P7.15

7.16 In Table P7.13, v_1 and v_2 represents the deflection in segment *AB* and *BC*. For the beam shown in Figure P7.16, identify all the conditions from Table P7.13 needed to solve for the deflection v(x) at any point on the beam.

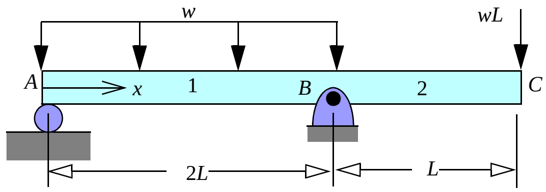


Figure P7.16

7.17 For the beam and loading shown in Figure P7.17, determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

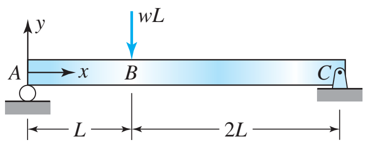


Figure P7.17

7.18 For the beam and loading shown in Figure P7.18, determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

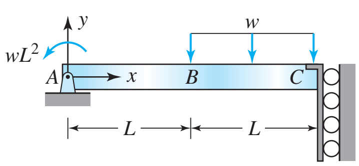


Figure P7.18

7.19 For the beam and loading shown in Figure P7.19 determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

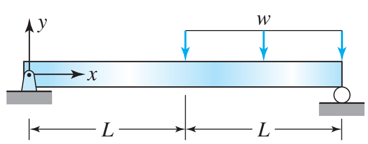


Figure P7.19

7.20 For the beam and loading shown in Figure P7.20 determine in terms of *w*, *L*, *E*, and *I* (a) the equation of the elastic curve; (b) the deflection at $x = L$.

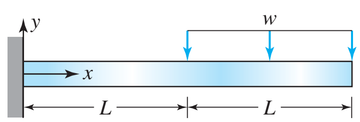


Figure P7.20

7.21 A simply supported beam in Figure P7.21 is acted upon by a distributed bending moment *m* per unit length. Determine (a) the elastic curve in terms of *m*, *E*, *I*, *L*, and *x*; (b) deflection at $x = L$.

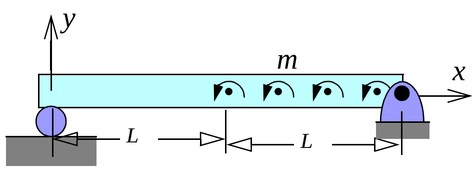


Figure P7.21

7.22 A diver weighing 200 lb stands at the edge of the diving board as shown in Figure 7.22. The diving board cross section is 16 in.x 1 in. and has a modulus of elasticity of 1500 ksi. Determine the maximum deflection in the diving board.

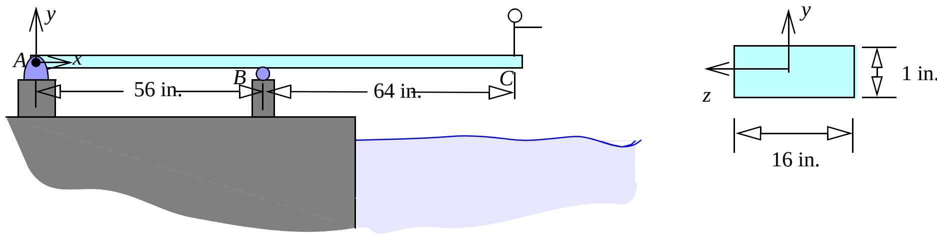


Figure P7.22

7.23 For the beam and loading shown in Figure P7.23, write the boundary-value problem for determining the deflection of the beam at any point x. Assume EI is constant. Do not integrate or solve.

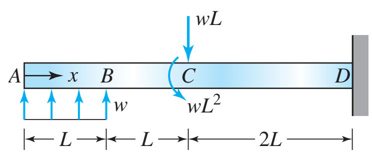


Figure P7.23

7.24 For the beam shown in Figure P7.24, write the boundary-value problem for determining the deflection of the beam at any point x. Assume EI is constant. Do not integrate or solve.

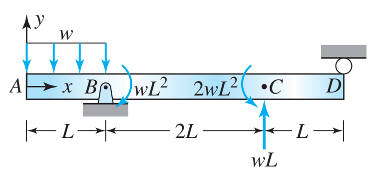


Figure P7.24

Variable area moment of inertia

7.25 A cantilever beam with variable depth $h(x)$ and constant width b is shown in Figure P7.25. The beam is to have a constant strength σ. In terms of b, L, E, x, and σ, determine (a) the variation of $h(x)$; (b) the maximum deflection.

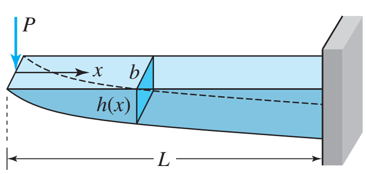


Figure P7.25

7.26 A cantilever tapered circular beam with variable radius $R(x)$ is shown in Figure P7.26.The beam is to have a constant strength σ. In terms of L, E, x, and σ, determine (a) the variation of $R(x)$; (b) the maximum deflection.

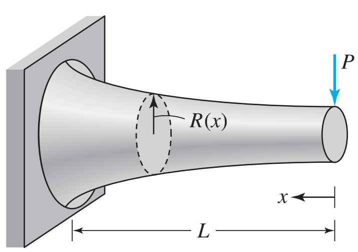


Figure P7.26

7.27 For the tapered beam shown in Figure P7.27, determine the maximum bending normal stress and the maximum deflection in terms of E, w, b, h_0, and L.

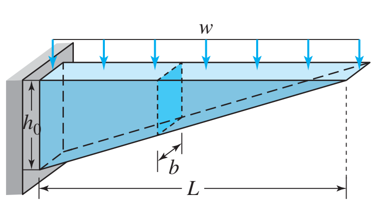


Figure P7.27

7.28 For the tapered circular beam shown in Figure P7.28, determine the maximum bending normal stress and the maximum deflection in terms of E, P, d_0, and L.

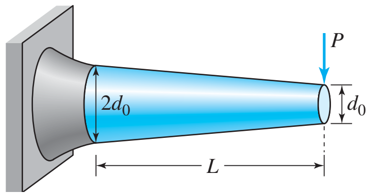


Figure P7.28

7.29 The 2-in. × 8-in. wooden beam of rectangular cross section shown in Figure P7.29 is braced at the support using 2-in. × 1-in. wooden pieces. The modulus of elasticity of wood is 2000 ksi. Determine the maximum bending normal stress and the maximum deflection.

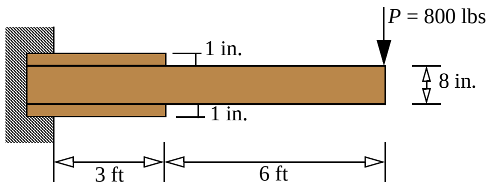


Figure P7.29

7.30 A 2 in. x 8 in. wooden rectangular cross-section beam is braced the near the load point using 2 in. x 1 in. wooden pieces as shown in Figure P7.30. The load is applied at the mid point of the beam. The modulus of elasticity of wood is 2,000 ksi. Determine the maximum bending normal stress and the maximum deflection. (Hint: Use symmetry about mid point to reduce calculations)

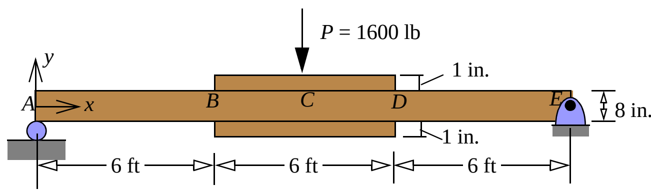


Figure P7.30

Stretch Yourself

7.31 To reduce weight of a metal beam the flanges are made of steel $E = 200$ GPa and the web of aluminum $E = 70$ GPa as shown in Figure P7.31. Determine the maximum deflection of the beam.

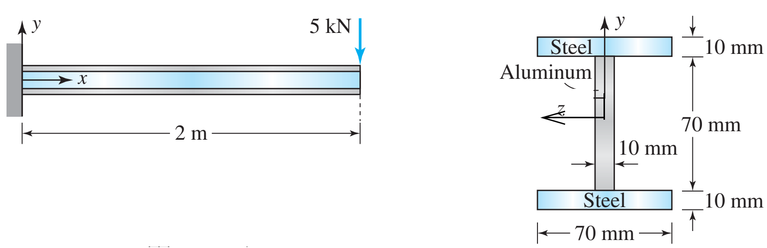


Figure P7.31

7.2 FOURTH-ORDER BOUNDARY-VALUE PROBLEM

We were able to solve for the deflection of a beam in Section 7.1 using second-order differential equations because we could find $\boldsymbol{M}_z$ as a function of x. In statically indeterminate beams, the internal moment determined from static equilibrium will contain some unknown reactions in the moment expression. Also, if the distributed load p_y is not uniform or linear but a more complicated function, then finding the internal moment $\boldsymbol{M}_z$ as a function of x may be difficult. In either case it may be preferable to start from an alternative equation. We can substitute Equation (7.1) into Equation (6.17), (that is, into $d\boldsymbol{M}_z/dx = -\boldsymbol{V}_y$) and then substitute the result into Equation (E6.18), (that is, $d\boldsymbol{V}_y/dx = -p_y$) to obtain

$$\boldsymbol{V}_y = -\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) \tag{7.4}$$

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = p_y \tag{7.5}$$

If the bending rigidity EI_{zz} is constant, then it can be taken outside the differentiation. However, if the beam is tapered, then I_{zz} is a function of x, and the form given in Equations (7.4) and (7.5) must be used.

7.2.3 Boundary Conditions

The deflection v(x) can be obtained by integrating Equation (7.5), but the fourth-order differential equation will generate four integration constants. To determine these constants, four boundary conditions are needed. The integration of Equation (7.5) will yield V_y of Equation (7.4), which on integration would yield M_z of Equation (7.1), which on integration would in turn yield v and dv/dx. Thus boundary conditions could be imposed on any of the four quantities v, dv/dx, M_z, and V_y.

To understand how these conditions are determined, we generalize a principle discussed in statics for determining the reaction force and/or moments. Recall how we determine reaction forces and moments at the supports in drawing free-body diagrams:

- If a point cannot move in a given direction, then a reaction force opposite to the direction acts at that support point.
- If a line cannot rotate about an axis in a given direction, then a reaction moment opposite to the direction acts at that support.

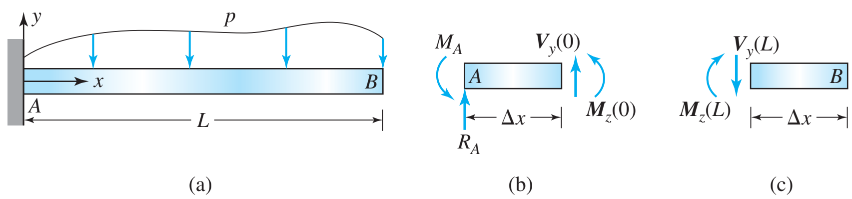


Figure 7.15 Example demonstrating grouping of boundary conditions.

Consider the cantilever beam with an arbitrarily varying distributed load shown in Figure 7.15*a*. We make an imaginary cut very close to the support at A (at an infinitesimal distance Δx) and draw the free-body diagram as shown in Figure 7.15*b*. The internal shear force and the internal moment are drawn according to our sign convention. Notice that the distributed force is not shown because as Δx goes to zero, the contribution of the distributed force will drop out from the equilibrium equations. By equilibrium we obtain $V_y(0) = -R_A$ and $M_z(0) = -M_A$. Thus if a point cannot move—that is, the deflection v is zero at a point—then the shear force is not known, because the reaction force is not known. Similarly if a line cannot rotate around an axis passing through a point, $dv/dx = 0$, and the internal moment is not known because the reaction moment is not known.

The reverse is equally true. Consider the free-body diagram constructed after making an imaginary cut at an infinitesimal distance from end B, as shown Figure 7.15*c*. By equilibrium we obtain $V_y(L) = 0$ and $M_z(L) = 0$. However, the free end can deflect and rotate by any amount dictated by the loading. Thus, when we specify a value of shear force, then we cannot specify displacement. And when we specify a value of internal moment at a point, then we cannot specify rotation. We can thus place the quantities v, dv/dx, M_z, and V_y:

- *Group 1:* At a boundary point either the deflection v can be specified or the internal shear force V_y can be specified, but not both.
- *Group 2:* At a boundary point either the slope dv/dx can be specified or the internal bending moment M_z can be specified, but not both.

Two conditions are specified at each end of the beam, generating four boundary conditions. One condition is chosen from each group. Stated succinctly, the boundary conditions at each end of the beam are

- *Group 1:* v or V_y

and (7.6)

- *Group 2:* $\frac{dv}{dx}$ or M_z

From Figure 7.15*c* we concluded that the shear force and the bending moment at the free end were zero. This conclusion can be reached by inspection without drawing a free-body diagram. If at the end there were a point force or a point moment, then clearly the magnitude of the shear force would equal the point force, and the magnitude of the internal moment would equal the point moment. Again, we can reach this conclusion without drawing a free-body diagram. But to get the correct sign of V_y and M_z we need a free-body diagram, with the internal quantities drawn according to our sign convention. We address the issue in Section 7.2.5.

7.2.4 Continuity and Jump Conditions

Suppose there is a point force or a point moment at x_j, or that the distributed force is given by different functions on the left and right of x_j. Then, again, the displacement will be represented by different functions on the left and right of x_j.

Thus we have two fourth-order differential equations, and their integration constants will require eight conditions:

- Four conditions are the boundary conditions discussed in Section 7.2.3.
- Two additional conditions are the continuity conditions at x_j discussed in Section 7.1.2.
- The remaining two conditions are the equilibrium equations on V_y and M_z at x_j.

The equilibrium conditions on V_y and M_z at x_j are jump conditions due to a point force or a point moment to be discussed in the next section.

7.2.5 Use of Template in Boundary Conditions or Jump Conditions

We discussed the concept of templates in drawing shear–moment diagrams in Section 6.4.2. Here we discuss it in determining the boundary conditions on V_y and M_z and jumps in these internal quantities due to a point force or a point moment.

Recall that a template is a small segment of a beam on which a point moment M_j and a point force F_j are drawn (Δx tends to zero in Figure 7.16). F_j and M_j could be applied or reactive forces and moments and their directions are arbitrary. The ends at $+\Delta x$ and $-\Delta x$ represent the imaginary cut just to the left and just to the right of the point forces and point moments. The internal shear force and the internal bending moment on these imaginary cuts are drawn according to our sign convention, as discussed in Section 6.2.6. Writing the equilibrium equations for this $2\Delta x$ segment of the beam, we obtain the template equations,

$$V_2(x_j) - V_1(x_j) = -F_j \tag{7.7.a}$$

$$M_2(x_j) - M_1(x_j) = M_j \tag{7.7.b}$$

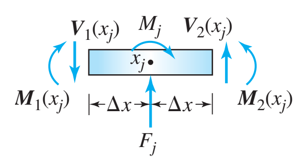


Figure 7.16 Template at x_j.

The moment equation does not contain the moment due to the forces because these moments will go to zero as Δx goes to zero.

If the point force on the beam is in the direction of F_j shown on the template, the template equation for force is used as given. If the point force on the beam is opposite to the direction of F_j shown on the template, then the template equation is used by changing the sign of F_j. The template equation for the moment is used in a similar fashion.

- If x_j is a *left* boundary point, then there is no beam left of x_j. Hence V_1 and M_1 are zero and we obtain the boundary conditions on V_y and M_z from V_2 and M_2.
- If x_j is a *right* boundary point, then there is no beam right of x_j. Hence V_2 and M_2 are zero and we obtain the boundary conditions on V_y and M_z from V_1 and M_1.
- If x_j is in between the ends of the beam, then the jump in shear force and internal moment is calculated using the template equations.

An alternative to the templates is to draw free-body diagrams after making imaginary cuts at an infinitesimal distance from the point force and writing equilibrium equations. Example 7.5 demonstrates the use of the template. Examples 7.6 and 7.7 demonstrate the use of free-body diagrams to determine the boundary conditions or the jump in internal quantities.

EXAMPLE 7.5

The bending rigidity of the beam shown in Figure 7.17 is 135 (10^6) lbs·in.^{2}, and the displacements of the beam in segments *AB* (v_1) and *BC* (v_2) are as given below. Determine (a) the reactions at the left wall at *A*; (b) the reaction force at *B* and the applied moment M_B.

$$v_1 = 5(x^3 - 20x^2)10^{-6} \text{ in.} \qquad 0 \le x \le 20 \text{ in.}$$

$$v_2 = 10(x^3 - 30x^2 + 200x)10^{-6} \text{ in.} \qquad 20 \text{ in.} \le x \le 40 \text{ in.}$$

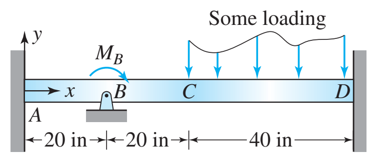


Figure 7.17 Beam in Example 7.5.

PLAN

By differentiating the given displacement functions and using Equations (7.1) and (7.4), we can find the bending moment M_z and the shear force V_y in segments *AB* and *BC*. (a) Using the template in Figure 7.16, we can find the reactions at *A* from the values of V_y and M_z at $x = 0$. (b) Using the template in Figure 7.16, we can find the reaction force and the applied moment at *B* from the values of V_y and M_z before and after $x = 20$ in.

SOLUTION

The shear force calculation requires the third derivative of the displacement functions. The functions v_1 and v_2 can be differentiated three times:

$$\frac{dv_1}{dx} = 5(3x^2 - 40x)(10^{-6}) \tag{E1}$$

$$\frac{d^2v_1}{dx^2} = 5(6x - 40)(10^{-6}) \text{ in.}^{-1} \tag{E2}$$

$$\frac{d^3v_1}{dx^3} = (5)(6)(10^{-6}) = 30(10^{-6}) \text{ in.}^{-2} \tag{E3}$$

$$\frac{dv_2}{dx} = 10(3x^2 - 60x + 200)(10^{-6}) \tag{E4}$$

$$\frac{d^2v_2}{dx^2} = 10(6x - 60)(10^{-6}) \text{ in.}^{-1} \tag{E5}$$

$$\frac{d^3v_2}{dx^3} = (10)(6)(10^{-6}) = 60(10^{-6}) \text{ in.}^{-2} \tag{E6}$$

From Equations (7.1), (E2), and (E5), the internal moment is

$$M_{z_1} = EI_{zz}\frac{d^2v_1}{dx^2} = [135(10^6) \text{ lbs.in.}^2][5(6x - 40)(10^{-6}) \text{ in.}^{-1}] = 675(6x - 40) \text{ in.}\cdot\text{lbs} \tag{E7}$$

$$M_{z_2} = EI_{zz}\frac{d^2v_2}{dx^2} = [135(10^6) \text{ lbs.in.}^2][10(6x - 60)(10^{-6}) \text{ in.}^{-1}] = 1350(6x - 60) \text{ in.}\cdot\text{lbs} \tag{E8}$$

From Equations (7.4), (E3), and (E6), the shear force is

$$V_{y_1} = EI_{zz}\frac{d^3v_1}{dx^3} = [135(10^6) \text{ lbs.in.}^2][30(10^{-6}) \text{ in.}^{-2}] = 4050 \text{ lbs} \tag{E9}$$

$$V_{y_2} = EI_{zz}\frac{d^3v_2}{dx^3} = [135(10^6) \text{ lbs.in.}^2][60(10^{-6}) \text{ in.}^{-2}] = 8100 \text{ lbs} \tag{E10}$$

The internal moment and shear force at *A* can be found by substituting $x = 0$ into Equations (E7) and (E9),

$$M_{z_1}(0) = 675(-40) = -27{,}000 \text{ in.}\cdot\text{lbs} \qquad V_{y_1}(0) = 4050 \; lbs \tag{E11}$$

The internal moment and shear force just before and after *B* can be found by substituting $x = 20$ into Equations (E7) through (E10),

$$M_{z_1}(20) = 675[(6)(20) - 40] = 54{,}000 \text{ in.}\cdot\text{lbs} \qquad M_{z_2}(20) = 1350[(6)(20) - 60] = 81{,}000 \text{ in.}\cdot\text{lbs} \tag{E12}$$

$$V_{y_1}(20) = 4050 \text{ lbs} \qquad V_{y_2}(20) = 8100 \text{ lbs} \tag{E13}$$

Figure 7.18 shows the free-body diagram of the entire beam. It also shows the template of Figure 7.16 for convenience.

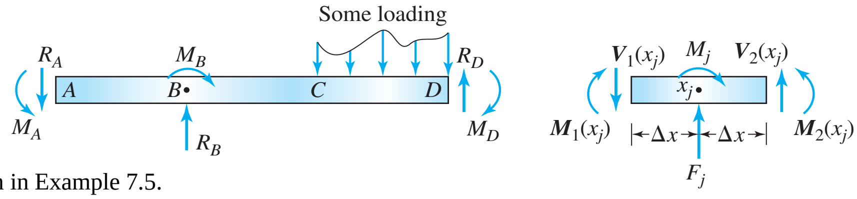


Figure 7.18 Free-body diagram of entire beam in Example 7.5.

If we compare the reaction force at A to F_j and the reaction moment to M_j in Figure 7.16, we obtain $F_j = -R_A$ and $M_j = -M_A$. As point A is the left end of the beam, $V_1(x_j)$ and $M_1(x_j)$ are zero on the template, and $V_2(x_j) = V_{y_1}(0)$ and $M_2(x_j) = M_{z_1}(0)$. From the template equation we obtain $R_A = V_{y_1}(0)$ and $M_A = -M_{z_1}(0)$. Substituting Equation (E11), we obtain

ANS. $R_A = 4050 \text{ lbs} \qquad M_A = 27{,}000 \text{ in.}\cdot\text{lbs}$

If we compare the reaction force at B to F_j and the applied moment to M_j in Figure 7.16, we obtain $F_j = R_B$ and $M_j = M_B$. Substituting for $x_j = 20$ in. and using Equations (E12) through (E13), we obtain R_B and M_B,

$$R_B = V_1(20) - V_{y_2}(20) = 4050 \text{ lbs} - 8100 \text{ lbs} = -4500 \text{ lbs} \tag{E14}$$

$$M_B = M_{z_2}(20) - M_{z_1}(20) = 81{,}000 \text{ in.}\cdot\text{lbs} - 54{,}000 \text{ in.}\cdot\text{lbs} = 27{,}000 \text{ in.}\cdot\text{lbs} \tag{E15}$$

ANS. $R_B = -4500 \text{ lbs} \qquad M_B = 27{,}000 \text{ in.}\cdot\text{lbs}$

COMMENTS

1. An alternative to the use of the template is to draw a free-body diagram after making imaginary cuts at an infinitesimal distance from the point forces, as shown in Figure 7.19. The internal forces and moments must be drawn according to our sign convention. By writing equilibrium equations the required quantities can be found.

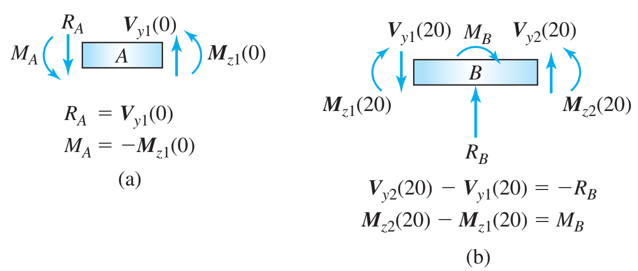


Figure 7.19 Alternative to template.

2. The free-body diagram of the entire beam in Figure 7.18 is not necessary. From the template equations, the force F_j and the moment M_j with the correct signs can be found. If F_j and M_j are positive, then R_B and M_B will be in the direction shown on the template. If these quantities are negative, then the direction is opposite.
3. This problem demonstrates how we (i) determine the conditions on shear force and bending moment and (ii) relating these internal quantities to the reaction forces and moments. The same basic principles apply when the displacement functions have to be determined first, as we see next.

EXAMPLE 7.6

In terms of E, I, w, L, and x, determine (a) the elastic curve; (b) the reaction force at A in Figure 7.20.

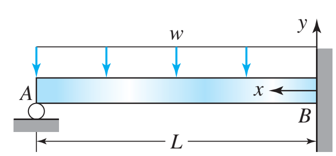


Figure 7.20 Beam and loading in Example 7.6.

METHOD 1 PLAN: FOURTH-ORDER DIFFERENTIAL EQUATION

(a) Noting that the distributed force is in the negative y direction, we can substitute $p_y = -w$ in Equation (7.5) and write the fourth-order differential equation. The two boundary conditions at A are zero deflection and zero moment, and the two boundary conditions at B are zero deflection and zero slope. We can solve the boundary-value problem and obtain the elastic curve. (b) We can draw a free-body diagram after making an imaginary cut just to the right of A and relate the reaction force to the shear force. We can find the shear force at point A by substituting $x = L$ in the solution obtained in part (a).

SOLUTION

(a) The boundary-value problem statement can be written below following the description in the Plan.

- **Differential equation:**

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -w \tag{E1}$$

- **Boundary conditions:**

$$v(0) = 0 \tag{E2}$$

$$\frac{dv}{dx}(0) = 0 \tag{E3}$$

$$v(L) = 0 \tag{E4}$$

$$EI_{zz}\frac{d^2v}{dx^2}(L) = 0 \tag{E5}$$

Integrating Equation (E1) twice,

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -wx + c_1 \tag{E6}$$

$$EI_{zz}\frac{d^2v}{dx^2} = -\frac{wx^2}{2} + c_1x + c_2 \tag{E7}$$

Substituting Equation (E7) into Equation (E5), we obtain

$$c_1L + c_2 = \frac{wL^2}{2} \tag{E8}$$

Integrating Equation (E7), we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{wx^3}{6} + c_1\frac{x^2}{2} + c_2x + c_3 \tag{E9}$$

Substituting Equation (E9) into Equation (E3), we obtain

$$c_3 = 0 \tag{E10}$$

Substituting Equation (E10) and integrating Equation (E9), we obtain

$$EI_{zz}v = -\frac{wx^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_4 \tag{E11}$$

Substituting Equation (E11) into Equation (E2), we obtain

$$c_4 = 0 \tag{E12}$$

Substituting Equations (E12) and (E11) into Equation (E4), we obtain

$$\frac{c_1L^3}{6} + \frac{c_2L^2}{2} = \frac{wL^4}{24} \tag{E13}$$

Solving Equations (E8) and (E13) simultaneously, we obtain

$$c_1 = \frac{5wL}{8} \qquad c_2 = -\frac{wL^2}{8} \tag{E14}$$

Substituting Equations (E12) and (E14) into Equation (E11) and simplifying, we obtain the elastic curve,

$$\text{ANS.} \qquad v(x) = -\frac{w}{48EI_{zz}}(2x^4 - 5Lx^3 + 3L^2x^2) \tag{E15}$$

Dimension check: Note that all terms in parentheses on the right-hand side of Equation (E15) have the dimension of length to the power of 4, or $O(L^4)$. Thus Equation (E15) is dimensionally homogeneous. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension:

$$w \to O\left(\frac{F}{L}\right) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{wx^4}{EI_{zz}} \to O\left(\frac{(F/L)L^4}{(F/L^2)O(L^4)}\right) \to O(L) \to \text{checks}$$

(b) We make an imaginary cut just to the right of point A (at an infinitesimal distance) and draw the free-body diagram of the left part using the sign convention in Section 6.2.6, as shown in Figure 7.21. By force equilibrium in the y direction, we can relate the shear force at A to the reaction force at A,

$$R_A = \boldsymbol{V}_A = \boldsymbol{V}_y(L) \tag{E16}$$

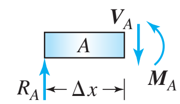


Figure 7.21 Infinitesimal equilibrium element at A in Example 7.6.

From Equations (7.4), (E6), and (E14), the shear force is

$$\boldsymbol{V}_y(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\left(EI_{zz}\frac{\mathrm{d}^2\mathrm{v}}{\mathrm{d}x^2}\right) = wx - \frac{5wL}{8} \tag{E17}$$

Substituting Equation (E17) into Equation (E16), we obtain the reaction at A.

ANS. $R_A = \dfrac{3wL}{8}$

METHOD 2 PLAN: Second-order differential equation

We can make an imaginary cut at some arbitrary location x and use the left part to draw the free-body diagram. The moment expression will contain the reaction force at A as an unknown. The second-order differential equation, Equation (7.1), would generate two integration constants, leading to a total of three unknowns. We need three conditions: the displacement at A is zero, and the displacement and slope at B are both zero. Solving the boundary-value problem, we can obtain the elastic curve and the unknown reaction force at A.

SOLUTION

We make an imaginary cut at a distance x from the right wall and take the left part of length $L - x$ to draw the free-body diagram using the sign convention for internal quantities discussed in Section 6.2.6 as shown in Figure 7.22.

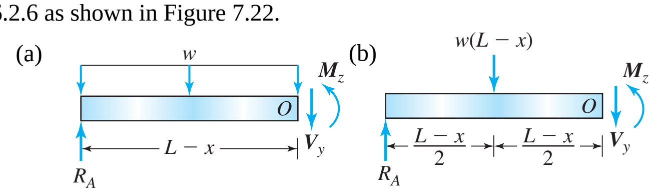


Figure 7.22 Free-body diagram in Example 7.6.

Balancing the moment at point O, we obtain the moment expression,

$$\boldsymbol{M}_z - R_A(L-x) + w\frac{(L-x)^2}{2} = 0 \qquad \text{or} \qquad \boldsymbol{M}_z = R_A(L-x) - \frac{w}{2}(L^2 + x^2 - 2Lx) \tag{E1}$$

Substituting into Equation (7.1) and writing the boundary conditions, we obtain the following boundary-value problem:

- **Differential equation:**

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} = R_A(L-x) - \frac{w}{2}(L^2 + x^2 - 2Lx) \tag{E2}$$

- **Boundary conditions:**

$$\mathrm{v}(0) = 0 \tag{E3}$$

$$\frac{d\mathrm{v}}{dx}(0) = 0 \tag{E4}$$

$$\mathrm{v}(L) = 0 \tag{E5}$$

Integrating Equation (E2), we obtain

$$EI_{zz}\frac{\mathrm{dv}}{\mathrm{d}x} = R_A\left(Lx - \frac{x^2}{2}\right) - \frac{w}{2}\left(L^2x + \frac{x^3}{3} - Lx^2\right) + c_1 \tag{E6}$$

Substituting Equation (E6) into Equation (E4), we obtain

$$c_1 = 0 \tag{E7}$$

Substituting Equation (E7) and integrating Equation (E6), we obtain

$$EI_{zz}\mathrm{v} = R_A\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) - \frac{w}{2}\left(\frac{L^2x^2}{2} + \frac{x^4}{12} - \frac{Lx^3}{3}\right) + c_2 \tag{E8}$$

Substituting Equation (E8) into Equation (E3), we obtain

$$c_2 = 0 \tag{E9}$$

Substituting Equations (E8) and (E9) into Equation (E5), we obtain

$$R_A\left(\frac{L^3}{2}-\frac{L^3}{6}\right)-\frac{w}{2}\left(\frac{L^4}{2}+\frac{L^4}{12}-\frac{L^4}{3}\right) = 0 \quad \text{or} \quad \text{ANS.} \quad R_A = \frac{3wL}{8} \tag{E10}$$

Substituting Equations (E9) and (E10) into Equation (E8) and simplifying, we obtain v(x).

$$\textbf{ANS.} \qquad v(x) = -\frac{w}{48EI_{zz}}(2x^4 - 5Lx^3 + 3L^2x^2) \tag{E11}$$

COMMENTS

1. Method 2 has less algebra than Method 1 and should be used whenever possible.
2. Suppose that in drawing the free-body diagram for calculating the internal moment, we had taken the right-hand part. Then we would have two unknowns rather than one—the wall reaction force and moment in the expression for moment. In such a case we would have to eliminate one of the unknowns using the static equilibrium equation for the entire beam. In other words, in statically indeterminate problems, the internal moment should contain a number of unknown reactions equal to the degree of static redundancy.
3. The moment boundary condition given by in Method 1 is implicitly satisfied. We can confirm this by substituting $x = L$ in Equation (E1).

EXAMPLE 7.7

A light pole is subjected to a wind pressure that varies as a quadratic function, as shown in Figure 7.23. In terms of E, I, w, L, and x, determine (a) the deflection at the top of the pole; (b) the ground reactions.

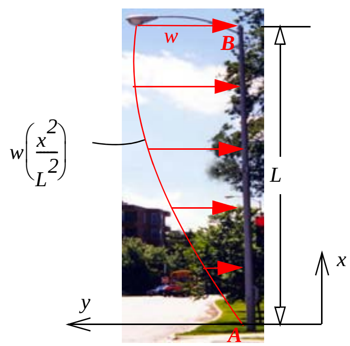


Figure 7.23 Beam and loading in Example 7.7.

PLAN

(a) Finding the moment as a function of x by static equilibrium is difficult for this statically determinate problem. We can use the fourth-order differential equation, Equation (7.5). We have four boundary conditions: the deflection and slope at A are zero, and the moment and shear force at B are zero. We can then solve the boundary-value problem and determine the elastic curve. By substituting $x = L$ in the elastic curve equation, we can obtain the deflection at the top of the pole. (b) By making an imaginary cut just above point A, we can relate the internal shear force and the internal moment at point A to the reactions at A. By substituting $x = 0$ in the moment and shear force expressions, we can obtain the shear force and moment values at point A.

SOLUTION

The boundary-value problem below can be written as described in the Plan.

- **Differential equation:**

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -w\left(\frac{x^2}{L^2}\right) \tag{E1}$$

- **Boundary conditions:**

$$v(0) = 0 \tag{E2}$$

$$\frac{dv}{dx}(0) = 0 \tag{E3}$$

$$EI_{zz}\frac{d^2v}{dx^2}\bigg|_{x=L} = 0 \tag{E4}$$

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right)\bigg|_{x=L} = 0 \tag{E5}$$

Integrating Equation (E1), we obtain

$$\frac{d}{dx}\left(EI_{zz}\frac{d^2v}{dx^2}\right) = -\frac{wx^3}{3L^2} + c_1 \tag{E6}$$

Substituting Equation (E6) into Equation (E5), we obtain

$$c_1 = \frac{wL}{3} \tag{E7}$$

Substituting Equation (E7) into Equation (E6) and integrating, we obtain

$$EI_{zz}\frac{d^2v}{dx^2} = -\frac{wx^4}{12L^2} + \frac{wL}{3}x + c_2 \tag{E8}$$

Substituting Equation (E8) into Equation (E4), we obtain

$$c_2 = -\frac{wL^2}{4} \tag{E9}$$

Substituting Equation (E9) into Equation (E8) and integrating, we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{wx^5}{60L^2} + \frac{wLx^2}{6} - \frac{wL^2x}{4} + c_3 \tag{E10}$$

Substituting Equation (E10) into Equation (E3), we obtain

$$c_3 = 0 \tag{E11}$$

Substituting Equation (E11) into Equation (E10) and integrating, we obtain

$$EI_{zz}\frac{dv}{dx} = -\frac{wx^6}{360L^2} + \frac{wLx^3}{18} - \frac{wL^2x^2}{8} + c_4 \tag{E12}$$

Substituting Equation (E12) into Equation (E2), we obtain

$$c_4 = 0 \tag{E13}$$

Substituting Equation (E13) into Equation (E12) and simplifying, we obtain

$$v(x) = -\frac{w}{360EI_{zz}L^2}(x^6 - 20L^3x^3 + 45L^4x^2) \tag{E14}$$

Dimension check: Note that all terms in parentheses on the right-hand side of Equation (E14) have the dimension of length to the power of 6, or, $O(L^6)$. Thus Equation (E14) is dimensionally homogeneous. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension:

$$w \to O\left(\frac{F}{L}\right) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{wx^6}{EI_{zz}L^2} \to O\left(\frac{(F/L)L^6}{(F/L^2)L^4L^2}\right) \to O(L) \to \text{checks}$$

(a) Substituting $x = L$ into , we obtain the deflection at the top of the pole.

ANS. $v(L) = -\frac{13wL^4}{180EI_{zz}}$

(b) We make an imaginary cut just above point A $(\Delta x \to 0)$ and take the bottom part to draw the free-body diagram shown in Figure 7.24. By equilibrium of forces and moments, we can relate the reaction force R_A and the reaction moment M_A to the internal shear force and the internal bending moment at point A,

$$\boldsymbol{M}_z(0) = -M_A \qquad \boldsymbol{V}_y(0) = -R_A \tag{E15}$$

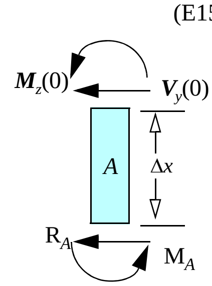


Figure 7.24 Free body diagram of an infinitesimal element at A in Example 7.7.

Substituting Equations (E7) and (E6) into Equation (7.4) and Equations (E9) and (E8) into Equation (7.1), we can obtain the shear force and bending moment expressions,

$$\boldsymbol{M}_z(x) = -\frac{wx^4}{12L^2} + \frac{wL}{3}x - \frac{wL^2}{4} \tag{E16}$$

$$V_y(x) = \frac{wx^3}{3L^2} - \frac{wL}{3} \tag{E17}$$

Substituting Equations (E16) and (E17) into Equation (E15), we obtain the reaction force and the reaction moment.

ANS. $R_A = \frac{wL}{3}$ $\quad M_A = \frac{wL^2}{4}$

COMMENTS

1. The directions of R_A and M_A can be checked by inspection, as these are the directions necessary for equilibrium of the externally distributed force.
2. The free-body diagram in Figure 7.24, the reaction force R_A and the reaction moment M_A can be drawn in any direction, but the internal quantities V_y and M_z must be drawn according to the sign convention in Section 6.2.6. Irrespective of the direction in which R_A and M_A are drawn, the final answer will be as given. The sign in the equilibrium equations, Equation (E15), will account for the assumed directions of the reactions.

PROBLEM SET 7.2

Fourth-order boundary-value problems

7.32 The displacement in the *y* direction in segment *AB*, shown in Figure P7.32, was found to be $v(x) = (20x^3 - 40x^2)10^{-6}$ in. If the bending rigidity is 135×10^6 lb·in.^{2}, determine the reaction force and the reaction moment at the wall at *A*.

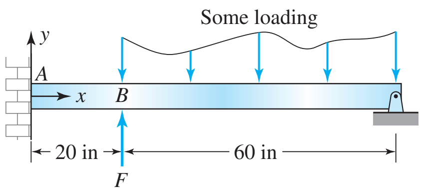


Figure P7.32

7.33 In Figure P7.33, the displacement in the *y* direction in section *AB*, is given by $v_1(x) = -3(x^4 - 20x^3)(10^{-6})$ in. and in *BC* by $v_2(x) = -8(x^2 - 100x + 1600)(10^{-3})$ in. If the bending rigidity is 135×10^6 lb·in.^{2}, determine: (a) the reaction force at *B* and the applied moment M_B; (b) the reactions at the wall at *A*.

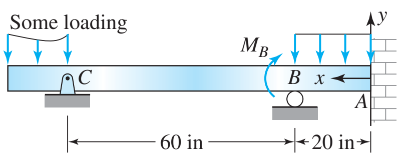


Figure P7.33

7.34 For the beam shown in Figure P7.34, determine the elastic curve and the reaction(s) at *A* in terms of *E*, *I*, *P*, *w*, and *x*.

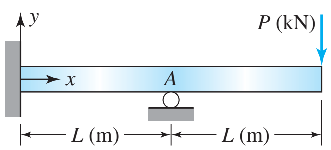


Figure P7.34

7.35 For the beam shown in Figure P7.35, determine the elastic curve and the reaction(s) at *A* in terms of *E*, *I*, *P*, *w*, and *x*.

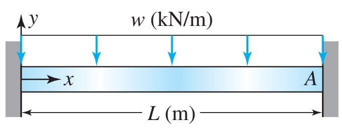


Figure P7.35

7.36 For the beam shown in Figure P7.36, determine the slope at $x = L$ and the reaction moment at the left wall in terms of E, I, w, and L.

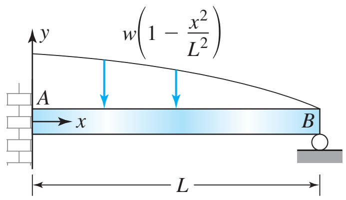


Figure P7.36

7.37 For the beam shown in Figure P7.37, determine the deflection and the moment reaction at $x = L$ in terms of E, I, w, and L.

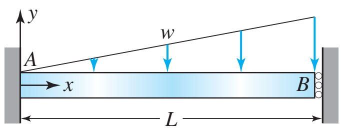


Figure P7.37

7.38 For the beam shown in Figure P7.38, determine the deflection and the slope at $x = L$ in terms of E, I, w, and L.

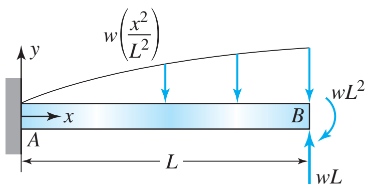


Figure P7.38

7.39 For the beam and loading shown in Figure P7.39, determine the deflection and slope at $x = L$ in terms of E, I, w, and L.

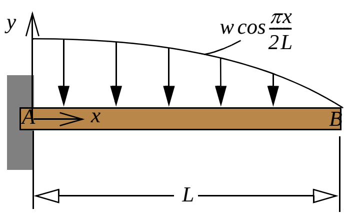


Figure P7.39

7.40 For the beam and loading shown in Figure P7.40, determine the slope at x = L and the reaction moment at the left wall in terms of E, I, w, and L.

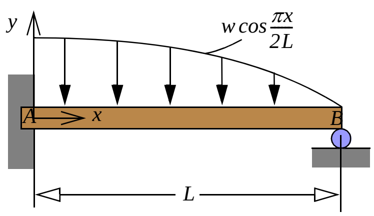


Figure P7.40

7.41 For the beam and loading shown in Figure P7.41, determine the maximum deflection in terms of E, I, w, and L.

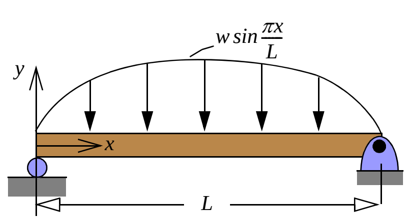


Figure P7.41

7.42 For the beam and loading shown in Figure P7.42, determine the maximum deflection in terms of E, I, w, and L.

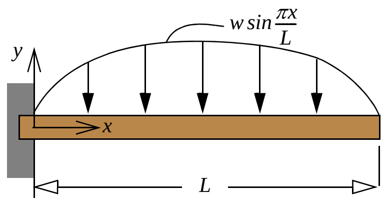


Figure P7.42

7.43 A cantilever beam under uniform load has a spring with a stiffness k attached to it at point A, as shown in Figure P7.43. The spring constant in terms of stiffness of the beam is written as $k = \alpha EI/L^3$, where α is a proportionality factor. Determine the compression of the spring in terms of α, w, E, I, and L.

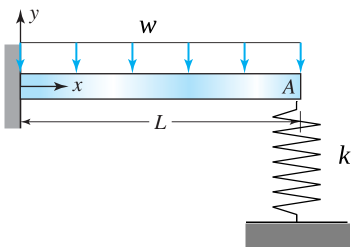


Figure P7.43

7.44 A linear spring that has a spring constant K is attached at the end of a beam, as shown in Figure P7.44. In terms of w, E, I, L, and K, write the boundary-value problem but do not integrate or solve.

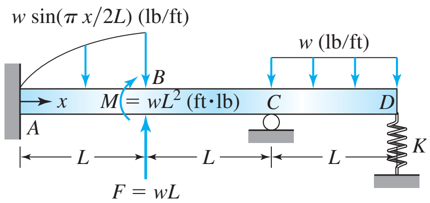


Figure P7.44

Historical problems

7.45 The beam and loading shown in Figure P7.45 was the first statically indeterminate beam for which a solution was obtained by Navier. Verify that Navier's solution for the reaction at A is given by the equation below.

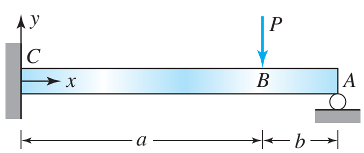


Figure P7.45

$$R_A = \frac{Pa^2(3L-a)}{2L^3} \qquad \text{where } L = a + b$$

7.46 Jacob Bernoulli incorrectly assumed that the neutral axis was tangent to the concave side of the curve in Figure P7.46 and obtained the equation given below. In the equation R is the radius of curvature of the beam at any location x. Derive this equation based on Bernoulli's assumption and show that it is incorrect by a factor of 4. (*Hint:* Follow the process in Section 6.1 and take the moment about point B.)

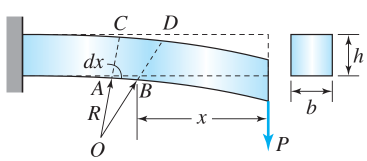


Figure P7.46

$$\frac{Ebh^3}{3}\left(\frac{1}{R}\right) = Px$$

7.47 Clebsch considered a beam loaded by several concentrated forces P_i placed at a location x_i, as shown in Figure P7.47. He obtained the second-order differential equation between the concentrated forces. By integration he obtained the slope and deflection as given and concluded that all C_i's are equal and all D_i's are equal. Show that his conclusion is correct. For $x_i \le x \le x_{i+1}$,

$$EI\frac{d^2v}{dx^2} = Rx - \sum_{i=1}^{i} P_j(x - x_j) \qquad EI\frac{dv}{dx} = R\frac{x^2}{2} - \sum_{i=1}^{i} P_i\frac{(x-x_i)^2}{2} + C_i \qquad EIv = R\frac{x^3}{6} - \sum_{i=1}^{i} P_i\frac{(x-x_i)^3}{6} + C_i x + D_i$$

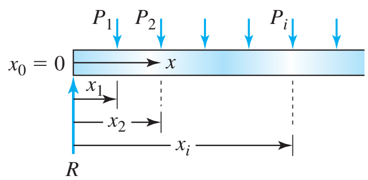
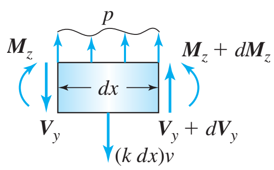


Figure P7.47

Stretch Yourself

7.48 A beam resting on an elastic foundation has a distributed spring force that depends on the deflections at a point acting as shown in Figure P7.48. Show that the differential equation governing the deflection of the beam is given by Equation (7.8), where k is the foundation modulus, that is, spring constant per unit length.



$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2v}{dx^2}\right) + kv = p \tag{7.8}$$

Figure P7.48 Elastic foundation effect.

7.49 To account for shear, the assumption of planes remaining perpendicular to the axis of the beam (Assumption 3 in Section 6.2) is dropped, and it is assumed that the plane rotates by the angle ψ from the vertical. This yields the following displacement equations:

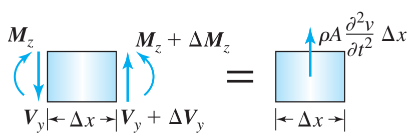
$$u = -y\psi(x) \qquad v = v(x)$$

The rest of the derivation[1] is as before. Show that the following equations apply:

$$\frac{d}{dx}\left[GA\left(\frac{dv}{dx} - \psi\right)\right] = -p \qquad \frac{d}{dx}\left(EI_{zz}\frac{d\psi}{dx}\right) = -GA\left(\frac{dv}{dx} - \psi\right) \tag{7.9}$$

where A is the cross-sectional area and G is the shear modulus of elasticity. Beams governed by these equations are called *Timoshenko beams*.

7.50 Figure P7.50 shows a differential element of a beam that is free to vibrate, where ρ is the material density, A is the cross-sectional area, and $\partial^2 v/\partial t^2$ is the linear acceleration. Show that the dynamic equilibrium is given by Equation (7.10).



$$\frac{\partial^2 v}{\partial t^2} + c^2\frac{\partial^4 v}{\partial x^4} = 0 \qquad \text{where } c = \sqrt{EI_{zz}/\rho A}. \tag{7.10}$$

Figure P7.50 Dynamic equilibrium.

7.51 Show by substitution that the following solution satisfies Equation (7.10):

$$v(x,t) = G(x)H(t) \qquad G(x) = A\cos\omega x + B\sin\omega x + C\cosh\omega x + D\sinh\omega x \qquad H(t) = E\cos(c\omega^2)t + D\sin(c\omega^2)t$$

7.52 Show by substitution that the following deflection solution satisfies the fourth order boundary value problem of the cantilever beam shown in Figure P7.52.

$$v(x) = \frac{1}{6EI}\left[R_A x^3 + 3M_A x^2 + \int_0^x (x-x_1)^3 p(x_1)\,dx_1\right] \qquad \text{where } R_A = -\int_0^L p(x_1)\,dx_1 \text{ and } M_A = \int_0^L x_1 p(x_1)\,dx_1. \tag{7.11}$$

[1]Use Equations (2.12a) and (2.12d) to get ε_{xx} and γ_{xy}. Use Hooke's law, the static equivalency equations [Equations (6.1) and (6.13)], and the equilibrium equations [Equations (6.17) and (6.18)].

Computer problems

7.53 Table P7.53 shows the value of distributed load at several point along the axis of a 10 ft long rectangular beam. Determine the slope and deflection at the free end using. Use modulus of elasticity as 2000 ksi.

TABLE P7.53 **Data in Problem 7.53**

x (ft)	$p(x)$ (lb/ft)	x (ft)	$p(x)$ (lb/ft)
0	275	6	377
1	348	7	316
2	398	8	233
3	426	9	128
4	432	10	0
5	416		

7.54 For the beam and loading given in Problem 7.53, determine the slope and deflection at the free end in the following manner. First represent the distributed load by $p(x) = a + bx + cx^2$ and, using the data in Table P7.53, determine constants *a*, *b*, and *c* by the least-squares method. Then using fourth-order differential equations solve the boundary-value problem. Use the modulus of elasticity as 2000 ksi.

7.55 Table P7.55 shows the measured radii of a solid tapered circular beam at several points along the axis, as shown in Figure P7.55. The beam is made of aluminum (E = 28 GPa) and has a length of 1.5 m. Determine the slope and deflection at point *B*.

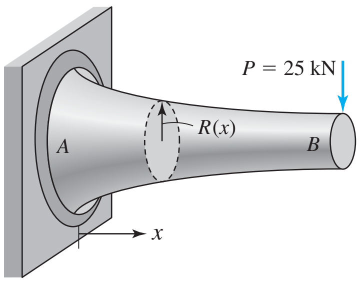


Figure P7.55

TABLE P7.55 **Data for Problem 7.55**

x (m)	$R(x)$ (mm)	x (m)	$R(x)$ (mm)
0.0	100.6	0.8	60.1
0.1	92.7	0.9	60.3
0.2	82.6	1.0	59.1
0.3	79.6	1.1	54.0
0.4	75.9	1.2	54.8
0.5	68.8	1.3	54.1
0.6	68.0	1.4	49.4
0.7	65.9	1.5	50.6

7.56 Let the radius of the tapered beam in Problem 7.55 be represented by the equation $R(x) = a + bx$. Using the data in Table P7.55, determine constants *a* and *b* by the least-squares method and then find the slope and deflection at point *B* by analytical integration.

MoM in Action: Skyscrapers

A skyscraper can be a monument to the builder’s pride or a *literal* monument, designed to attract tourists and tenants to the city’s, the country’s, or the world’s tallest building. When John Roscoe, in 1930, wanted a taller building than Walter Chrysler’s, he pushed for his own, completed just one year after the Chrysler building More than 40 years later, his Empire State Building (Figure 7.25*a*). was still the world’s tallest building, at 1250 ft. Even today, it is surely the most famous skyscraper ever. New construction is also driven by the same social forces as those behind the boom in Chicago, New York, and London at the end of 19th century. Businesses then and now want to be near a city’s commercial center and emerging economies are seeing a movement of the population from villages to cities. As this edition goes to press, the tallest building is Taipei 101 in Taiwan (Figure 7.25*b*). Built in 2003, it stands 1671 feet tall.

Figure 7.25 (a) Empire State in New York. (b) Taipei 101 in Taipei, Taiwan. (c) Joint construction.

Social forces, then, have pushed skyscrapers higher and higher, but technological advances have made that possible. Early high-rise buildings had a pyramid design: the building cross-section decreased with height to avoid excessive stresses at the bottom. The height of these building was limited by the strength of masonry materials and by the difficulty of getting water to higher stories. Besides, renters did not want to climb too many stairs! With the advent of steel beams, reinforced concrete, glass, electric water pumps, and elevators, however, the human imagination was freed to build tall. If one thinks of a high-rise building as an axial column, then skyscrapers are like cantilevered beams subject to bending loads in the wind. A proper variation of both axial and bending rigidity with height is important in design. Skyscrapers must be strong enough to withstand hurricane winds in excess of 140 mph. With an increase in height, too, the bearing stresses at the base increase, often requiring digging deep to bedrock.

In addition to the stresses, the deflection of a skyscraper increases with height. By welding and bolting the horizontal girders to steel columns, the rigidity of the joint (Figure 7.25*c*) is increased, which helps reduce the sway. Skyscraper designs often have columns on the outer perimeter, which are connected to the central core columns. The outer columns act like flanges to resist most of the wind load, while the inner columns carry most of the weight. In modern skyscrapers, computer-controlled masses of hundreds of tons, called *tuned mass dampers*, move to counter the building sway. Today skyscrapers are also designed to move *with* earthquakes rather then stress the building frames.

The terrorist attack on World Trade Center (see page 525) has highlighted the need for better fireproofing of steel beams, and technology is once more providing the solution. But terrorist acts do not deter the human spirit, which like the skyscrapers themselves still soars. In the sands of United Arab Emirates, the next tallest skyscraper is rising. Called *Burj Dubai*, or the *Dubai tower*, it will be twice the height of the Empire State Building.

7.3* SUPERPOSITION

The assumptions and limitations that were imposed in deriving the simplest theory for beam bending ensured that we have a linear theory. As a consequence, the differential equations governing beam deflection, Equations (7.1) and (7.5), are linear differential equations, and hence the principle of superposition can be applied to beam deflection.

The leftmost beam in Figure 7.26 is loaded with a uniformly distributed load w and a concentrated load P_1. The superposition principle says that the deflection of a beam with uniform load w and point force P_1 is equal to the sum of the deflections calculated by considering each load separately, as shown on the right two beams in Figure 7.26. Although the example in Figure 7.26 demonstrates the principle of superposition, there is no intrinsic gain in calculating the deflection of each load separately and adding to find the final answer. But if the solutions to basic cases are tabulated, as in Table C.3, then the principle of superposition becomes a very useful tool to obtain results quickly. Thus the maximum deflection of the beam on the left can be found using the results of cases 1 and 3 in Table C.3. Comparing the loading of the two beams in Figure 7.26 to those shown for cases 1 and 3, we note that $P = -P_1 = -wL$ and $p_0 = -w$, $a = L$, and $b = 0$. Substituting these values into v_{max} given in Table C.3 and adding, we obtain

$$v_{max} = -\frac{(wL)L^3}{3EI} - \frac{wL^4}{8EI} = -\frac{11wL^4}{24EI} \tag{7.12.a}$$

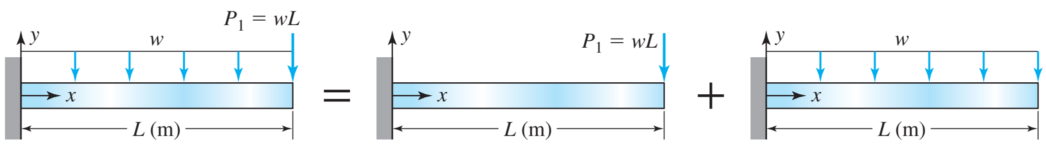


Figure 7.26 Example of superposition principle.

Another very useful application of superposition is the deflection of statically indeterminate beams. Consider a beam built in at one end and simply supported at the other end with a uniformly distributed load, as shown in Figure 7.27. The support at A can be replaced by a reaction force, and once more the total loading can be shown as the sum of two individual loads, as shown at right in Figure 7.27. Comparing the loading of the two beams in Figure 7.26 to those shown for cases 1 and 3 in Table C.3, we note that $P = R_A$, $p_0 = -w$, $a = L$, and $b = 0$. v_{max} is at point A in both cases. Substituting these values into v_{max} given in Table C.3 and adding, we obtain

$$v_A = \frac{R_A L^3}{3EI} - \frac{wL^4}{8EI} \tag{7.12.b}$$

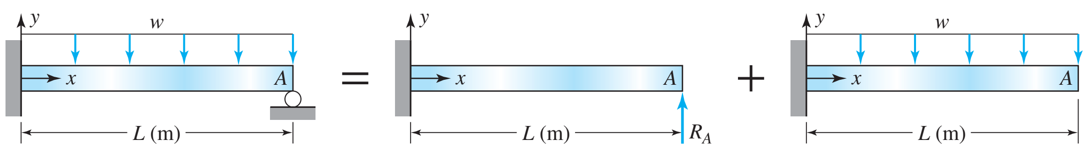


Figure 7.27 Example of use of superposition principle in solving statically indeterminate beam deflection.

But the deflection at A must be zero in the original beam. Thus we can solve for the reaction force as $R_A = 3wL/8EI$. Now the solution of $v(x)$ given in Table C.3 can be superposed to obtain

$$v(x) = \frac{R_A x^2}{6EI}(3L - x) + \frac{(-w)x^2}{24EI}(x^2 - 4Lx + 6L^2) \tag{7.12.c}$$

Substituting for R_A and simplifying, the solution for the elastic curve is

$$v(x) = \frac{wx^2(-2x^2 + 5Lx - 3L^2)}{48EI} \tag{7.12.d}$$

EXAMPLE 7.8

For the beam shown in Figure 7.28, using the principle of superposition and Table C.3, determine (a) the reactions at *A*; (b) the maximum deflection.

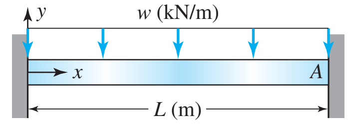


Figure 7.28 Beam in Example 7.8.

PLAN

(a) The wall at *A* can be replaced by a force reaction R_A and a moment reaction M_A. Thus the beam would be a cantilever beam with a uniformly distributed load, a point force at the end, and a point moment at the end, corresponding to the first three cases in Table C.3. Superposing the slope and deflection values from Table C.3 and equating the result to zero would generate two equations in the two unknowns R_A and M_A which give the reactions at *A*. (b) From the symmetry of the problem, we can conclude that the maximum deflection will occur at the center. Substituting $x = L/2$ in the elastic curve equation of Table C.3 and adding the results, we can find the maximum deflection of the beam.

SOLUTION

(a) The right wall at *A* can be replaced by a reaction force and a reaction moment, as shown at left in Figure 7.29. The total loading on the beam can be considered as the sum of the three loadings shown at right in Figure 7.29.

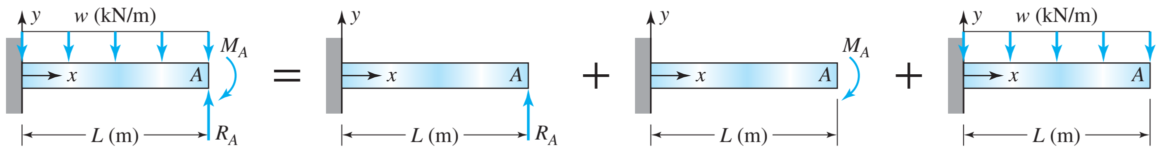


Figure 7.29 Superposition of three loadings in Example 7.8.

Comparing the three beam loadings in Figure 7.29 to that shown for cases 1 through 3 in Table C.3, we obtain $P = R_A$, $M = -M_A$, and $p = -w$, $a = L$, and $b = 0$. Noting that v_{max} and θ_{max} shown in Table C.3 for the cantilever beam occur at point *A*, we can substitute the load values and superpose to obtain the deflection v_A and the slope at θ_A. Noting that at the wall at *A* the deflection v_A and the slope at θ_A must be zero, we obtain two simultaneous equations in R_A and M_A,

$$v_A = \frac{R_A L^3}{3EI} + \frac{(-M_A)L^2}{2EI} + \frac{(-w)L^4}{8EI} = 0 \qquad \text{or} \qquad 8R_A L - 12M_A = 3wL^2 \tag{E1}$$

$$\theta_A = \frac{R_A L^2}{2EI} + \frac{(-M_A)L}{EI} + \frac{(-w)L^3}{6EI} = 0 \qquad \text{or} \qquad 3R_A L - 6M_A = wL^2 \tag{E2}$$

Equations (E1) and (E2) can be solved to obtain R_A and M_A.

$$\textbf{ANS.} \qquad R_A = \frac{wL}{2} \qquad M_A = \frac{wL^2}{12} \tag{E3}$$

(b) The maximum deflection would occur at the center of the beam. Substituting $x = L/2$, $P = R_A = wL/2$, $M = -M_A = -wL/12$, and $p = -w$ in the equation of the elastic curve for cases 1 through 3 in Table C.3 and superposing the solution, we obtain

$$v_{max} = v\left(\frac{L}{2}\right) = \frac{(wL/2)(L/2)^2}{6EI}\left(3L - \frac{L}{2}\right) + \frac{-(wL^2/12)(L/2)^2}{2EI} + \frac{(-w)(L/2)^2}{24EI}\left[\left(\frac{L}{2}\right)^2 - 4L\left(\frac{L}{2}\right) + 6L^2\right] \qquad \text{or} \tag{E4}$$

$$v_{max} = \frac{5wL^4}{96EI} - \frac{wL^4}{96EI} - \frac{17wL^4}{384EI} \tag{E5}$$

$$\textbf{ANS.} \qquad v_{max} = -\frac{wL^4}{384EI}$$

COMMENTS

1. All terms in Equations (E1) and (E2) have the same dimension, as they should. If this were not the case, then we would need to examine the equations obtained using superposition and the subsequent simplifications carefully to ensure dimensional homogeneity.
2. By symmetry we know that the reaction forces at each wall must be equal. Hence the value of the reaction forces should be $wL/2$, as calculated in Equation (E3).

EXAMPLE 7.9

The end of one cantilever beam rests on the end of another cantilever beam, as shown in Figure 7.30. Both beams have length L and bending rigidity EI. Determine the deflection at A and the wall reactions at B and C in terms of w, L, E, and I.

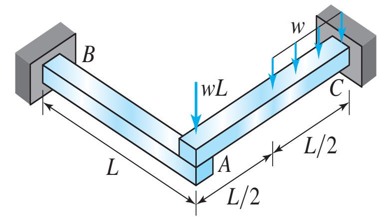


Figure 7.30 Two cantilever beams in Example 7.9.

PLAN

The two beams can be separated by putting an unknown force R_A that is equal but opposite in direction on each beam at point A. From case 1 in Table C.3 for beam AB, the deflection at A can be found in terms of R_A. From cases 1 and 3 for beam AC, the deflection at A can be found by superposition. By equating the deflection at A for the two beams, we can find the force R_A. (a) Once R_A is known, the deflection at A is found from the equation written for beam AB. (b) The reactions at B and C can be found using equilibrium equations on each beam's free-body diagram.

SOLUTION

(a) The assembly of the beams shown in Figure 7.30 can be represented by two beams with a force R_A that acts in equal but opposite directions, as shown in Figure 7.31*a* and *b*. The loading on the beam in Figure 7.31*b* can be represented as the sum of the two loadings shown in Figure 7.31*c* and *d*.

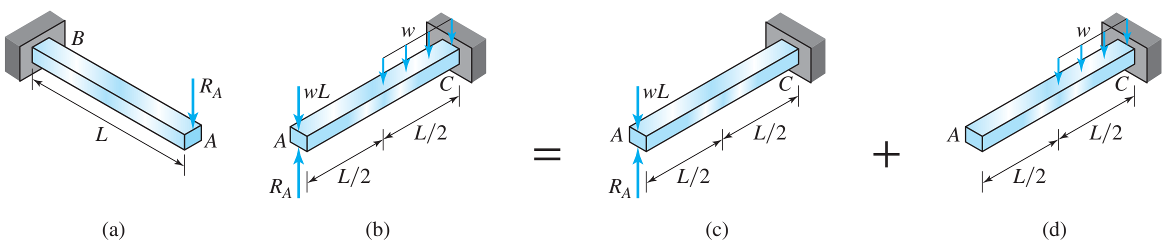


Figure 7.31 Analysis of beam assembly by superposition in Example 7.9.

Comparing the beam of Figure 7.31*a* to that shown in case 1 in Table C.3, we obtain $P = -R_A$, $a = L$, and $b = 0$. Noting that v_{max} in case 1 occurs at A, the deflection at A can be written as

$$\mathrm{v}_A = \frac{(-R_A)L^2}{6EI}2L = -\frac{R_A L^3}{3EI} \tag{E1}$$

Comparing the beam in Figure 7.31*c* to that of case 1 in Table C.3, we obtain $P = R_A - wL$, $a = L$, and $b = 0$. Comparing the beam in Figure 7.31*d* to that of case 3 in Table C.3, we obtain $p_0 = -w$, $a = L/2$, and $b = L/2$. Since v_{max} for both cases occurs at A, by superposition the deflection at A can be written as

$$\mathrm{v}_A = \frac{(R_A - wL)L^2}{6EI}2L + \frac{(-w)(L/2)^3(3L/2 + 4L/2)}{24EI} = \frac{(R_A - wL)L^3}{3EI} - \frac{7wL^4}{384EI} \tag{E2}$$

Equating Equations (E1) and (E2), give the reaction R_A:

$$-\frac{R_A L^3}{3EI} = \frac{(R_A - wL)L^3}{3EI} - \frac{7wL^4}{384EI} \qquad \text{or} \qquad R_A = \frac{135wL}{256} \tag{E3}$$

Substituting Equation (E3) into Equation (E1), we obtain the deflection at A.

ANS. $\mathrm{v}_A = -\dfrac{45wL^4}{256EI}$

(b) The reactions at the wall can be found from the free-body diagrams of each beam, as shown in Figure 7.32. By equilibrium of forces in the y direction and the moments about B in Figure 7.32*a*, the reactions at B can be found,

$$R_B = R_A \qquad M_B = R_A L \tag{E4}$$

ANS. $R_B = \dfrac{135wL}{256}$ $\qquad M_B = \dfrac{135wL^2}{256}$

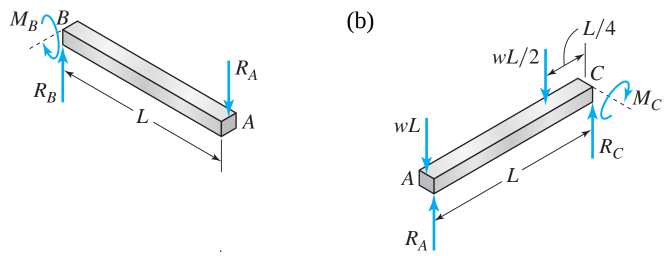


Figure 7.32 Free-body diagrams in Example 7.9.

By equilibrium of forces in the y direction and the moments about C in Figure 7.32b, the reactions at C can be found,

$$R_C = wL + \frac{wL}{2} - R_A = \frac{249wL}{256} \tag{E5}$$

$$M_C = wL(L) + \frac{wL}{2}\left(\frac{L}{4}\right) + R_A L = \frac{153wL^2}{256} \tag{E6}$$

ANS. $R_C = \dfrac{249wL}{256}$ $\quad M_C = \dfrac{153wL^2}{256}$

COMMENT

1. This example demonstrates how the principle of superposition can significantly simplify the analysis and design of structures. Handbooks now document an extensive number of cases for which beam deflections are known. These apply to a wide variety of beam assemblies. But to develop a list of formulas (as in Table C.3) requires a knowledge of the methods described in Sections 7.1 and 7.2.

7.4* DEFLECTION BY DISCONTINUITY FUNCTIONS

Thus far, we have used different functions to represent the distributed load p_y, or moment $\boldsymbol{M}_z$, for different parts of the beam. We then had to determine the integration constants that satisfy the continuity conditions and equilibrium conditions at the junctions x_j. These tedious and algebraically intensive tasks, may be unavoidable for a complicated distributed loading function. But for many engineering problems, where the distributed loads either are constant or vary linearly, there is an alternative method that avoids the algebraic tedium. The method is based on the concept of *discontinuity functions*.

7.4.1 Discontinuity Functions

Consider a distributed load p and an equivalent load $P = p\varepsilon$, as shown in Figure 7.33. Suppose we now let the intensity of the distributed load increase continuously to infinity. At the same time, we decrease the length over which the distributed force is applied to zero so that the area $p\varepsilon$ remains a finite quantity. We then obtain a concentrated force P applied at $x = a$. Mathematically,

$$P = \lim_{p \to \infty} \lim_{\varepsilon \to 0} (p\varepsilon)$$

Rather than write the limit operations, we can represent a concentrated force with, $P\langle x - a \rangle^{-1}$.

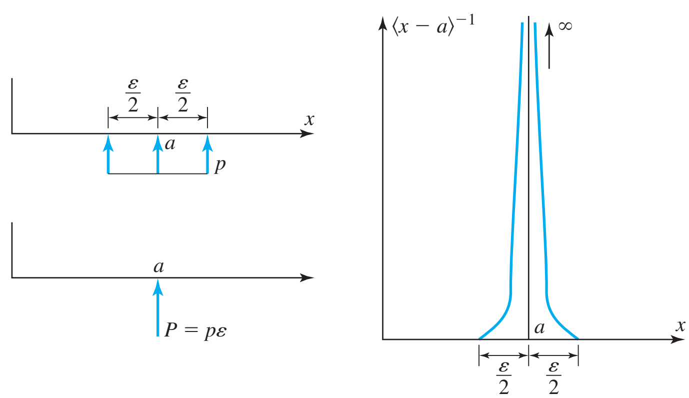


Figure 7.33 Delta function.

The function $\langle x-a \rangle^{-1}$ is called the *Dirac delta function*, or **delta function**. The delta function is zero except in an infinitesimal region near *a*. As *x* tends toward *a*, the delta function tends to infinity, but the area under the function is equal to 1. Mathematically, the delta function is defined as

$$\langle x-a \rangle^{-1} = \begin{Bmatrix} 0, & x \neq a \\ \infty, & x \to a \end{Bmatrix} \qquad \int_{a-\varepsilon}^{a+\varepsilon} \langle x-a \rangle^{-1}\, dx = 1 \tag{7.13}$$

Now consider the following integral of the delta function:

$$\int_{-\infty}^{x} \langle x-a \rangle^{-1}\, dx$$

The lower limit of minus infinity emphasizes that the point is before *a*. If $x < a$, then in the interval of integration, the delta function is zero at all points; hence the integral value is zero. If $x > a$, then the integral can be written as the sum of three integrals,

$$\int_{-\infty}^{a-\varepsilon} \langle x-a \rangle^{-1}\, dx + \int_{a-\varepsilon}^{a+\varepsilon} \langle x-a \rangle^{-1}\, dx + \int_{a+\varepsilon}^{x} \langle x-a \rangle^{-1}\, dx$$

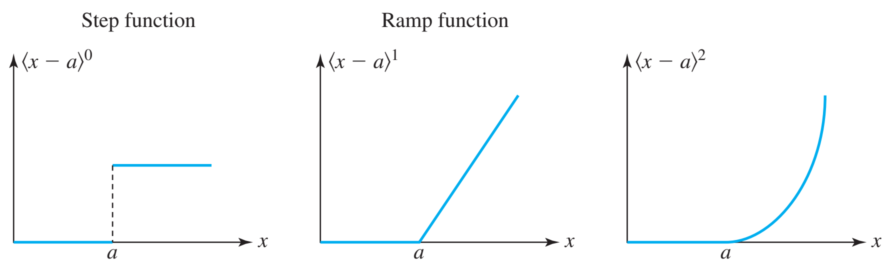


Figure 7.34 Discontinuity functions.

The first and third integrals are zero because the delta function is zero at all points in the interval of integration, whereas the second integral is equal to 1 as per Equation (7.13). Thus the integral $\int_{-\infty}^{x} \langle x-a \rangle^{-1}\, dx$ is zero before *a* and one after *a*. It is called the **step function** as shown in Figure 7.34 and is represented by the notation $\langle x-a \rangle^{0}$.

$$\langle x-a \rangle^{0} = \int_{-\infty}^{x} \langle x-a \rangle^{-1}\, dx = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases} \tag{7.14}$$

Now consider the integral of the step function,

$$\int_{-\infty}^{x} \langle x-a \rangle^{0}\, dx$$

If $x < a$, then in the interval of integration the step function is zero at all points. Hence the integral value is zero. If $x > a$, then we can write the integral as the sum of two integrals,

$$\int_{-\infty}^{a} \langle x-a \rangle^{0}\, dx + \int_{a}^{x} \langle x-a \rangle^{0}\, dx$$

The first integral is zero because the step function is zero at all points in the interval of integration, whereas the second integral value is $x - a$. The integral $\int_{-\infty}^{x} \langle x-a \rangle^{0}\, dx$ is called the **ramp function**. It is represented by the notation $\langle x-a \rangle^{1}$ and is shown in Figure 7.34. Proceeding in this manner we can define an entire class of functions, which are represented mathematically as follows:

$$\langle x-a \rangle^{n} = \begin{cases} 0, & x \le a \\ (x-a)^{n}, & x > a \end{cases} \tag{7.15}$$

We can also generate the following integral formula from Equation (7.15):

$$\int_{-\infty}^{x} \langle x-a \rangle^n \, dx = \frac{\langle x-a \rangle^{n+1}}{n+1}, \qquad n \geq 0 \tag{7.16}$$

We define one more function, called the **doublet function**. It is represented by the notation $\langle x-a \rangle^{-2}$ and is defined mathematically as

$$\langle x-a \rangle^{-2} = \begin{cases} 0, & x \neq a \\ \infty, & x \to a \end{cases} \qquad \int_{-\infty}^{x} \langle x-a \rangle^{-2} \, dx = \langle x-a \rangle^{-1} \tag{7.17}$$

The delta function $\langle x-a \rangle^{-1}$ and the doublet function $\langle x-a \rangle^{-2}$ become infinite at $x = a$, that is, they are singular at $x = a$ and are referred to as **singularity functions**. The entire class of functions $\langle x-a \rangle^n$ for positive and negative n are called **discontinuity functions**.

The discontinuity functions are zero if the argument is negative. By differentiating Equations (7.14), (7.16), and (7.17) we can obtain the following formulas:

$$\frac{d\langle x-a \rangle^{-1}}{dx} = \langle x-a \rangle^{-2} \qquad \frac{d\langle x-a \rangle^{0}}{dx} = \langle x-a \rangle^{-1} \tag{7.18}$$

$$\frac{d\langle x-a \rangle^{n}}{dx} = n\langle x-a \rangle^{n-1}, \qquad n \geq 1 \tag{7.19}$$

7.4.2 Use of Discontinuity Functions

Before proceeding to develop a method for solving for the elastic curve using discontinuity functions, we discuss the process by which the internal moment $\boldsymbol{M}_z$ and the distributed force p_y can be written using the discontinuity functions. We will develop the procedure using a simple example of a cantilever beam subject to different types of loading, as shown in Figure 7.35. Then we will generalize the procedure to more general loading and types of support.

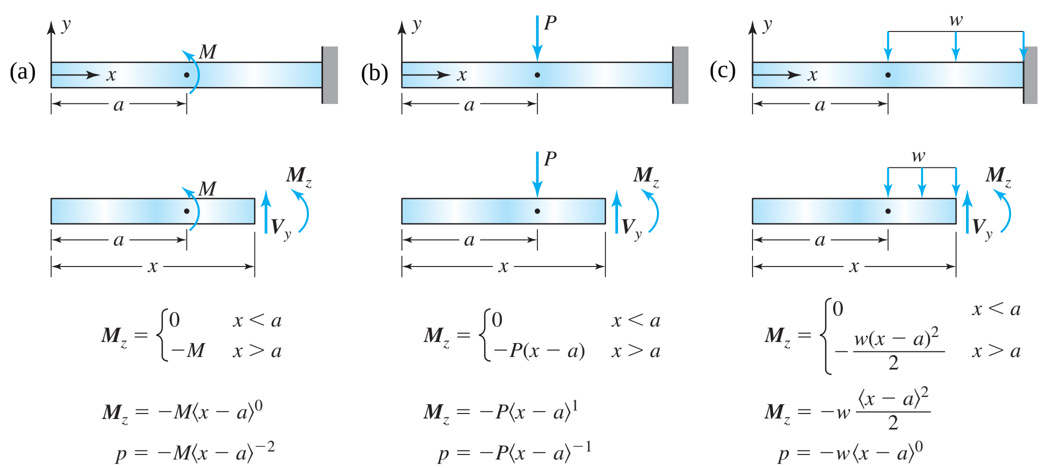


Figure 7.35 Use of discontinuity functions.

When we make an imaginary cut before $x = a$ in the cantilever beams shown in Figure 7.35, the internal moment $\boldsymbol{M}_z$ will be zero. If the imaginary cut is made after $x = a$, then the internal moment $\boldsymbol{M}_z$ will not be zero and can be determined using a free-body diagram. Once the moment expression is known, then it can be rewritten using the discontinuity functions. This moment expression can be used to find displacement using the second-order differential equation, Equation (7.1). However, if the fourth-order differential equation, Equation (7.5), has to be solved, then the expression of the distributed force p_y is needed. Now the distributed force can be obtained from the moment expression using the identity that is obtained by substituting Equation (6.18) into Equation (6.17), or $d^2\boldsymbol{M}_z/dx^2 = p_y$. By using Equation (7.19) we can obtain the distributed force expression from the moment expression, as shown in Figure 7.35.

If the distributed load is as shown in Figure 7.35*c*, then the expression for it can be obtained directly, without the free-body diagram, and the moment expression can be obtained by integrating twice. For the concentrated force and moment also, it is not difficult to recognize the type of discontinuity function that will be used in the representation. The difficulty lies in obtaining the correct sign in the expression for the internal moment $\boldsymbol{M}_z$. We shall overcome this problem by using a template to guide us.

A template is created by making an imaginary cut beyond the applied load. On the imaginary cut the internal moment is drawn according to the sign convention discussed in Section 6.2.6. A moment equilibrium equation is written. If the applied load is in the assumed direction on the template, then the sign used is the sign in the moment equilibrium equation. If the direction of the applied load is opposite to that on the template, then the sign in the equilibrium equation is changed. The beams shown in Figure 7.35 are like templates for the given coordinate systems.

EXAMPLE 7.10

Write the moment and distributed force expressions using discontinuity functions for the three templates shown in Figure 7.36.

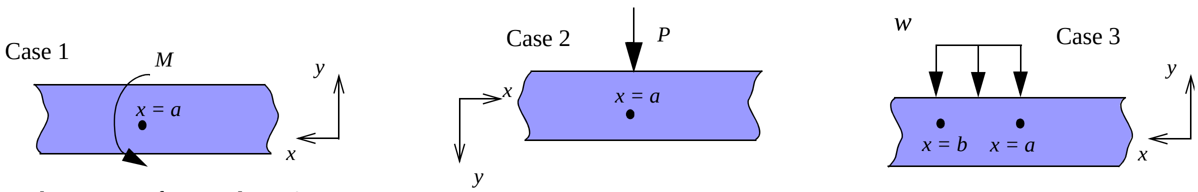


Figure 7.36 Three cases of Example 7.10.

PLAN

For cases 1 and 2 we can make an imaginary cut after $x = a$ and draw the shear force and bending moment according to the sign convention in Section 6.2.6. By equilibrium we can obtain the moment expression and rewrite it using discontinuity functions. By differentiating twice, we can obtain the distributed force expression. For case 3 we can write the expression for the distributed force using discontinuity functions and integrate twice to obtain the moment expression.

SOLUTION

Case 1: We make an imaginary cut at $x > a$ and draw the free-body diagram using the sign convention in Section 6.2.6 as shown in Figure 7.37*a*. By equilibrium we obtain

$$\boldsymbol{M}_z = M \tag{E7}$$

is valid only after $x > a$. Using the step function we can write the moment expression, and by differentiating twice as per Equation (7.19) we obtain our result.

ANS. $\boldsymbol{M}_z = M\langle x-a\rangle^0 \qquad p_y = M\langle x-a\rangle^{-2}$

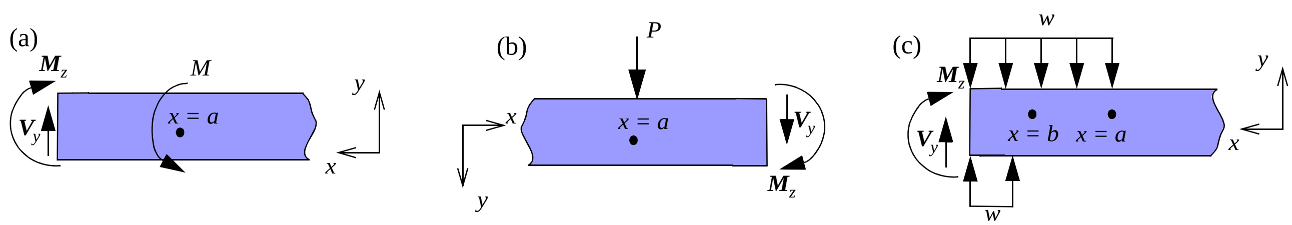


Figure 7.37 (a) Case 1, (b) Case 2, (c) Case 3 in Example 7.10.

Case 2: We make an imaginary cut at $x > a$ and draw the free-body diagram using the sign convention in Section 6.2.6 as shown in Figure 7.37*b*. By equilibrium we obtain

$$\boldsymbol{M}_z = P(x-a) \tag{E8}$$

is valid only after $x > a$. Using the ramp function we can write the moment expression, and by differentiating twice as per Equation (7.19) we obtain our result.

ANS. $\boldsymbol{M}_z = P\langle x-a\rangle^1 \qquad p_y = P\langle x-a\rangle^{-1}$

Case 3: The distributed force is in the negative y direction. Its start can be represented by the step function at $x = a$. The end of the distributed force can also be represented by a step function using a sign opposite to that used at the start as shown in Figure 7.37*b*.

ANS. $$p_y = -w\langle x-a\rangle^0 + w\langle x-b\rangle^0 \tag{E9}$$

Integrating Equation (E9) twice and using Equation (7.16), we obtain the moment expression

ANS. $$M_z = -\frac{w}{2}\langle x-a\rangle^2 + \frac{w}{2}\langle x-b\rangle^2 \qquad \text{(E10)}$$

COMMENTS

1. The three cases shown could be part of a beam with more complex loading. But the contribution for each of the loads would be calculated as shown in the example.
2. In obtaining Equation (E10) we did not yet write integration constants. When we integrate for displacements, we will determine these from boundary conditions.
3. In case 3 we did not have to draw the free-body diagram. This is an advantage when the distributed load changes character over the length of the beam. Even for statically determinate beams, it may be advantageous to start with the fourth-order, rather than the second-order differential equation.

EXAMPLE 7.11

Using discontinuity functions, determine the equation of the elastic curve in terms of E, I, L, P, and x for the beam shown in Figure 7.38.

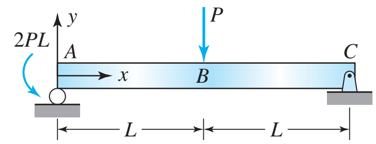


Figure 7.38 Beam and loading in Example 7.11.

PLAN

Two templates can be created, one for an applied moment and one for the applied force. With the templates as a guide, the moment expression in terms of discontinuity functions can be written. The second-order differential equation, Equation (7.1), can be written and solved using the zero deflection boundary conditions at A and C to obtain the elastic curve.

SOLUTION

Figure 7.39 shows two templates. By equilibrium, the moment expressions for the two templates can be written

$$M_z = M\langle x-a\rangle^0 \qquad M_z = F\langle x-a\rangle^1 \qquad \text{(E1)}$$

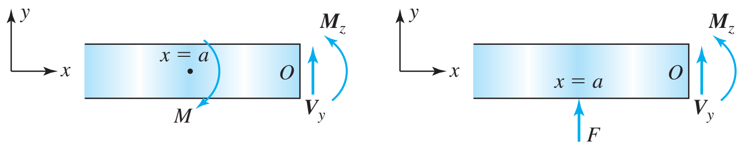


Figure 7.39 Templates for Example 7.11.

Figure 7.40 shows the free-body diagram of the beam. By equilibrium of moment at C, the reaction at A can be found as $R_A = 3P/2$. We can write the moment expressions using the templates in Figure 7.39 to guide us. The reaction force is in the same direction as the force in the template. Hence the term in Equation (E2) will have the same sign as shown in the template equation.

The applied moment at point A has an opposite direction to that shown in the template in Figure 7.39. Hence the term in the moment expression in Equation (E2) will have a negative sign to that shown in the template equation. The force P at B has an opposite sign to that shown on the template, and hence the term in the moment expression will have a negative sign, as shown in Equation (E2).

$$M_z = \frac{3P}{2}\langle x\rangle^1 - 2PL\langle x\rangle^0 - P\langle x-L\rangle^1 \qquad \text{(E2)}$$

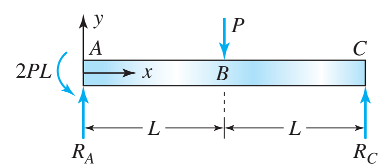


Figure 7.40 Free-body diagram in Example 7.11.

Substituting Equation (E2) into Equation (7.1) and writing the zero deflection conditions at A and C, we obtain the boundary-value problem:

- **Differential equation:**

$$EI_{zz}\frac{d^2 v}{dx^2} = \frac{3P}{2}\langle x\rangle^1 - 2PL\langle x\rangle^0 - P\langle x-L\rangle^1, \qquad 0 \le x < 2L \qquad \text{(E3)}$$

- **Boundary conditions:**

$$v(0) = 0 \tag{E4}$$

$$v(2L) = 0 \tag{E5}$$

Integrating Equation (E3) twice using Equation (7.16), we obtain

$$EI_{zz}\frac{dv}{dx} = \frac{3P}{4}\langle x\rangle^2 - 2PL\langle x\rangle^1 - \frac{P}{2}\langle x-L\rangle^2 + c_1 \tag{E6}$$

$$EI_{zz}v = \frac{P}{4}\langle x\rangle^3 - PL\langle x\rangle^2 - \frac{P}{6}\langle x-L\rangle^3 + c_1x + c_2 \tag{E7}$$

Substituting Equation (E7) into Equation (E4), we obtain the constant c_2:

$$\frac{P}{4}\langle 0\rangle^3 - PL\langle 0\rangle^2 - \frac{P}{6}\langle -L\rangle^3 + c_2 = 0 \qquad \text{or} \qquad c_2 = 0 \tag{E8}$$

Substituting Equation (E7) into Equation (E5), we obtain the constant c_1:

$$\frac{P}{4}\langle 2L\rangle^3 - PL\langle 2L\rangle^2 - \frac{P}{6}\langle L\rangle^3 + c_1(2L) = 0 \qquad \text{or} \qquad c_1 = \frac{13}{12}PL^2 \tag{E9}$$

Substituting Equations (E8) and (E9) into Equation (E7), we obtain the elastic curve.

$$\textbf{ANS.} \quad v(x) = \frac{P}{12EI_{zz}}[3\langle x\rangle^3 - 12L\langle x\rangle^2 - 2\langle x-L\rangle^3 + 13L^2x] \tag{E10}$$

Dimension check: All terms in brackets are dimensionally homogeneous as all have the dimensions of length cubed. But we can also check whether the left-hand side and any one term of the right-hand side have the same dimension,

$$P \to O(F) \qquad x \to O(L) \qquad E \to O\left(\frac{F}{L^2}\right) \qquad I_{zz} \to O(L^4) \qquad v \to O(L) \qquad \frac{Px^3}{EI_{zz}} \to O\left(\frac{FL^3}{(F/L^2)L^4}\right) \to O(L) \to \text{checks}$$

COMMENTS

1. Comparing the boundary-value problem in this example with that of Example 7.2, we note the following: (i) There is only one differential equation here representing the two differential equations of Example 7.2. (ii) There are no continuity equations at $x = L$ as there were in Example 7.2. The net impact of these two features is a significant reduction in the algebra in this example compared to the algebra in Example 7.2.
2. Equation (E10) represents the two equations of the elastic curve in Example 7.2. We note that $\langle x-L\rangle^3 = 0$ for $0 \le x < L$. Hence Equation (E10) can be written $v(x) = P(3x^3 - 12Lx^2 + 13L^2x)/12EI_{zz}$, which is same as Equation (E18) in Example 7.2. For $L \le x < 2L$, the term $\langle x-L\rangle^3 = (x-L)^3$. Hence Equation (E10) can be written $v(x) = P[3x^3 - 12Lx^2 + 13L^2x - 2(x-L)^3]/12EI_{zz}$, which is same as Equation (E19) in Example 7.2.

EXAMPLE 7.12

A beam with a bending rigidity $EI = 42{,}000\ \text{N}\cdot\text{m}^2$ is shown in Figure 7.41. Determine: (a) the deflection at point B; (b) the moment and shear force just before and after B.

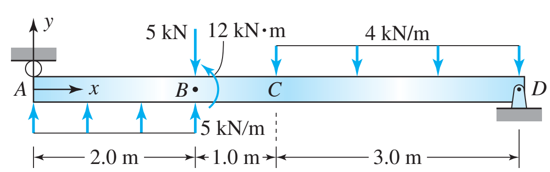


Figure 7.41 Beam and loading in Example 7.12.

PLAN

The coordinate system in this example is the same as in Example 7.11, and hence we can use the templates in Figure 7.39. Differentiating the template equations twice, we obtain the template equation for the distributed forces, and we write the distributed force expression in terms of discontinuity functions. Using Equation (7.5) and the boundary conditions at A and D, we can write the boundary-value problem and solve it to obtain the elastic curve. (a) Substituting $x = 2$ m in the elastic curve, we can obtain the deflection at B. (b) Substituting $x = 2.5$ in the shear force expression, we can obtain the shear force value.

SOLUTION

(a) The templates of Example 7.11 are repeated in Figure 7.42. The moment expression is differentiated twice to obtain the template equations for the distributed force,

$$\boldsymbol{M}_z = M\langle x-a\rangle^0 \qquad \boldsymbol{M}_z = F\langle x-a\rangle^1 \tag{E1}$$

$$p = M\langle x-a\rangle^{-2} \qquad p = F\langle x-a\rangle^{-1} \tag{E2}$$

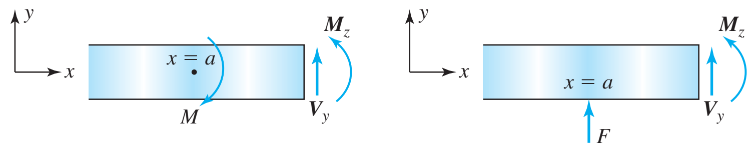


Figure 7.42 Templates for Example 7.12.

We note that the distributed force in segment *AB* is positive, starts at zero, and ends at $x = 2$. The distributed force in segment *CD* is negative, starts at $x = 3$, and is over the rest of the beam. Using the template equations and Figure 7.42, we can write the distributed force expression,

$$p = 5\langle x\rangle^0 - 5\langle x-2\rangle^0 - 4\langle x-3\rangle^0 - 5\langle x-2\rangle^{-1} - 12\langle x-2\rangle^{-2} \tag{E3}$$

Substituting Equation (E3) into Equation (7.5) and writing the boundary conditions, we obtain the boundary-value problem:

- **Differential equation:**

$$\frac{d^2}{dx^2}\left(EI_{zz}\frac{d^2\mathrm{v}}{dx^2}\right) = 5\langle x\rangle^0 - 5\langle x-2\rangle^0 - 4\langle x-3\rangle^0 - 5\langle x-2\rangle^{-1} - 12\langle x-2\rangle^{-2}, \qquad 0 \le x < 6 \tag{E4}$$

- **Boundary conditions:**

$$\mathrm{v}(0) = 0 \tag{E5}$$

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2}(0) = 0 \tag{E6}$$

$$\mathrm{v}(6) = 0 \tag{E7}$$

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2}(6) = 0 \tag{E8}$$

Integrating Equation (E4) twice, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(EI_{zz}\frac{d^2\mathrm{v}}{dx^2}\right) = 5\langle x\rangle^1 - 5\langle x-2\rangle^1 - 4\langle x-3\rangle^1 - 5\langle x-2\rangle^0 - 12\langle x-2\rangle^{-1} + c_1 \tag{E9}$$

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} = \frac{5}{2}\langle x\rangle^2 - \frac{5}{2}\langle x-2\rangle^2 - 2\langle x-3\rangle^2 - 5\langle x-2\rangle^1 - 12\langle x-2\rangle^0 + c_1 x + c_2 \tag{E10}$$

Substituting Equation (E10) into Equation (E5), we obtain

$$c_2 = 0 \tag{E11}$$

Substituting Equation (E10) into Equation (E8), we obtain

$$\frac{5}{2}\langle 6\rangle^2 - \frac{5}{2}\langle 4\rangle^2 - 2\langle 3\rangle^2 - 5\langle 4\rangle^1 - 12\langle 4\rangle^0 + c_1(6) = 0 \qquad \text{or} \qquad c_1 = 0 \tag{E12}$$

Substituting Equations (E11) and (E12) into Equation (E10) and integrating twice, we obtain

$$EI_{zz}\frac{d\mathrm{v}}{dx} = \frac{5}{6}\langle x\rangle^3 - \frac{5}{6}\langle x-2\rangle^3 - \frac{2}{3}\langle x-3\rangle^3 - \frac{5}{2}\langle x-2\rangle^2 - 12\langle x-2\rangle^1 + c_3 \tag{E13}$$

$$EI_{zz}\mathrm{v} = \frac{5}{24}\langle x\rangle^4 - \frac{5}{24}\langle x-2\rangle^4 - \frac{2}{12}\langle x-3\rangle^4 - \frac{5}{6}\langle x-2\rangle^3 - 6\langle x-2\rangle^2 + c_3 x + c_4 \tag{E14}$$

Substituting Equation (E14) into Equation (E5), we obtain

$$c_4 = 0 \tag{E15}$$

Substituting Equation (E14) into Equation (E6), we obtain

$$\frac{5}{24}\langle 6\rangle^4 - \frac{5}{24}\langle 4\rangle^4 - \frac{2}{12}\langle 3\rangle^4 - \frac{5}{6}\langle 4\rangle^3 - 6\langle 4\rangle^2 + c_3(6) = 0 \qquad \text{or} \qquad c_3 = -\frac{323}{36} = -8.97 \tag{E16}$$

Substituting Equations (E15) and (E16) into Equation (E14) and simplifying, we obtain the elastic curve,

$$\mathrm{v} = \frac{1}{72EI_{zz}}[15\langle x\rangle^4 - 15\langle x-2\rangle^4 - 12\langle x-3\rangle^4 - 60\langle x-2\rangle^3 - 432\langle x-2\rangle^2 - 646x] \tag{E17}$$

Substituting $x = 2$ into Equation (E17), we obtain the deflection at point *B*,

$$\mathrm{v}(2) = \frac{1}{72[(42)(10^3)]}[15\langle 2\rangle^4 - 15\langle 0\rangle^4 - 12\langle -1\rangle^4 - 60\langle 0\rangle^3 - 432\langle 0\rangle^2 - 646(6)] \tag{E18}$$

ANS. $\mathrm{v}(2) = -1.2$ mm

(b) As stated in Equation (7.1), the moment M_z can be found from Equation (E10). And as seen in Equation (7.4), the shear force V_y is the negative of the expression given in Equation (E11). Noting that the constants c_1 and c_2 are zero, we obtain the expressions for M_z and V_y:

$$M_z(x) = \left[\frac{5}{2}\langle x\rangle^2 - \frac{5}{2}\langle x-2\rangle^2 - 2\langle x-3\rangle^2 - 5\langle x-2\rangle^1 - 12\langle x-2\rangle^0\right] \text{ kN}\cdot\text{m} \tag{E19}$$

$$V_y(x) = [-5\langle x\rangle^1 + 5\langle x-2\rangle^1 + 4\langle x-3\rangle^1 + 5\langle x-2\rangle^0 + 12\langle x-2\rangle^{-1}] \text{ kN} \tag{E20}$$

Point B is at $x = 2$. Just after point B, that is, at $x = 2–$, all terms except the first term in Equations (E19) and (E20) are zero.

ANS. $M_z(2-) = 10 \text{ kN}\cdot\text{m} \qquad V_y(2-) = -10 \text{ kN}$

Just after point B, that is, at $x = 2+$, the step function $\langle x-2\rangle^0$ is equal to 1. Hence this term along with the first term are the nonzero terms in Equations (E19) and (E20).

$$M_z(2+) = (10-12) = -2 \text{ kN}\cdot\text{m} \qquad V_y(2+) = (-10+5) = -5 \text{ kN} \tag{E21}$$

ANS. $M_z(2+) = -2 \text{ kN}\cdot\text{m} \qquad V_y(2+) = -5 \text{ kN}$

COMMENT

1. We note that $M_z(2+) - M_z(2-) = -12 \text{ kN}\cdot\text{m}$ and $V_y(2+) - V_y(2-) = 5 \text{ kN}\cdot\text{m}$, which are the values of the applied moment and applied shear force. Thus the jump in the internal shear force and internal moment difference is captured by the step function.

7.5* AREA-MOMENT METHOD

One last method is especially useful in finding the deflection or the slope of the beam is to be found at a specific point. Called the *area-moment method*, it is based on graphical interpretation of the integrals that are generated by integration of Equation (7.1).

Equation (7.1) can be written

$$\frac{d}{dx}v'(x) = \frac{M_z}{EI_{zz}} \tag{7.20.a}$$

where $v'(x) = dv(x)/dx$ represents the slope of the elastic curve. Integrating the equation from any point A to any other point x, we obtain

$$\int_{v'(x_A)}^{v'(x)} dv'(x) = \int_{x_A}^{x} \frac{M_z}{EI_{zz}}\, dx_1 \quad \text{or}$$

$$v'(x) = v'(x_A) + \int_{x_A}^{x} \frac{M_z}{EI_{zz}}\, dx_1 \tag{7.20.b}$$

Integrating Equation (7.20.b) between point A and any point x, we obtain

$$v(x) = v(x_A) + v'(x_A)(x - x_A) + \int_{x_A}^{x}\left(\int_{x_A}^{x} \frac{M_z}{EI_{zz}}\, dx_1\right) dx \tag{7.20.c}$$

The last integral[2] can be written as

$$v(x) = v(x_A) + v'(x_A)(x - x_A) + \int_{x_A}^{x} (x - x_1)\frac{M_z}{EI_{zz}}\, dx_1 \tag{7.20.d}$$

Assume EI_{zz} is a constant for the beam. From Equations 7.20.b and 7.20.d, the slope and the deflection at point B can be written

[2]By integrating by parts, it can be shown that

$$\int_{x_A}^{x}\left[\int_{x_A}^{x} f(x_1)dx_1\right]dx = \int_{x_A}^{x}(x - x_1)f(x_1)\, dx_1$$

Letting $f(x_1) = M_z/EI_{zz}$, we can obtain Equation (7.20.d) from Equation (7.20.c).

$$\boxed{v'(x_B) = v'(x_A) + \frac{1}{EI_{zz}}\int_{x_A}^{x_B} \boldsymbol{M}_z\, dx} \tag{7.21}$$

$$\boxed{v(x_B) = v(x_A) + v'(x_A)(x_B - x_A) + \frac{1}{EI_{zz}}\int_{x_A}^{x_B} (x_B - x)\,\boldsymbol{M}_z\, dx} \tag{7.22}$$

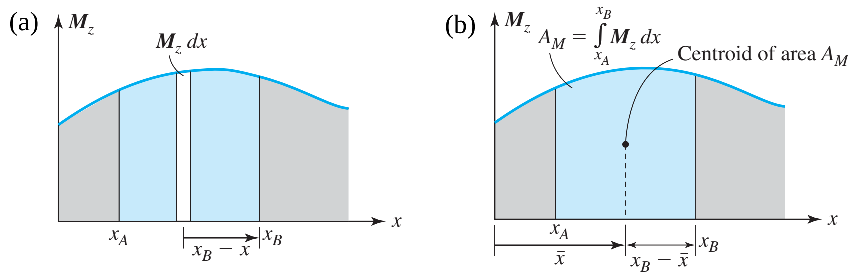


Figure 7.43 Graphical interpretation of integrals in area-moment method.

The integral in Equation (7.21) can be interpreted as the area under the bending moment curve, as shown in Figure 7.43. The moment diagram can be constructed as discussed in Section 6.4. Thus, if the slope $v'(x_A)$ at point *A* is known, then by adding the area under the moment curve, we can obtain the slope at point *B*. The area A_M will be considered positive if the moment curve is in the upper plane and negative if it is in the lower plane.

From Figure 7.43*a*, we see that the integral in Equation (7.22) is the first moment of the area under the moment curve about point *B*. This first moment of the area can be found by taking the distance of the centroid from *B* and multiplying by the area,

$$\int_{x_A}^{x_B} (x_B - x)\boldsymbol{M}_z\, dx = (x_B - \bar{x})A_M \tag{7.23.a}$$

With this interpretation the deflection of *B* can be found from Equation (7.22). Table C.2 in the Appendix lists the areas and the centroids of the areas under various curves. These values can be used in calculating the integrals in Equations 7.21 and 7.22.

Consider the cantilever beam in Figure 7.44*a* and the associated bending moment diagram. At point *A* the slope and the deflection at *A* are zero. Hence $v'(x_A) = 0$ and $v(x_A) = 0$ in Equations 7.21 and 7.22. The area A_M, representing the integral in Equation (7.21), is $-PL(L)/2$. Thus the slope at *B* is $v'(x_B) = -PL^2/2EI$. Since distance of the centroid from *B* is $x_B - \bar{x} = 2L/3$, $(x_B - \bar{x})A_M = (2L/3)(-PL^2/2)$ is the value of the integral equation, Equation (7.22). Thus the deflection at *B* is $v(x_B) = -PL^3/3EI$.

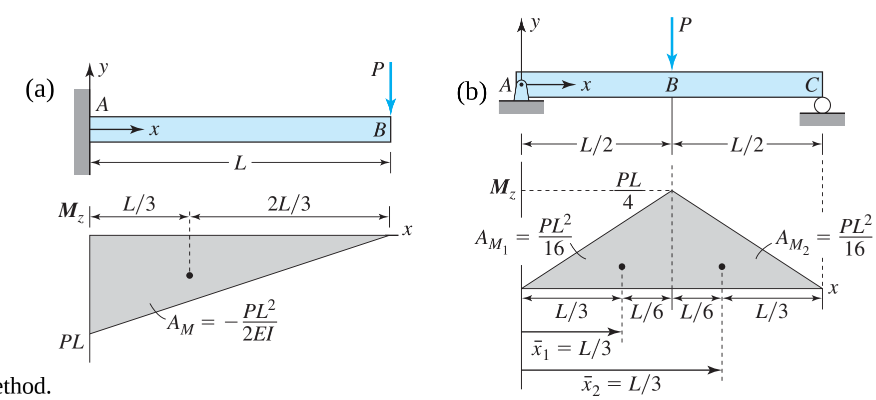


Figure 7.44 Application of area-moment method.

Now consider the simply supported beam and the associated bending moment in Figure 7.44*b*. The value of the slope is not known at any point on the beam. Thus before the deflection and slope at *B* can be determined, the slope at *A* must be found. The deflection at *A* is zero. Treating the slope at *A* as an unknown constant, we equate the deflection at *C* from Equation (7.22) to zero and obtain the slope at *A*,

$$v(x_C) = v(x_A) + v'(x_A)(x_C - x_A) + \frac{1}{EI}[A_{M_1}(x_C - \bar{x}_1) + A_{M_2}(x_C - \bar{x}_2)] = 0 \text{ or} \tag{7.23.b}$$

$$v'(x_A)(L) + \frac{1}{EI}\left[\frac{PL^2}{16}\left(\frac{2L}{3}\right) + \frac{PL^2}{16}\left(\frac{L}{3}\right)\right] = 0 \qquad \text{or} \qquad v'(x_A) = -\frac{PL^2}{16EI} \tag{7.23.c}$$

Using Equation (7.22) once more, we find the deflection at point B:

$$v(x_B) = v(x_A) + v'(x_A)(x_B - x_A) + \frac{1}{EI}[A_{M_1}(x_B - \bar{x}_1)]$$

$$= -\frac{PL^2}{16EI}\left(\frac{L}{2}\right) + \frac{1}{EI}\left[\frac{PL^2}{16}\left(\frac{L}{6}\right)\right] = -\frac{PL^3}{48} \tag{7.23.d}$$

EXAMPLE 7.13

In terms of E, I, w, and L, determine the deflection and slope at point B for the beam and loading shown in Figure 7.45.

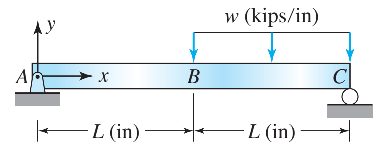


Figure 7.45 Beam and loading in Example 7.13.

PLAN

The reaction forces at A and C can be found and then the shear–moment diagram can be drawn as discussed in Section 6.4. The area under the moment curve and the location of the centroids can then be determined. Because the deflection at A is zero, the deflection at C can be written in terms of the unknown slope at A using Equation (7.22). Equating the deflection at C to zero, then gives the slope at A. Slope and deflection at B can now be found using Equations 7.21 and 7.22, respectively.

SOLUTION

From the free-body diagram of the entire beam, the reaction forces at the supports can be found and the shear–moment diagram drawn, as shown in Figure 7.46. The moment curve in region BC is a quadratic, and the areas under the curves is the sum of three area. Table C.2 in the Appendix lists the formulas for the areas and centroids.

$$A_1 = \frac{L}{2}\left(\frac{wL^2}{4}\right) = \frac{wL^3}{8} \qquad \bar{x}_1 = \frac{2}{3}(L) \tag{E1}$$

$$A_2 = \frac{L}{4}\left(\frac{wL^2}{4}\right) = \frac{wL^3}{16} \qquad \bar{x}_2 = L + \frac{L}{8} = \frac{9L}{8} \tag{E2}$$

$$A_3 = \frac{2}{3}\left(\frac{L}{4}\right)\left(\frac{wL^2}{32}\right) = \frac{wL^3}{192} \qquad \bar{x}_3 = L + \frac{5}{8}\left(\frac{L}{4}\right) = \frac{37L}{32} \tag{E3}$$

$$A_4 = \frac{2}{3}\left(\frac{3L}{4}\right)\left(\frac{9wL^2}{32}\right) = \frac{9wL^3}{64} \qquad \bar{x}_4 = 2L - \frac{5}{8}\left(\frac{3L}{4}\right) = \frac{49L}{32} \tag{E4}$$

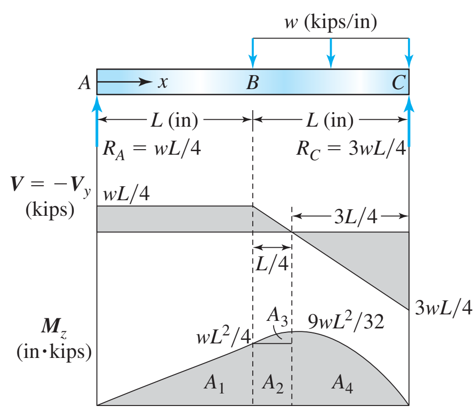


Figure 7.46 Shear–moment diagram in Example 7.13.

The deflection at C can be written as

$$v(x_C) = v(x_A) + v'(x_A)(x_C - x_A) + \frac{1}{EI}[A_1(x_C - \bar{x}_1) + A_2(x_C - \bar{x}_2) + A_3(x_C - \bar{x}_3) + A_4(x_C - \bar{x}_4)] \quad \text{(E5)}$$

The deflections at the support are zero. Substituting $v(x_C) = 0$ and $v(x_A) = 0$ and the values of the areas and centroids in Equation (E5), we can find the slope at A.

$$v'(x_A)(2L) + \frac{1}{EI}\left[\frac{wL^3}{8}\left(2L - \frac{2L}{3}\right) + \frac{wL^3}{16}\left(2L - \frac{9L}{8}\right) + \frac{wL^3}{192}\left(2L - \frac{37L}{32}\right) + \frac{9wL^3}{64}\left(2L - \frac{49L}{32}\right)\right] = 0 \quad \text{or} \quad v'(x_A) = -\frac{7wL^3}{48EI} \quad \text{(E6)}$$

The deflection at B can be written as

$$v(x_B) = v(x_A) + v'(x_A)(x_B - x_A) + \frac{1}{EI}[A_1(x_B - \bar{x}_1)] \quad \text{(E7)}$$

Substituting the calculated values we obtain the deflection at B,

$$v(x_B) = -\frac{7wL^3}{48EI}(L) + \frac{1}{EI}\left(\frac{wL^3}{8}\right)\left(L - \frac{2L}{3}\right) \quad \text{(E8)}$$

ANS. $v(x_B) = -\frac{5wL^4}{48EI}$

COMMENTS

1. The example demonstrates uses of the area moment method for finding slopes and deflection at a point in the beam.
2. If the elastic curve needs to be determined for an indeterminate beam, we can use the area moment method to determine the reactions and then the second-order differential equation to solve the problem. But if this approach is to have any computational advantage over using the fourth-order differential equations, then it must be possible to draw the moment diagram quickly by inspection.

PROBLEM SET 7.3

Superposition

7.57 Determine the deflection at the free end of the beam shown in Figure P7.20.

7.58 Determine the reaction force at support A in Figure P7.34.

7.59 Determine the deflection at point A on the beam shown in Figure P7.59 in terms of w, L, E, and I.

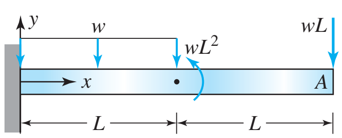


Figure P7.59

7.60 Determine the reaction force and the slope at A for the beam shown in Figure P7.60, using superposition.

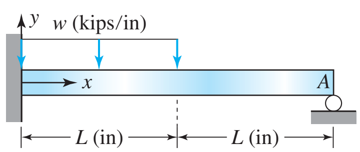


Figure P7.60

7.61 Two beams of length L and bending rigidity EI, shown in Figure P7.61, are simply supported at the ends and are in contact at the center. Determine the deflection at the center in terms of P, L, E, and I.

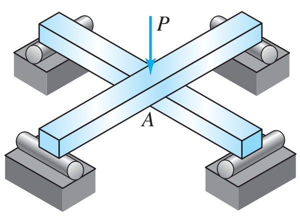


Figure P7.61

7.62 Two beams of length L and bending rigidity EI, shown in Figure P7.62, are simply supported at the ends and are in contact at the center. Determine the deflection at the center in terms of w, L, E, and I.

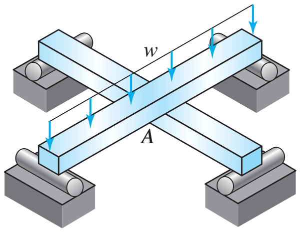


Figure P7.62

7.63 A cantilever beam's end rests on the middle of a simply supported beam, as shown in Figure P7.63. Both beams have length L and bending rigidity EI. Determine the deflection at A and the reactions at the wall at C in terms of P, L, E, and I.

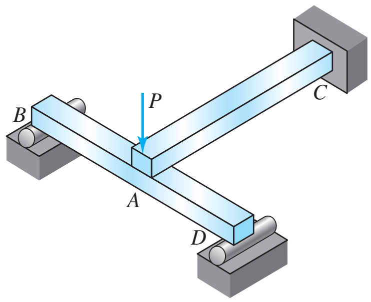


Figure P7.63

7.64 A cantilever beam's end rests on the middle of a simply supported beam, as shown in Figure P7.64. Both beams have length L and bending rigidity EI. Determine the deflection at A and the reactions at the wall at C in terms of w, L, E, and I.

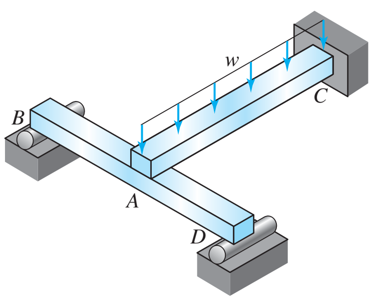


Figure P7.64

7.65 The end of one cantilever beam rests on the end of another cantilever beam, as shown in Figure P7.65. Both beams have length L and bending rigidity EI. Determine the deflection at A and the reactions at the wall at C in terms of w, L, E, and I.

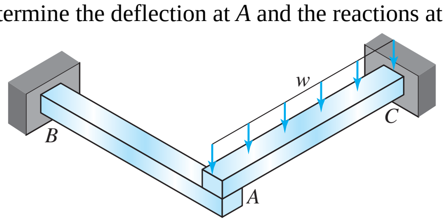


Figure P7.65

Discontinuity functions

7.66 A gymnast with a mass of 60 kg stands in the middle on a wooden balance beam as shown in Figure P7.66. The modulus of elasticity of the wood is 12.6 GPa. To bracket the elasticity of the support, two models are to be considered: (a) the supports are simply supported; (b) the supports are built in ends. Determine the maximum deflection of the beam for both the cases.

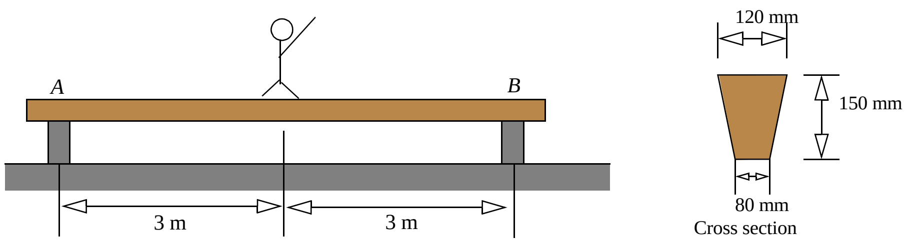


Figure P7.66

7.67 Solve Problem 7.17 using discontinuity functions.

7.68 Solve Problem 7.18 using discontinuity functions.

7.69 Solve Problem 7.19 using discontinuity functions.

7.70 Solve Problem 7.20 using discontinuity functions.

7.71 (a) Solve for the elastic curve for the beam and loading shown in Figure P7.23. (b) Determine the slope and deflection at point *C*.

7.72 Solve Problem 7.34 using discontinuity functions.

7.73 A beam is supported and loaded as shown in Figure P7.73. The spring constant in terms of beam stiffness is written as $k = \alpha EI/L^3$, where α is a proportionality factor. Determine the extension of the spring in terms of α, *w*, *E*, *I*, and *L*.

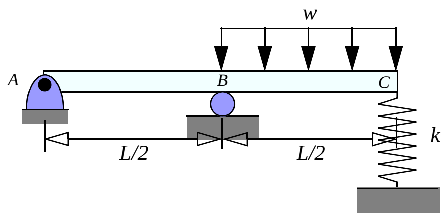


Figure P7.73

Area-moment method

7.74 Using the area-moment method, determine the deflection in the middle for the beam shown in Figure P7.2.

7.75 Using the area-moment method, determine the deflection in the middle of the beam shown in Figure P7.3.

7.76 Using the area-moment method, determine the deflection and slope at the free end of the beam shown in Figure P7.4.

7.77 Using the area-moment method, determine the slope at $x = 0$ and deflection at $x = L$ of the beam shown in Figure P7.6.

7.78 Using the area-moment method, determine the slope at $x = 0$ and deflection at $x = L$ of the beam shown in Figure P7.17.

7.79 Using the area-moment method, determine slope at $x = 0$ and deflection at $x = L$ of the beam shown in Figure P7.18.

7.80 Using the area-moment method, determine the slope at the free end of the beam shown in Figure P7.20.

Stretch Yourself

7.81 To improve the load carrying capacity of a wooden beam (E_W = 2000 ksi) a steel strip (E_s = 30,000 ksi) is securely fastened to it as shown in Figure P7.81. Determine the deflection at $x = L$.

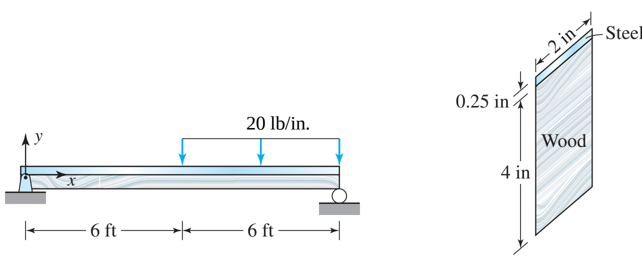


Figure P7.81

*7.6 CONCEPT CONNECTOR

Compared to a theory for the deflection of beams, our understanding of the strength of beams developed more intuitively, as described in Section 6.7. The very term *elastic curve* for the deflection of a beam reflects the early impact of mathematics.

7.6.1 History: Beam Deflection

In the seventeenth century, procedures were in use to draw tangents (similar to differentiation) and areas swept by curves (similar to integration). Issac Newton (1642-1727) in 1666 realized that the two procedures were inverse of each other and developed a method called *fluxional method*, which he circulated among some of his friends but did not publish. Newton's method did not receive much attention. Nine years later in Germany, Gottfried Wilhelm Leibniz (1646-1716) developed nearly the same method independently. Leibniz's notation caught on, especially in Europe, and so did his name for the method, *differential calculus*. Members of the Bernoulli family were in the forefront of finding applications for this new mathematical tool. One of the applications they considered was the determination of the elastic curve.

Daniel Bernoulli

Leonard Euler

Figure 7.47 Pioneers of beam deflection theories.

Jacob Bernoulli (1646–1716) and John Bernoulli (1667–1748) won acclaim for their mathematical work, which the French Academy of Science recognized by making the brothers members in 1699. Daniel Bernoulli (1700–1782), John's son, made important contributions to hydrodynamics, while Leonard Euler (1707–1782), John's pupil, introduced analytic methods used today in practically every area of mathematics. His name is also associated with buckling theory, as we shall see in Chapter 11. Both Daniel Bernoulli and Euler were pioneers in the theory of the elastic curve.

Jacob Bernoulli had started with Mariotte's assumption that the neutral axis is tangent to the bottom (the concave side) of the curve in a cantilever beam. From this he obtained a relationship between the curvature of the beam at any point and the applied load, as described in Problem 7.46. Although Mariotte's assumption proved incorrect, Bernoulli's result was correct—except for the value of the bending rigidity. Euler, on the suggestion of Daniel Bernoulli, approached the same problem by minimizing the strain energy in a beam, which yielded the correct relationship. Euler called the constant relating moment and curvature the *moment of stiffness* (rather than bending rigidity), but he recognized that it had to be determined experimentally. As we saw in Section 3.12.1, much later Thomas Young made a similar observation concerning axial rigidity, and the modulus of elasticity is named after him. Such are the quirks of history.

Claude-Louis Navier (1785–1836), whose work on the concept of stress we met in Section 1.5, was the first to solve for the deflection of statically indeterminate beams. Navier carried the extra unknown reactions in the second-order differential equation and determined these reactions from conditions on the deflection and slopes at the support (see Problem 7.45).

Jean Claude Saint-Venant, whose work we have seen in several chapters, analyzed the deflection of a cantilever beam due to a force at the free end. He was the first to realize that it can be found without formally integrating the differential equations. This was the beginning of the area-moment method that we studied in Section 7.5*. Alfred Clebsch (1833–1872), in his 1862 book on elasticity, considered the deflection of a beam under concentrated forces (see Problem 7.47). His approach later evolved into the discontinuity method, discussed in Section 7.4*. The English mathematician W. H. Macaulay formally introduced the discontinuity functions in 1919.

Each aspect of beam theory thus had its own development. The normal stress in bending was relatively intuitive; shear stress in bending was guided by experiment; and beam deflection was guided by mathematics. Together, they highlight the importance of intuition, experimental evidence, and mathematical formalization. Engineers need them all to understand nature.

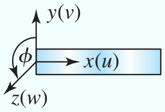
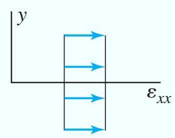
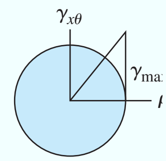
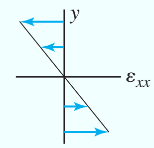
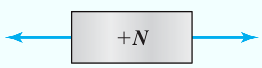
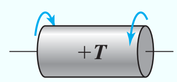
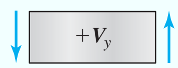
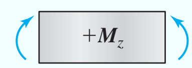
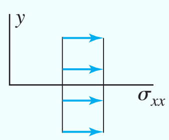
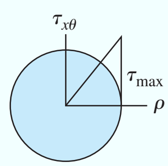
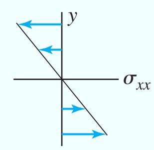
7.7 CHAPTER CONNECTOR

In this chapter we saw several methods for determining the deflection of beams. The preferred approach treats beam deflection as a second-order boundary-value problem. However, other approaches may be needed if the distributed loads on the beam are complicated functions or if we have only experimentally measured values for the distributed load. With the discontinuity function, a single differential equation can represent the loading on the entire beam. This method should be used if the beam loading changes in a discrete manner across the beam. The area-moment method, a graphical technique, can yield quick solutions for the beam deflection and slope at a point *if* the moment diagram can be constructed easily. The superposition method is another versatile design tool. It can be used for determinate or indeterminate beams, provided we know the beam deflection and slope. Handbooks supply these values for many basic cases.

Chapter 7 concludes the second major part of this book. Table 7.1 offers a synopsis of one-dimensional structural elements, described in Chapters 4 through 7. The table highlights the essential elements common to these theories. They allow us to obtain deformation, strains, and stresses at any point in a one-dimensional structural element. In the next three chapters we will use this information in many ways.

In order to determine whether a structure will break under a given load, we need *failure theories*, which we study in Chapter 10. To apply failure theories, we first need to determine the maximum normal and shear stresses at a point. Chapter 8, on *stress transformation*, describes how to obtain these stresses from our one-dimensional theories. Only experiment, however, can render the final verdict on designs based on the one-dimensional theory. One popular experimental technique is to measure strains using *strain gages*. And this technique requires a relationship between the strains obtained from one-dimensional theory and the strains in any given direction. That relationship, known as *strain transformation*, is the topic of Chapter 9. Chapter 10 is the culmination of the first nine chapters. Here we study stresses and strains in structural elements subject to combined axial, torsional, and bending loads. We also address the design and failure of structures and machine elements.

Table 7.1 Synopsis of one-dimensional structural theories.

	Axial (Rods)	Torsion (Shafts)	Symmetric Bending (Beams)
Displacements/ deformation	$u(x, y, z) = u(x)$ $v = 0 \quad w = 0$	$u = 0 \quad v = 0 \quad w = 0$ $\phi(x, y, z) = \phi(x)$	$u(x, y, z) = -y\dfrac{dv}{dx}$ $v = v(x) \quad w = 0$
Strains	$\varepsilon_{xx} = \dfrac{du}{dx}$ 	$\gamma_{x\theta} = \rho\dfrac{d\phi}{dx}$ 	$\varepsilon_{xx} = -y\dfrac{d^2v}{dx^2}$ 
Stresses	$\sigma_{xx} = E\varepsilon_{xx} = E\dfrac{du}{dx}$	$\tau_{x\theta} = G\gamma_{x\theta} = \rho\dfrac{d\phi}{dx}$	$\sigma_{xx} = E\varepsilon_{xx} = -Ey\dfrac{d^2v}{dx^2} \qquad \tau_{xy} \neq 0$
Internal forces and moments	$N = \int_A \sigma_{xx}\, dA$	$T = \int_A \rho\tau_{x\theta}\, dA$	$\int_A \sigma_{xx}\, dA = 0$ ...Locates neutral axis $M_z = -\int_A y\sigma_{xx}\, dA \qquad V_y = \int_A \tau_{xy}\, dA$
Sign convention			 
Stress formulas	$\sigma_{xx} = \dfrac{N}{A}$ 	$\tau_{x\theta} = \dfrac{T\rho}{J}$ 	$\sigma_{xx} = -\dfrac{M_z y}{I_{zz}}$  $\tau_{sx} = \tau_{xs} = -\dfrac{V_y Q_z}{I_{zz}t}$
Deformation formulas	$\dfrac{du}{dx} = \dfrac{N}{EA}$ $u_2 - u_1 = \dfrac{N(x_2 - x_1)}{EA}$ EA = axial rigidity	$\dfrac{d\phi}{dx} = \dfrac{T}{GJ}$ $\phi_2 - \phi_1 = \dfrac{T(x_2 - x_1)}{GJ}$ GJ = torsional rigidity	$\dfrac{d^2v}{dx^2} = \dfrac{M_z}{EI_{zz}}$ $v = \int\left(\int \dfrac{M_z}{EI_{zz}}\, dx\right) dx + C_1 x + C_2$ EI_{zz} = bending rigidity

POINTS AND FORMULAS TO REMEMBER

- The deflected curve of a beam represented by v(x) is called *elastic curve*.
- $\boldsymbol{M}_z = EI_{zz}\dfrac{d^2\mathrm{v}}{dx^2}$ (7.1) $\quad \boldsymbol{V}_y = -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(EI_{zz}\dfrac{d^2\mathrm{v}}{dx^2}\right)$ (7.4) $\quad \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI_{zz}\dfrac{d^2\mathrm{v}}{dx^2}\right) = p$ (7.5)
- where v is the deflection of the beam at any x and is positive in the positive y direction; $\boldsymbol{M}_z$ is the internal bending moment; $\boldsymbol{V}_y$ is the internal shear force; p is the distributed force on the beam and is positive in the positive y direction; EI_{zz} is the bending rigidity of the beam; and $d^2\mathrm{v}/dx^2$ is the curvature of the beam.
- The mathematical statement listing all the differential equations and all the conditions necessary for solving for v(x) is called *boundary-value problem* for beam deflection.
- Boundary conditions for second-order differential equations:

  Built-in end at x_A $\mathrm{v}(x_A) = 0 \qquad \dfrac{\mathrm{dv}}{\mathrm{d}x}(x_A) = 0$

  Simple support at x_A $\mathrm{v}(x_A) = 0$

  Smooth slot at x_A $\dfrac{\mathrm{dv}}{\mathrm{d}x}(x_A) = 0$

- Continuity conditions at x_j:

  $$\mathrm{v}_1(x_j) = \mathrm{v}_2(x_j) \qquad \frac{d\mathrm{v}_1}{dx}(x_j) = \frac{d\mathrm{v}_2}{dx}(x_j)$$

- Boundary conditions for fourth-order differential equations are determined at each boundary point by specifying:

  $(\mathrm{v} \text{ or } \boldsymbol{V}_y)$ and $(d\mathrm{v}/dx \text{ or } \boldsymbol{M}_z)$ (7.6)
- In fourth-order boundary-value problems, at each point x_j where the differential equation changes, the continuity conditions and equilibrium conditions must be specified.
- The superposition method is a versatile design tool that can be used for solving problems of determinate and indeterminate beams provided the beam deflection and slope values are available for many basic cases, such as in a handbook.
- In the discontinuity function method a single differential equation and conditions on deflection and slopes at support describe the complete boundary-value problem.
- Discontinuity functions:

  $$\langle x-a\rangle^{-n} = \begin{cases} 0 & x \neq a \\ \infty & x \to a \end{cases} \qquad \langle x-a\rangle^{n} = \begin{cases} 0 & x \le a \\ (x-a)^n & x > a \end{cases}$$

- Differentiation formulas:

  $$\frac{d\langle x-a\rangle^{-1}}{dx} = \langle x-a\rangle^{-2} \qquad \frac{d\langle x-a\rangle^{0}}{dx} = \langle x-a\rangle^{-1} \qquad \frac{d\langle x-a\rangle^{n}}{dx} = n\langle x-a\rangle^{n-1} \quad n \ge 1$$

- Integration formulas:

  $$\int_{-\infty}^{x}\langle x-a\rangle^{-2}\,dx = \langle x-a\rangle^{-1} \qquad \int_{-\infty}^{x}\langle x-a\rangle^{-1}\,dx = \langle x-a\rangle^{0} \qquad \int_{-\infty}^{x}\langle x-a\rangle^{n}dx = \frac{\langle x-a\rangle^{n+1}}{n+1} \quad n \ge 0$$

- The area-moment method is a graphical technique that can yield quick solutions of beam deflection and slope at a point, *if* the moment diagram can be constructed easily.

  $$\mathrm{v}'(x_B) = \mathrm{v}'(x_A) + \frac{1}{EI_{zz}}\underbrace{\int_{x_A}^{x_B}\boldsymbol{M}_z\,dx}_{A_M} \qquad (7.21)$$

  $$\mathrm{v}(x_B) = \mathrm{v}(x_A) + \mathrm{v}'(x_A)(x_B - x_A) + \frac{1}{EI_{zz}}\underbrace{\int_{x_A}^{x_B}(x_B - x)\boldsymbol{M}_z\,dx}_{(x_B - \bar{x})A_M} \qquad (7.22)$$

CHAPTER EIGHT
STRESS TRANSFORMATION

Learning objectives

1. Learn the equations and procedures of relating stresses in different coordinate systems (on different planes) at a point.
2. Visualize planes passing through a point on which stresses are given or are being found, in particular the planes of maximum normal and shear stress.

Figure 8.1 shows failure surfaces of aluminum and cast iron members under axial and torsional loads. Why do different materials under similar loading produce different failure surfaces? If we had a combined loading of axial and torsion, then what would be the failure surface, and which stress component would cause the failure? The answer to this question is critical for the successful design of structural members that are subjected to combined axial, torsional, and bending loads. In Chapter 10 we will study combined loading and failure theories that relate maximum normal and shear stresses to material strength. In this chapter we develop procedures and equations that transform stress components from one coordinate system to another at a *given point*.

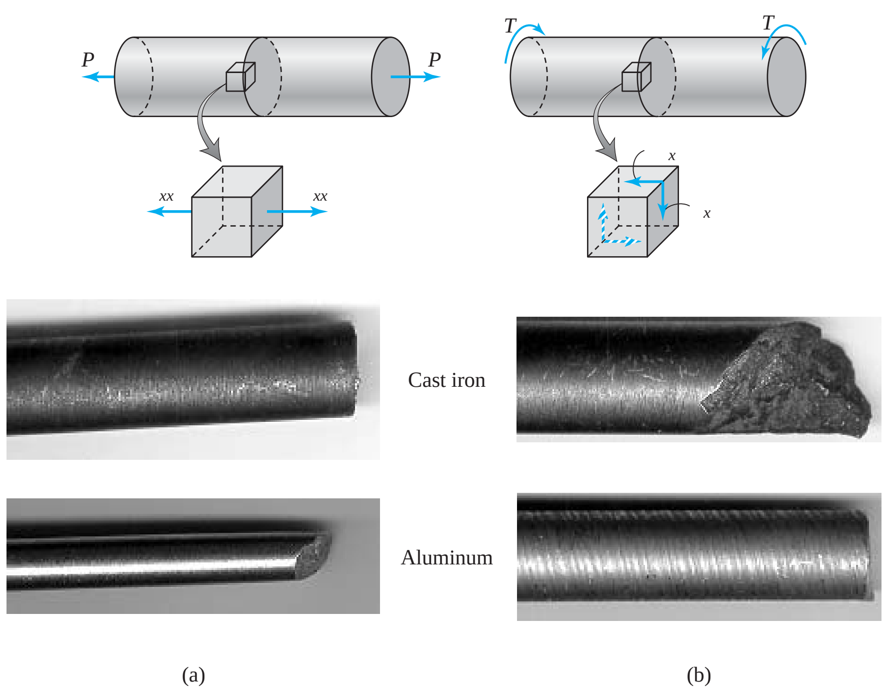


Figure 8.1 Failure surfaces. (*a*) Axial load. (*b*) Torsional load. (Specimens courtesy Professor J. B. Ligon.)

Stress transformation can also be viewed as relating stresses on different planes that pass through a point. The outward normals of the planes form the axes of a coordinate system. Thus relating stresses on different planes is equivalent to relating stresses in different coordinate systems. We will use both viewpoints in this chapter of stress transformation.

8.1 PRELUDE TO THEORY: THE WEDGE METHOD

In this chapter we will study three methods of stress transformation. The *wedge method*, described in this section, is used to derive stress transformation equations in the next section. The stress transformation equations are then manipulated to generate a graphical procedure called *Mohr's circle*, which is described in Section 8.3.

Two coordinate systems will be used in this chapter. First, the entire problem is described in a fixed reference coordinate system called the **global coordinate system**. We usually relate internal forces and moments to external forces and moments in the global coordinate system. The internal quantities are then used to obtain stresses, such as axial stress, torsional shear stress, and bending normal and shear stresses. And second, a **local coordinate system** that can be fixed at any point on the body. The orientation of the local coordinate system is defined with respect to the global coordinate. In all two-dimensional problems in this book, the local coordinate system will be the *n*, *t*, *z* coordinate system.

- The *n* direction is the direction of the *outward normal* to the plane on which we are finding the stresses.
- The *z* direction is identical to the *z* direction of the global coordinate system.
- The tangent *t* direction can be found from the right-hand rule, as shown in Figure 8.2. Just as *x* cross *y* yields *z*, in a similar manner *n* cross *t* yields *z*. With the thumb of the right hand pointed in the known *z* direction, the curl of the fingers is from the known *n* direction toward the *t* direction.

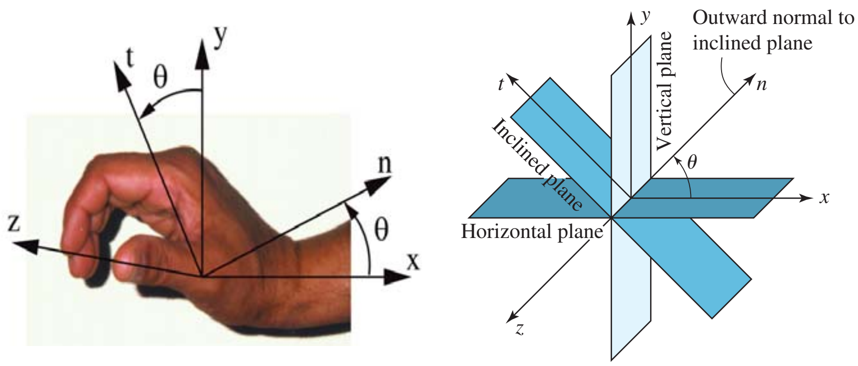


Figure 8.2 Local and global coordinate systems.

Alternatively, the positive *t* direction can be found by curling the fingers of the right hand from the *z* direction toward the *n* direction. The positive *t* direction is then given by the direction of the thumb. With the *n* direction as positive in the outward normal direction, positive shear stress τ_{nt} is in the positive tangent direction and negative τ_{nt} will be in the negative tangent direction.

In this section we restrict ourselves to plane stress problems (see Sections 1.3.2 and 3.6). We will consider only those inclined planes that can be obtained by rotation about the *z* axis, as shown in Figure 8.2*b*.

8.1.1 Wedge Method Procedure

The wedge method has five steps shown below, and elaborated by applications of Examples 8.1 and 8.2.

Step 1: A stress cube with the plane on which stresses are to be found, or are given, is constructed.

Step 2: A wedge is constructed from the following three planes:

1. A vertical plane that has an outward normal in the *x* direction.
2. A horizontal plane that has an outward normal in the *y* direction.
3. The specified inclined plane on which we either seek or are given the stresses.

Establish a local *n*, *t*, *z* coordinate system using the outward normal of the inclined plane as the *n* direction. All the known and unknown stresses are shown on the wedge. The diagram so constructed will be called a *stress wedge.*

Step 3: Multiply the stress components by the area of the planes on which the stress components are acting, to obtain the forces acting on that plane. The wedge with the forces drawn will be referred to as the *force wedge*.

Step 4: Balance forces in *any* two directions to determine the unknown stresses. We can write equilibrium equations on the force wedge because the wedge represents a point on a body that is in equilibrium.

Step 5: Check the answer intuitively by considering each stress component individually. By inspection, we decide whether the stress component will produce tensile or compressive normal stress on the incline and whether it will produce positive or negative shear stress on the incline.

EXAMPLE 8.1

A steel beam in a bridge was repaired by welding along a line that is 35º to the axis of the beam. The normal stress near the bottom of the beam is estimated using beam theory and is shown on the stress cube. Determine the normal and shear stress on the plane containing the weld line.

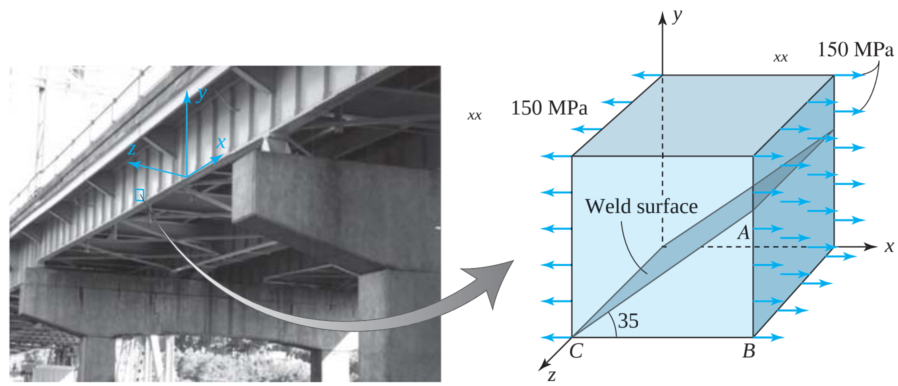


Figure 8.3 Stress cube at a point on a bridge.

PLAN

Step 1 of the procedure outlined in Section 8.1.1 is complete, as shown in Figure 8.3. We follow the remaining steps to solve the problem.

SOLUTION

Step 2: We construct a wedge from the horizontal, vertical, and inclined plane, as shown in Figure 8.4*a*. The outward normal to the inclined plane is drawn, and knowing the positive *z* direction, we establish the positive *t* direction using the right-hand rule for the *n*, *t*, *z* coordinates. On the inclined plane we can show the normal stress σ_{nn} and the shear stress τ_{nt}. From triangle *ABC* we note that $\Delta y = \Delta t \sin 35^{\circ}$.

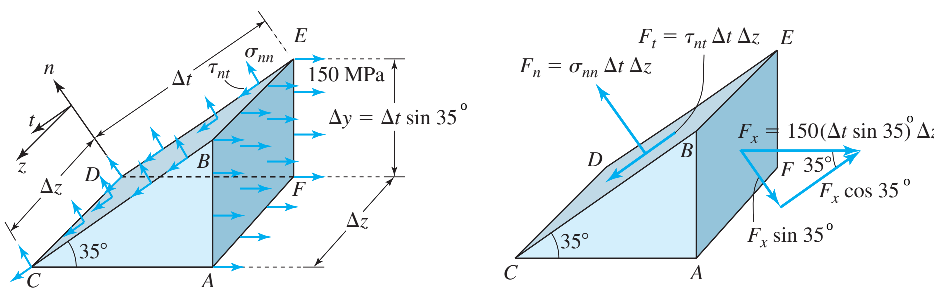


Figure 8.4 Wedges in Example 8.1.

(a) Stress wedge (b) Force wedge

Step 3: We multiply the stresses σ_{nn} and τ_{nt} by the area of the incline *BCDE* to obtain the forces in the *n* and *t* directions, respectively. Similarly, we multiply the stress of 150 MPa by the area of the plane *ABEF* to obtain the force in the *x* direction. These forces are shown on the force wedge in Figure 8.4*b*.

Step 4: As the unknowns are in the *n* and *t* directions, we balance the forces in the *n* and *t* directions. The components of force F_x in the *n* and *t* directions are shown on the force wedge in Figure 8.4*b*. Balancing forces in the *n* direction, we obtain

$$\sigma_{nn}\,\Delta t\,\Delta z - (150\,\Delta t \sin 35^{\circ}\,\Delta z)\sin 35^{\circ} = 0 \qquad \text{or} \qquad (\sigma_{nn} - 150\sin 35^{\circ}\sin 35^{\circ})\,\Delta t\,\Delta z = (\sigma_{nn} - 49.35)\,\Delta t\,\Delta z = 0 \tag{E1}$$

ANS. $\sigma_{nn} = 49.35\text{ MPa (T)}$

In a similar manner, balancing the forces in the *t* direction, we obtain

$$\tau_{nt}\,\Delta t\,\Delta z - (150\,\Delta t \sin 35^{\circ}\,\Delta z)\cos 35^{\circ} = 0 \qquad \text{or} \qquad (\tau_{nt} - 150\sin 35^{\circ}\cos 35^{\circ})\,\Delta t\,\Delta z = (\tau_{nt} - 70.48)\,\Delta t\,\Delta z = 0 \tag{E2}$$

ANS. $\tau_{nt} = 70.48\text{ MPa (T)}$

Step 5: We check the answer using intuitive arguments. The surface *ABC* in Figure 8.3 tends to move away from the rest of the cube. Hence the material resistance opposing it results in a tensile stress, as seen. A more visual way is to imagine the inclined plane in Figure

8.3 as a glued surface. Because of σ_{xx}, the two surfaces on either side of the glue are pulled apart; hence the glue is put into tension. Similar σ_{xx} will cause the wedge *ABC* to slide upward relative to the rest of the cube; hence the material resistance (like friction) will be downward, resulting in a positive shear stress, as seen.

COMMENTS

1. In Equations (E1) and (E2) the dimensions Δt and Δz were common factors and did not affect the final answer. In other words, the dimensions of the stress cube are immaterial. This is not surprising, as the stress cube is a visualization aid symbolically representing a point. Only the relative orientation of the plane is important.
2. The stress cube in Figure 8.3 and the stress and force wedges in Figure 8.4 can be represented in two dimensions, as shown in Figure 8.5. These are easier to draw and work with. But once more it must be emphasized that stress is a distributed force and not a vector, as depicted in Figure 8.5*b*. Force equilibrium can be done only on the force wedge

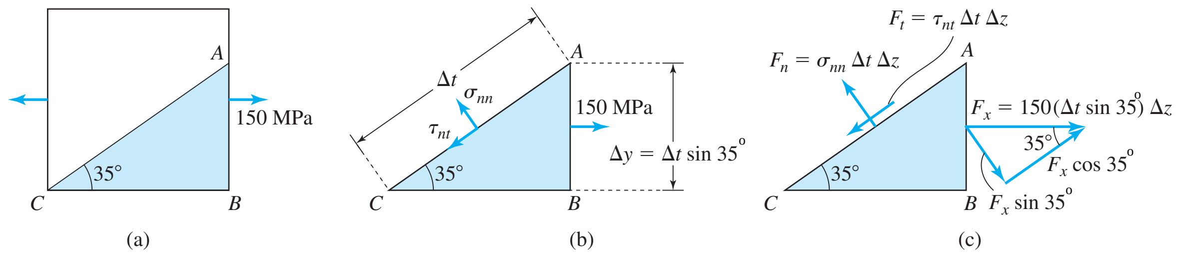


Figure 8.5 Wedge method in Example 8.1. (*a*) stress cube; (*b*) stress wedge; (*c*) force wedge.

3. In constructing the stress wedge we took the lower wedge. An alternative approach is to take the upper wedge, as shown in Figure 8.6. This is possible as the dimensions of the stress cube are immaterial. Only the orientation of the planes is important.

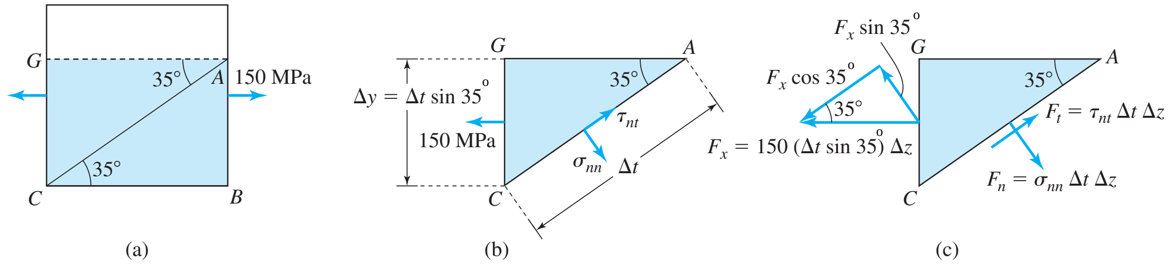


Figure 8.6 Alternative approach in Example 8.1. (*a*) stress cube; (*b*) stress wedge; (*c*) force wedge.

4. Some writing could be saved by taking $\Delta t = 1$ and $\Delta z = 1$, as these terms always drop out. But the geometric visualization may become more difficult in the process.

EXAMPLE 8.2

Fibers are oriented at 30° to the *x* axis in a lamina of a composite[1] plate, as shown Figure 8.7. Stresses at a point in the lamina were found by the finite-element method[2] as

$$\sigma_{xx} = 30 \text{ MPa (T)} \qquad \sigma_{yy} = 60 \text{ MPa (C)} \qquad \tau_{xy} = 50 \text{ MPa}$$

In order to assess the strength of the interface between the fiber and the resin, determine the normal and shear stresses on the plane containing the fiber.

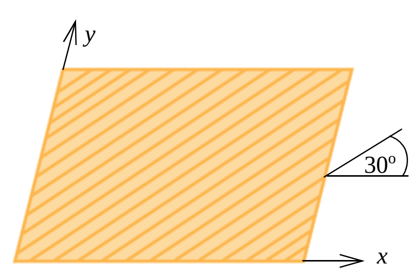


Figure 8.7 Stresses in lamina in Example 8.2.

PLAN

In Step 1 of the procedure outlined Section 8.1.1, we can draw the stress cube with an plane inclined at 30° and then follow the remaining steps of the procedure.

[1]See Section 3.12.3 for a brief description of composite materials.
[2]See Section 4.8 for a brief description of the finite-element method.

SOLUTION

Step 1: We draw a stress cube in two dimensions for the given state of stress, and the plane inclined at 30° counterclockwise to the x axis, as shown in Figure 8.8*a*

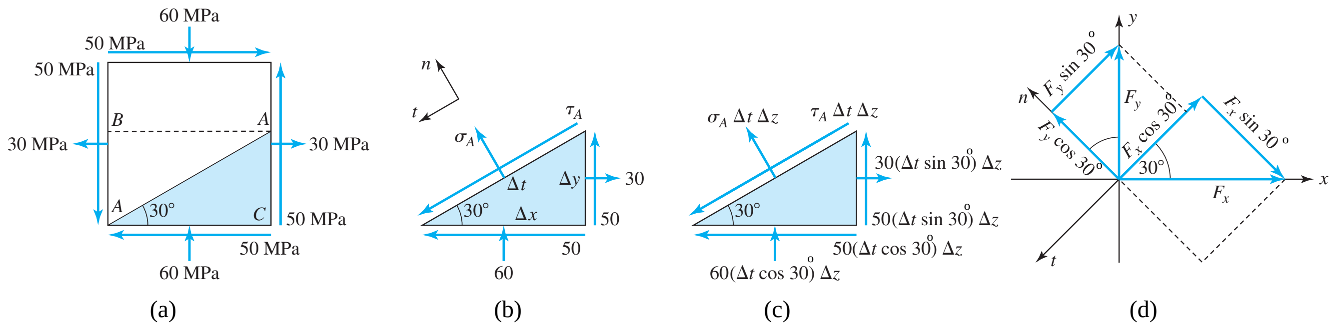


Figure 8.8 (*a*) Stress cube. (*b*) Stress wedge. (*c*) Force wedge (d) Resolution of force components.

Step 2: We can choose wedge *ACA* or wedge *ABA* as a stress wedge. Figure 8.8*b* shows the stress wedge *ACA* with a local n, t, z coordinate system.

Step 3: We assume the length of the inclined plane to be Δt. From geometry we see that $\Delta x = \Delta t \cos 30^o$ and $\Delta y = \Delta t \sin 30^o$. If we assume that the dimension of the cube out of the paper is Δz, we get the following areas: inclined plane $\Delta t\, \Delta z$, vertical plane $\Delta y\, \Delta z$, and horizontal plane $\Delta x\, \Delta z$. The stresses are converted into forces by multiplying by the area of the plane, and a force wedge is drawn as shown in Figure 8.8*c*.

Step 4: We can balance forces in any two directions. We choose to balance forces in the n and t directions as the unknowns are in the n and t directions. Figure 8.8*d* shows the resolution of the forces in the x, y coordinates to n, t coordinates.

The forces in the x and y directions in Figure 8.8*c* that need resolution are:

$$F_x = [(30\ \text{MPa})(\Delta t \sin 30°)\Delta z - (50\ \text{MPa})(\Delta t \cos 30°)\Delta z] = -(28.301\ \text{MPa})(\Delta t\, \Delta z) \tag{E1}$$

$$F_y = [(60\ \text{MPa})(\Delta t \cos 30°)\Delta z + (50\ \text{MPa})(\Delta t \sin 30°)\,\Delta z] = (76.961\ \text{MPa})(\Delta t\, \Delta z) \tag{E2}$$

From Figure 8.8c and Figure 8.8*d*, the equilibrium of forces in the n direction yields,

$$\sigma_A(\Delta t\, \Delta z) - F_x \sin 30° + F_y \cos 30° = 0 \quad \text{or}$$

$$\sigma_A(\Delta t\, \Delta z) - [-(28.301\ \text{MPa})(\Delta t\, \Delta z)] \sin 30° + [(76.961\ \text{MPa})(\Delta t\, \Delta z)] \cos 30° = 0 \quad \text{or}$$

$$\sigma_A(\Delta t\, \Delta z) + (14.15\ \text{MPa})(\Delta t\, \Delta z) + (66.65\ \text{MPa})(\Delta t\, \Delta z) = [\sigma_A + 80.8\text{MPa}](\Delta t\, \Delta z) = 0 \tag{E3}$$

ANS. $\sigma_A = 80.8\ \text{MPa (C)}$

The shear stress can be similarly calculated by equilibrium in the t direction,

$$\tau_A(\Delta t\ \Delta z) - F_x \cos 30° - F_y \sin 30° = 0 \quad \text{or}$$

$$\tau_A(\Delta t\ \Delta z) - [-(28.301\ \text{MPa})(\Delta t\, \Delta z)] \cos 30° - [(76.961\ \text{MPa})(\Delta t\, \Delta z)] \sin 30° = 0 \quad \text{or}$$

$$\tau_A(\Delta t\ \Delta z) + (24.509\ \text{MPa})(\Delta t\ \Delta z) - (38.481\ \text{MPa})\,(\Delta t\ \Delta z) = [\tau_A - (13.971\text{MPa})](\Delta t\ \Delta z) = 0 \tag{E4}$$

ANS. $\tau_A = 14.0\ \text{MPa}$

Step 5: We can check the answers intuitively. Consider each stress component individually and visualize the inclined plane as a glue line. The rectangles shown in Figure 8.9 are for purposes of explanation. One can go through the arguments mentally without drawing these rectangles.

Figure 8.9 shows that the right surface (wedge) and the left surface will move:

- Apart due to σ_{xx}—putting the glue in *tension*
- Into each other due to σ_{yy}—putting the glue in *compression*
- Into each other due to τ_{xy}—putting the glue in *compression*
- Into each other due to τ_{yx}—putting the glue in *compression*.

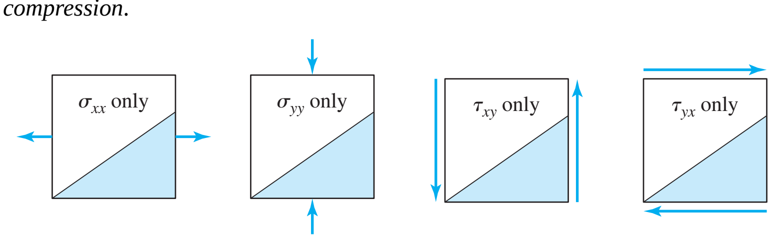


Figure 8.9 Intuitive check.

Thus the normal stress in the glue (on the inclined plane) is expected to be in compression, which is consistent with our answer.

Figure 8.9 shows that the right surface (shaded wedge), with respect to the left surface, will slide:

- Upward due to σ_{xx}; therefore the shaded wedge will have a *positive* (downward) shear stress
- Upward due to σ_{yy}; therefore the right wedge will have a *positive* (downward) shear stress
- Upward due to τ_{xy}; therefore the right wedge will have a *positive* (downward) shear stress

- Downward due to τ_{yx}; therefore the right wedge will have a *negative* (upward) shear stress.

Thus the shear stress on the incline is expected to be positive, which is consistent with our answer.

COMMENTS

1. In the intuitive check, three of the components gave one answer, whereas the fourth gave an opposite answer. What happens if two intuitive deductions are positive, two intuitive deductions are negative, and the stress components are nearly equal in magnitude? The question emphasizes that intuitive reasoning is a quick and important check on results, but one must be cautious with the conclusions.
2. We could have balanced forces in the x and y directions, in which case we have to find the x and y components of the normal and tangential forces on the force wedge. After removing the common factors $\Delta t\,\Delta z$ we would obtain

$$\sigma_A \sin 30° + \tau_A \cos 30° = -(50\ \text{MPa})\cos 30° + (30\ \text{MPa})\sin 30° = -28.30\ \text{MPa}$$

$$\sigma_A \cos 30° - \tau_A \sin 30° = -(50\ \text{MPa})\sin 30° - (60\ \text{MPa})\cos 30° = -76.96\ \text{MPa}$$

Solving these two equations, we obtain the values of σ_A and τ_A as before. By balancing forces in the n, t directions we generated one equation per unknown but did extra computation in finding components of forces in the n and t directions. By balancing forces in the x and y directions, we did less work finding the components of forces, but we did extra work in solving simultaneous equations. This shows that the important point is to balance forces in any two directions, and the direction chosen for balancing the forces is a matter of preference.

3. Figure 8.8 is useful in reducing the algebra when forces are balanced in the n and t directions. But you may prefer to resolve components of individual forces, as shown in Figure 8.10, and then write the equilibrium equations. The method is a little more tedious, but has the advantage that the intuitive check can be conducted as one writes the equilibrium equations as follows.

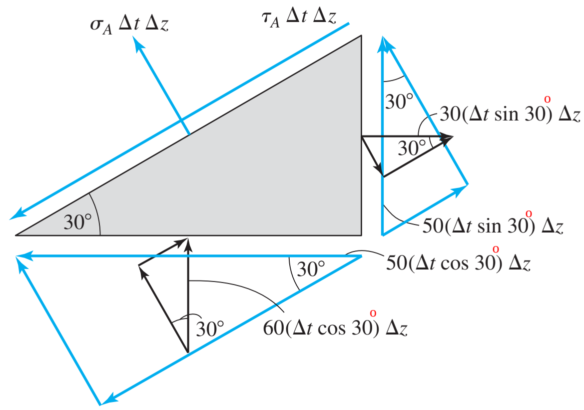


Figure 8.10 Alternative force resolution.

The normal stress σ_A on the incline will be:

- Tensile due to σ_{xx}; Compressive due to σ_{yy}; Compressive due to τ_{xy};Compressive due to τ_{yx}.

As σ_{xx} is the smallest stress component, it is not surprising that the total result is a compressive normal stress on the inclined plane.

The shear stress on the incline will be:

- Positive due to σ_{xx};Positive due to σ_{yy}; Positive due to τ_{xy};Negative due to τ_{yx}.

We expect the net result to be positive shear stress on the incline.

PROBLEM SET 8.1

Stresses by inspection

8.1 In Figure P8.1, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

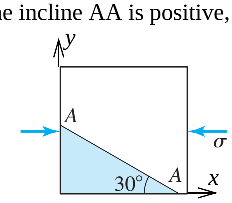


Figure P8.1

8.2 In Figure P8.2 determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

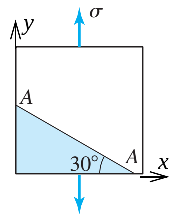


Figure P8.2

8.3 In Figure P8.3, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

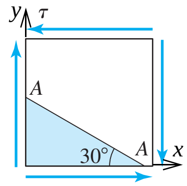


Figure P8.3

8.4 In Figure P8.4, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

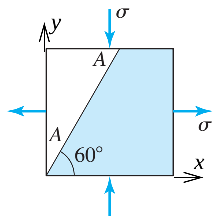


Figure P8.4

8.5 In Figure P8.5, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

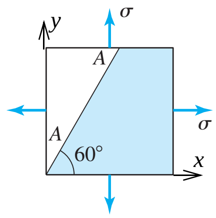


Figure P8.5

8.6 In Figure P8.6, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

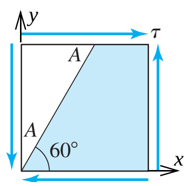


Figure P8.6

8.7 In Figure P8.7, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

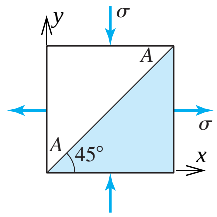


Figure P8.7

8.8 In Figure P8.8, determine by inspection (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

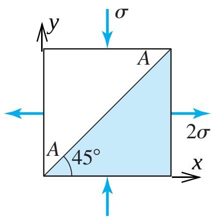


Figure P8.8

8.9 In Figure P8.9, determine by inspection: (a) if the normal stress on the incline AA is in tension, compression, or cannot be determined; (b) if the shear stress on the incline AA is positive, negative, or cannot be determined.

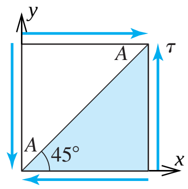


Figure P8.9

8.10 Determine the normal and shear stresses on plane *AA* in Problem 8.1 for $\sigma = 10$ ksi.

8.11 Determine the normal and shear stresses on plane *AA* in Problem 8.4 for $\sigma = 10$ ksi.

8.12 Determine the normal and shear stresses on plane *AA* in Problem 8.6 for $\tau = 10$ ksi.

8.13 Determine the normal and shear stresses on plane *AA* in Problem 8.7 for $\sigma = 60$ MPa.

8.14 Determine the normal and shear stresses on plane *AA* in Problem 8.9 for $\tau = 60$ MPa.

8.15 A shaft is adhesively bonded along the seam as shown in Figure P8.15. By inspection determine whether the adhesive will be in tension or in compression.

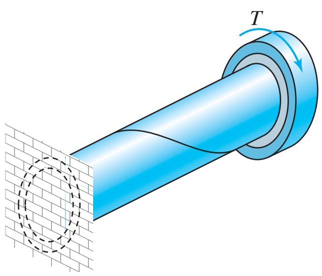


Figure P8.15

8.16 A shaft is adhesively bonded along the seam as shown in Figure P8.16. By inspection determine whether the adhesive will be in tension or in compression .

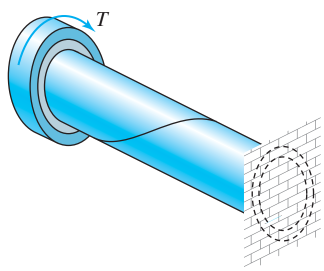


Figure P8.16

8.17 A shaft is adhesively bonded along the seam as shown in Figure P8.17. By inspection determine whether the adhesive will be in tension or in compression.

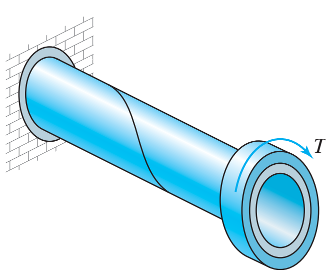


Figure P8.17

8.18 A shaft is adhesively bonded along the seam as shown in Figure P8.18. By inspection determine whether the adhesive will be in tension or in compression.

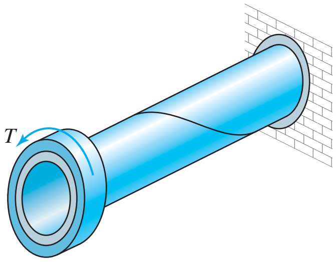


Figure P8.18

8.19 Determine the normal and shear stresses on plane AA shown in Figure P8.19.

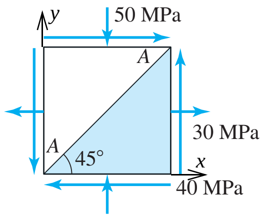


Figure P8.19

8.20 Determine the normal and shear stresses on plane AA shown in Figure P8.20.

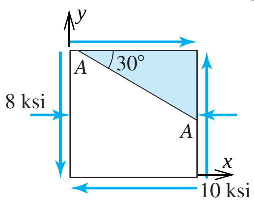


Figure P8.20

8.21 Determine the normal and shear stresses on plane AA shown in Figure P8.21.

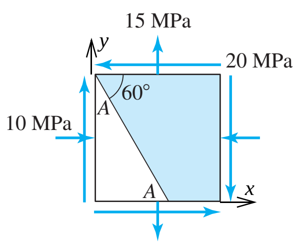


Figure P8.21

8.22 The stresses at a point in plane stress are σ_{xx} = 45 MPa (T), σ_{yy} = 15 MPa (T), and τ_{xy} = –20 MPa. Determine the normal and shear stresses on a plane passing through the point at 28° counterclockwise to the x axis.

8.23 The stresses at a point in plane stress are σ_{xx} = 45 MPa (T), σ_{yy} = 15 MPa (C), and τ_{xy} = –20 MPa. Determine the normal and shear stresses on a plane passing through the point at 38° clockwise to the x axis.

8.24 The stresses at a point in plane stress are σ_{xx} = 10 ksi (C), σ_{yy} = 20 ksi (C), and τ_{xy} = 30 ksi. Determine the normal and shear stresses on a plane passing through the point that is 42° counterclockwise to the x axis.

8.25 A cast-iron shaft of 25-mm diameter fractured along a surface that is 45° to the axis of the shaft. The shear stress τ due to torsion is as shown in Figure P8.25. If the ultimate normal stress for the brittle cast-iron material is 330 MPa (T), determine the torque that caused the fracture.

Figure P8.25

Design problems

8.26 In a wooden structure a member was adhesively bonded along a plane 40° to the horizontal plane, as shown in Figure P8.26. The stresses at a point on the bonded plane due to a load P on the structure, were estimated as shown, where P is in lb. If the adhesive strength in tension is 500 psi and its strength in shear is 200 psi, determine the maximum permissible load the structure can support without breaking the adhesive joint.

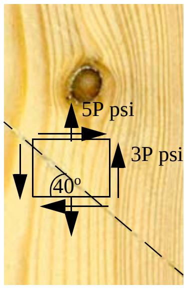


Figure P8.26

Stretch yourself

In three dimensions, the area of the inclined plane A can be related to the areas of the surfaces of the stress cube using the direction cosines of the outward normals, as shown in Figure P8.26.

$$n_x = \cos\theta_x \qquad A_x = n_x A \qquad n_y = \cos\theta_y \qquad A_y = n_y A \qquad n_z = \cos\theta_z \qquad A_z = n_z A$$

These relationships can be used to convert the stress wedge into a force wedge. Using this information, solve Problems 8.27 and 8.28. (Hint: A component of a vector in a given direction can be found by taking the dot [scalar] product of the vector with a unit vector in the given direction.)

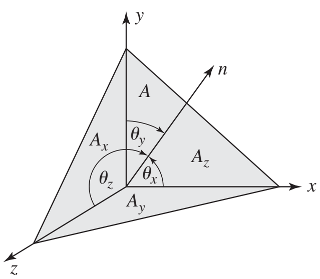


Figure P8.26

8.27 The stresses at a point are σ_{xx} = 8 ksi (T), σ_{yy} = 12 ksi (T), and σ_{zz} = 8 ksi (C). Determine the normal stress on a plane that has outward normals at 60°, –60°, and 45° to the *x*, *y*, and *z* directions, respectively.

8.28 The stresses at a point are τ_{xy} = 125 MPa and τ_{xz} = –150 MPa. Determine the normal stress on a plane that has outward normals at 72.54°, 120°, and 35.67° to the *x*, *y*, and *z* directions, respectively.

8.2 STRESS TRANSFORMATION BY METHOD OF EQUATIONS

We follow the wedge method procedure described in Section 8.1.1 with variables in place of numbers to develop equations that relate the stresses in the Cartesian coordinate system to the stresses on an arbitrary inclined plane. We once more consider only those planes that can be obtained by rotating about the *z* axis, as shown in Figure 8.2*a*. The *outward normal* to the inclined plane makes an angle θ with the *x* axis. The angle θ is considered *positive counterclockwise from the x axis*, as shown in Figure 8.11*a*.

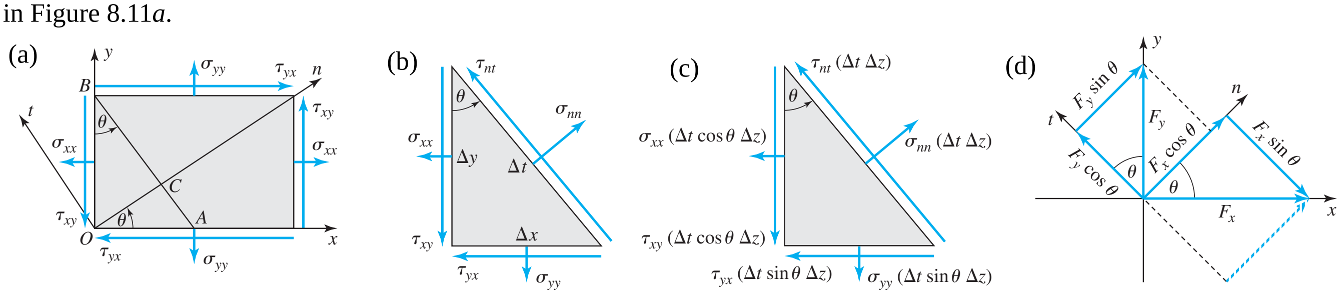


Figure 8.11 (*a*) Stress cube. (*b*) Stress wedge. (*c*) Force wedge. (*d*) Resolution of force components.

Step 1: We draw the stress cube with all positive stress components as shown in Figure 8.11*a*. From triangle *OAC* in Figure 8.11*a* we deduce that the angle *OAC* is 90° − θ. From triangle *OAB* we conclude that the angle *OBA* is θ.

Step 2: The stress wedge *OAB* is drawn as shown in Figure 8.11*b*. Positive normal σ_{nn} and shear stress τ_{nt} are drawn on the inclined plane.

Step 3: We obtain the force wedge by multiplying the stresses by the areas of the planes on which they act as shown in Figure 8.11*c*.

Step 4: To write the equilibrium equations we use Figure 8.11*d* for resolving forces from the *x* and *y* direction to *n* and *t* direction. The forces in the *x* and *y* direction in Figure 8.11*c* that need resolving are:

$$F_x = -\sigma_{xx}\cos\theta\,\Delta t\,\Delta z - \tau_{yx}\sin\theta\,\Delta t\,\Delta z = -(\sigma_{xx}\cos\theta + \tau_{yx}\sin\theta)(\Delta t\Delta z)$$

$$F_y = -\sigma_{yy}\sin\theta\,\Delta t\,\Delta z - \tau_{xy}\cos\theta\,\Delta t\,\Delta z = -(\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)(\Delta t\Delta z)$$

By equilibrium of forces in the *n* direction on the force wedge in Figure 8.11*c*, we obtain

$$\sigma_{nn}(\Delta t\,\Delta z) + F_x\cos\theta + F_y\sin\theta = 0 \quad \text{or}$$

$$\sigma_{nn}(\Delta t\,\Delta z) + [-(\sigma_{xx}\cos\theta + \tau_{yx}\sin\theta)(\Delta t\Delta z)]\cos\theta + [-(\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)(\Delta t\Delta z)]\sin\theta = 0$$

Because $\Delta t\,\Delta z$ is a common factor, these equations simplify to

$$\boxed{\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta} \tag{8.1}$$

Similarly by equilibrium of forces in the *t* direction on the force wedge in Figure 8.11*c*, we obtain the shear stress,

$$\tau_{nt}\Delta t\,\Delta z - F_x\sin\theta + F_y\cos\theta = 0 \quad \text{or}$$

$$\tau_{nt}\Delta t\,\Delta z - [-(\sigma_{xx}\cos\theta + \tau_{yx}\sin\theta)(\Delta t\Delta z)]\sin\theta + +[-(\sigma_{yy}\sin\theta + \tau_{xy}\cos\theta)(\Delta t\Delta z)]\cos\theta = 0 \quad \text{, or}$$

$$\boxed{\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)} \tag{8.2}$$

We can find σ_{tt} by substituting $90 + \theta$ in place of θ into Equation (8.1),

$$\boxed{\sigma_{tt} = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\cos\theta\sin\theta} \tag{8.3}$$

Equations (8.1) through (8.3) transform stresses from the *x* and *y* coordinate system into *n* and *t* coordinate system that is obtained by rotating by an angle θ in the counterclockwise direction.

8.2.1 Maximum Normal Stress

In Section 3.1we observed that a brittle material usually ruptures when the maximum tensile normal stress exceeds the ultimate tensile stress of the material. Cracks in the material propagate due to tensile stress. Adhesively bonded material debonds due to tensile normal stress, which is called *peel stress*. Similarly, failure may occur due to a maximum compressive normal stress because of the phenomenon called *buckling*, which is discussed in Chapter 11. In this section we develop equations for maximum tensile and compressive normal stresses.

In Equation (8.1) the stresses σ_{xx}, σ_{yy}, and τ_{xy} are assumed known. Thus Equation (8.1) expresses σ_{nn} as a function of θ. From calculus we know that the maximum or minimum of a function exists where the first derivative is zero. Before performing the differentiation we rewrite Equations (8.1) and (8.2) in terms of the double angles of 2θ[3] as

$$\sigma_{nn} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \tag{8.4}$$

$$\tau_{nt} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \tag{8.5}$$

Let $\theta = \theta_p$ be the angle of the outward normal of the plane on which the maximum or minimum normal stress exists. Differentiating Equation (8.4), we obtain

$$\left.\frac{d\sigma_{nn}}{d\theta}\right|_{\theta=\theta_p} = \left.-2\frac{(\sigma_{xx}-\sigma_{yy})\sin 2\theta}{2} + 2\tau_{xy}\cos 2\theta\right|_{\theta=\theta_p} = 0 \quad \text{or}$$

$$\boxed{\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}} \tag{8.6}$$

[3] $\cos^2\theta = (1+\cos 2\theta)/2$, $\sin^2\theta = (1-\cos 2\theta)/2$, $\cos^2\theta - \sin^2\theta = \cos 2\theta$, and $\cos\theta\sin\theta = (\sin 2\theta)/2$.

We note that tan $(180 + 2\theta_p) = \tan 2\theta_p$. Thus there are two angles—180° apart—that satisfy Equation (8.6), as shown in Figure 8.12: $\theta_1 = \theta_p$ and $\theta_2 = 90 + \theta_p$. Then from Figure 8.12 we obtain the following where the plus sign is taken with subscript 1 and the minus sign with subscript 2:

$$R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \sin 2\theta_{1,2} = \pm\tau_{xy}/R \qquad \cos 2\theta_{1,2} = \pm\frac{\sigma_{xx}-\sigma_{yy}}{2}/R$$

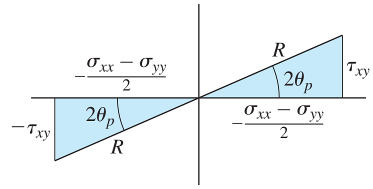


Figure 8.12 Two angles of principal planes.

Let the normal and shear stresses on planes with outward normals in the θ_1 and θ_2 directions be represented by σ_1, τ_1, and σ_2, τ_2, respectively. Substituting the sines and cosines of $2\theta_1$ and $2\theta_2$ into Equations (8.4) and (8.5), we obtain

$$\boxed{\sigma_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}} \tag{8.7}$$

$$\tau_{1,2} = 0 \tag{8.8}$$

where $\sigma_{1,2}$ represents the two stresses σ_1 and σ_2 with the plus sign to be taken with σ_1 and the minus sign with σ_2. Equation (8.8) shows that the planes on which σ_1 and σ_2 act are planes with zero shear stress. Planes with zero shear stresses are called **principal planes**. The normal direction to the principal planes is referred to as the **principal direction** or *principal axis*, and the angles the principal directions makes with the global coordinate system are called **principal angles**.

The normal stress on a principal plane is called the **principal stress** and the greatest principal stress is called **principal stress 1**. In defining greatest principal stress both the magnitude and the sign are considered. A stress of –2 MPa is greater than –10 MPa. Alternatively, if normal stresses are shown on an axis with negative values to the left of the origin and positive values to the right, then the rightmost normal stress is principal stress 1 denoted by σ_1.

The stresses in Equation (8.7) represent the maximum or minimum normal stress at a point. This implies that principal stresses are the maximum and minimum normal stresses at a point. Furthermore, the plane of principal stress 1 (θ_1) is 90° away from the plane of principal stress 2 (θ_2). In other words, principal planes are *orthogonal*. Adding Equation (8.1), Equation (8.3), and the principal stresses in Equation (8.7), we obtain

$$\boxed{\sigma_{nn} + \sigma_{tt} = \sigma_{xx} + \sigma_{yy} = \sigma_1 + \sigma_2} \tag{8.9}$$

Equation (8.9) shows that the sum of the normal stresses in an orthogonal coordinate system at a point does not depend on the orientation of the coordinate system in other words is *invariant*.

In summary:

- The sum of the normal stresses is invariant with the coordinate transformation.
- Principal stresses are maximum or minimum normal stresses
- Principal planes and principal directions are orthogonal.

8.2.2 Procedure for determining principal angle and stresses

Equation (8.6) will give us either θ_1 or θ_2. Thus it is not clear whether the principal angle found from Equation (8.6) is associated with σ_1 or σ_2. The problem can be resolved by the following procedure:

Step 1: Find θ_p from Equation (8.6).
Step 2: Substitute θ_p in Equation (8.1) to find a principal stress.
Step 3: Find the other principal stress from Equation (8.9).
Step 4: Decide which of the two principal stresses is principal stress 1.
Step 5: If the stress obtained from substituting θ_p into Equation (8.1) yields principal stress 1, then we report θ_p as principal angle 1 θ_1, otherwise we subtract (or add) 90° from θ_p and report the result as principal angle 1.

Step 6: Use Equation (8.7) as a check on the results.

From the definition of plane stress,[4] the plane with the outward normal in the z direction has zero shear stress. Therefore this plane is a principal plane and the normal stress σ_{zz} is a principal stress of zero value. In plane strain, the shear strain and hence shear stresses with subscript z are also zero. Hence in plane strain too, σ_{zz} is the third principal stress, but it is not zero. Using Figure 3.27, we can summarize as

$$\boxed{\sigma_3 = \sigma_{zz} = \begin{cases} 0 & \text{plane stress} \\ \nu(\sigma_{xx} + \sigma_{yy}) = \nu(\sigma_1 + \sigma_2) & \text{plane strain} \end{cases}} \tag{8.10}$$

The value of the third principal stress affects the maximum shear stress at a point, as will be seen in the next two sections.

8.2.3 In-Plane Maximum Shear Stress

Ductile materials yield when the maximum shear stress exceeds the yield stress. In bonded members, such as lap joints, the loads are transferred from one member to another through shear and are designed on the basis of the shear strength of the adhesive. In this section we develop equations for maximum shear stress.

In determining the maximum shear stress from Equation (8.2) we are considering only planes that can be obtained from rotation about the z axis, as shown in Figure 8.2. Thus we are not considering all possible planes that may pass through the point. The maximum shear stress on a plane that can be obtained by rotating about the z axis is called **in-plane maximum shear stress.** Let $\theta = \theta_s$ be the plane at which the in-plane maximum shear stress exists. By differentiating Equation (8.5) we get

$$\left.\frac{d\tau_{nt}}{d\theta}\right|_{\theta=\theta_s} = -2\frac{(\sigma_{xx} - \sigma_{yy})\cos 2\theta}{2} - 2\tau_{xy}\sin 2\theta\Big|_{\theta=\theta_s} = 0 \qquad \text{or} \qquad \tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}} \tag{8.11}$$

Once more, two angles can satisfy Equation (8.11). Letting $\bar{\theta}_1 = \theta_s$ and $\bar{\theta}_2 = 90 + \theta_s$, then from Figure 8.13 we obtain

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \sin 2\bar{\theta}_{1,2} = \mp\frac{\sigma_{xx} - \sigma_{yy}}{2}/R \qquad \cos 2\bar{\theta}_{1,2} = \pm\tau_{xy}/R$$

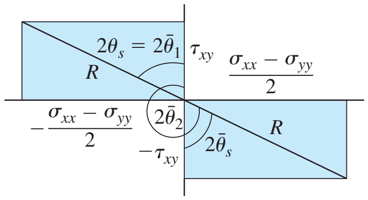


Figure 8.13 Two angles of maximum shear stress planes.

Let τ_{12} and τ_{21} be the shear stresses on the two planes defined by the angles $\bar{\theta}_1$ and $\bar{\theta}_2$. We can find the sines and cosines of $2\bar{\theta}_1$ and $2\bar{\theta}_2$, as shown in Figure 8.13, and substitute these quantities into Equations (8.4) and (8.5) to obtain

$$\sigma_{\text{av}} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \qquad |\tau_p| = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \tag{8.12}$$

where τ_p is the in-plane maximum shear stress obtained from the magnitude of the equation $\tau_{12} = -\tau_{21} = R$.

From Equations (8.6) and (8.11) we can obtain

$$\tan 2\theta_s = \frac{-1}{\tan 2\theta_p} = \tan(90° + 2\theta_p)$$

Therefore $\theta_s = 45 + \theta_p$. In other words, maximum in-plane shear stress exists on two planes, each of which is 45° away from the principal planes.

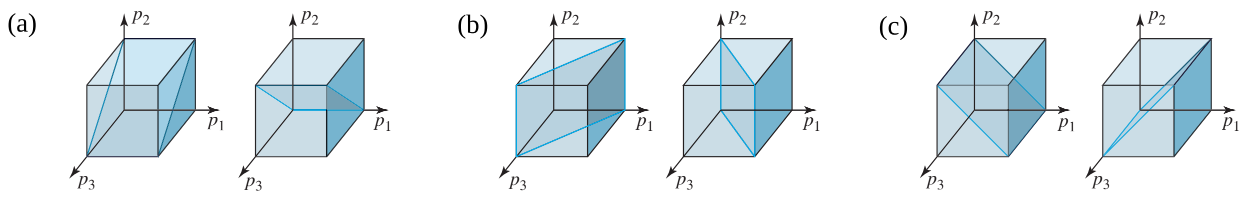
[4]See Section 1.3.2 for a definition of plane stress and Section 2.5.1 for a definition of plane strain. See Section 3.6 for the difference between plane stress and plane strain.

8.2.4 Maximum Shear Stress

The **maximum shear stress** at a point is the *absolute* maximum shear stress that acts on any plane passing through the point.

In the previous section we saw that as we rotate the coordinate system about the *z* axis (the third principal axis), the shear stress varies from a zero value, at a principal plane. It reaches a maximum value, given by Equation (8.10), on a plane that is 45° to a principal plane. Will this observation also be true if we rotate about principal axis 1 or 2? The answer is yes, because there is no distinction between the three principal planes passing through a point. On a cube six possible diagonal surfaces are at 45° to the cube surfaces. We consider each of the three rotations and show all the possibilities of maximum shear stress on the stress cube in Figures 8.14(a) through 8.14(c).

Figure 8.14 shows the six possible planes that are 45° to principal planes on which the maximum shear stress may exist if rotation is restricted about one of the three principal axis.



$$\tau_{23} = -\tau_{32} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{31} = -\tau_{13} = \frac{\sigma_3 - \sigma_1}{2} \qquad \tau_{21} = -\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}$$

Figure 8.14 Planes of maximum shear obtained by (a) rotating about principal axis 1. (b) by rotating about principal axis 2. (c) by rotating about principal axis 3.

The maximum shear stress at a point is the largest in magnitude of the three values obtained from Figures 8.14. It is written conveniently as

$$\boxed{\tau_{max} = \max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right)} \tag{8.13}$$

Equation (8.13) shows that the maximum shear stress value depends on principal stress 3. Equation (8.10) shows that the value of principal stress 3 depends on whether a plane stress or plane strain exists. In other words, the maximum shear stress value may be different in plane stress and in plane strain.

EXAMPLE 8.3

Solve Example 8.2 using Equations (8.1) and (8.2). Also determine the principal stresses, principal angle 1, and the maximum shear stress at the point.

PLAN

(a) We can determine the angle of the outward normal to the inclined plane containing the fiber. We can then use Equations (8.1) and (8.2) to find the normal and shear stresses on the plane. (b) Principal stress 3 is zero because the point is in plane stress. We follow the procedure in Section 8.2.2 to determine the principal angle 1 and principal stresses 1 and 2. (c) We can find the maximum shear stress from Equation (8.13).

SOLUTION

(a) The plane *AA* containing the fiber is at an angle of 30° from the *x* axis. Hence the direction of the outward normal is $\theta = 120^o$, as shown in Figure 8.15. Substituting in Equations (8.1) and (8.2), we obtain

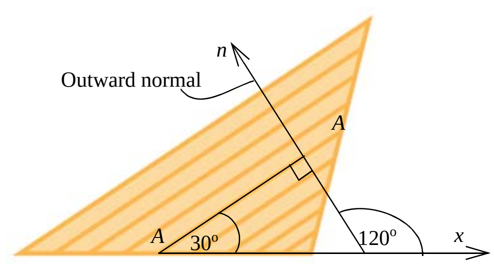


Figure 8.15 Outward normal to plane in Example 8.3.

$$\sigma_A = (30\ \text{MPa})\cos^2 120° + (-60\ \text{MPa})\sin^2 120° + 2(50\ \text{MPa})\sin 120°\cos 120° = -80.80\ \text{MPa} \qquad \text{(E1)}$$

$$\tau_A = -(30\ \text{MPa})\sin 120°\cos 120° + (-60\ \text{MPa})\sin 120°\cos 120° + (50\text{MPa})(\cos^2 120° - \sin^2 120°) = 13.97\ \text{MPa}$$

ANS. $\sigma_A = 80.8\ \text{MPa}\ (C) \qquad \tau_A = 14.0\ \text{MPa}$

(b) We follow procedure in Section 8.2.2.
Step 1: Find the principal angle from Equation (8.6),

$$\theta_p = \frac{1}{2}\arctan\left(\frac{50\ \text{MPa}}{[30\ \text{MPa} - (-60\ \text{MPa})]/2}\right) = \frac{1}{2}\arctan\left(\frac{50}{45}\right) = 24.01° \qquad \text{(E2)}$$

Step 2: Substitute the principal angle into Equation (8.1) to obtain one of the principal stresses,

$$\sigma_p = (30\ \text{MPa})\cos^2 24.01° + (-60\ \text{MPa})\sin^2 24.01° + 2(50\ \text{MPa})\sin 24.01°\cos 24.01° = 52.26\ \text{MPa} \qquad \text{(E3)}$$

Step 3: Note that $\sigma_{xx} + \sigma_{yy}$ = 30 MPa– 60 MPa= –30 MPa. From Equations (8.9) and (E3) the other principal stress is –82.26 MPa.
Step 4: The principal stress in Equation (E3) is greater than the stress in *step 3*, therefore it is principal stress 1.
Step5: The angle in Equation (E2) is principal angle 1.
Step 6: We can check our calculations of σ_1 and σ_2 using as shown,

$$\sigma_{1,2} = \frac{30\text{MPa} + (-60\ \text{MPa})}{2} \pm \sqrt{\left[\frac{30\ \text{MPa} - (-60\ \text{MPa})}{2}\right]^2 + (50\ \text{MPa})^2} = -15\ \text{MPa} \pm 67.26\ \text{MPa} \quad \text{or}$$

$$\sigma_1 = -15\ \text{MPa} + 67.26\ \text{MPa} = 52.26\ \text{MPa} \qquad \sigma_2 = -15\ \text{MPa} - 67.26\ \text{MPa} = -82.26\ \text{MPa}\text{-----Checks}$$

We report our answers as

ANS. $\sigma_1 = 52.3\ \text{MPa (T)} \qquad \sigma_2 = 82.3\ \text{MPa (C)} \qquad \sigma_3 = 0 \qquad \theta_1 = 24.0°\ \text{ccw}$

(c) The maximum shear stress at the point is half the maximum difference between the principal stresses, as per Equation (8.13), which in this problem is between σ_1 and σ_2,

$$\tau_{max} = \frac{52.26\ \text{MPa} - (-82.26\ \text{MPa})}{2} = 67.26\ \text{MPa}$$

ANS. $\tau_{max} = 67.3\ \text{MPa}$

COMMENTS

1. If the principal stress in step 3 was greater than the stress in Equation (E3), then it would be principal stress 1 and we could either add or subtract 90° from θ_p in Equation (E2) to report the principal angle one θ_1.
2. In finding normal stress σ_A and shear stress τ_A on the inclined plane, we substituted $\theta = 120°$ as the angle of the outward normal. It can be checked that if we substituted $\theta = 300°$, $\theta = -60°$, or $\theta = -240°$ into Equations (8.1) and (8.2), we would obtain the same values of σ_A and τ_A. This is illustrated in Figure 8.16. A plane passing through a point has two sides. The stresses on either side are the same, and hence the outward normal direction to either side can be used for computing normal and shear stresses on the plane. The direction of the outward normal can be measured by going counterclockwise (positive direction) or by going clockwise (negative direction) from the x axis. Equations (8.1) and (8.2) reflect this observation, as substitution of $(\theta + 180°)$, $(\theta - 180°)$, $(\theta + 360°)$, or $(\theta - 360°)$ in place of θ in Equations (8.1) and (8.2) results in the same expressions for the two equations. In other words, the values of the *stresses on a plane through a point are unique* and depend on the orientation of the plane only and not on how its orientation is described or measured.

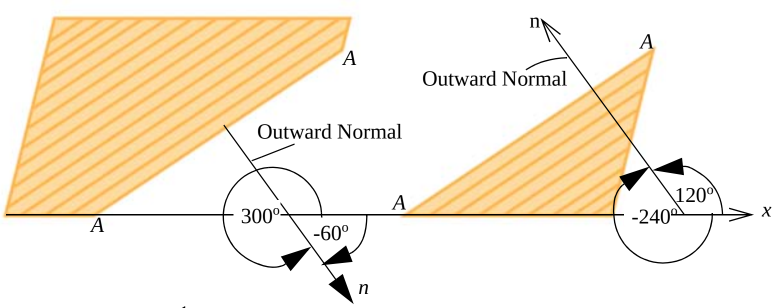


Figure 8.16 Different values of θ in Example 8.3.

3. If the point were in plane strain on a material with a Poisson ratio of $\frac{1}{3}$, then the third principal stress would be $\sigma_3 = \nu(\sigma_1 + \sigma_2) = -10$ MPa. Thus in this problem $\sigma_1 > \sigma_3 > \sigma_2$, and hence for this problem the maximum shear stress would be unaffected. But if the third principal stress value were not in between principal stresses 1 and 2, then by Equation (8.12) the maximum stress value would be affected.

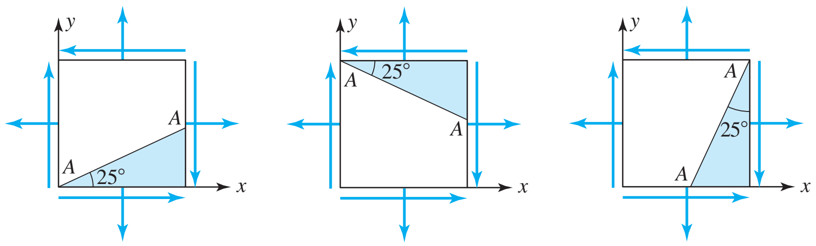
QUICK TEST 8.1

Time: 15 minutes/Total: 20 points

Each question is worth 2 points. Use the solutions given in Appendix E to grade yourself.

In Questions 1 through 3, what is the value of θ you would substitute in the stress transformation equations to find the normal and shear stresses on plane *AA*?

1. **2.** **3.**



4. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (T) and 20 ksi (C). What is the value of principal stress 1?

5. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (C) and 20 ksi (C). What is the value of principal stress 1?

6. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (T) and 20 ksi (T). What is the value of the maximum shear stress at that point?

7. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (T) and 20 ksi (C). What is the value of the maximum shear stress at that point?

8. At a point in plane stress, the principal stresses from the equations were found to be 5 ksi (C) and 20 ksi (C). What is the value of the maximum shear stress at that point?

In Questions 9 and 10, the angle θ_p from $\tan 2\theta_p = 2\tau_{xy}/(\sigma_{xx} - \sigma_{yy})$ was found to be –35°. On substituting this value into Equation (8.1), the normal stress was found to be 100 MPa (T). If the other principal stress is as given, then what is the value of principal angle 1?

9. 125 MPa (T).

10. 125 MPa (C).

8.3 STRESS TRANSFORMATION BY MOHR'S CIRCLE

In this section we develop a graphical technique for determining stresses on different planes passing through a point. Squaring Equations (8.4) and (8.5) and adding the result to eliminate θ, we obtain

$$\left(\sigma_{nn} - \frac{\sigma_{xx} + \sigma_{yy}}{2}\right)^2 + \tau_{nt}^2 = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2 \tag{8.14}$$

Consider the equation of a circle: $(x - a)^2 + y^2 = R^2$. We thus see that Equation (8.14) represents a circle with center coordinates $(a, 0)$ and radius R, where $a = (\sigma_{xx} + \sigma_{yy})/2$ and $R = \sqrt{[(\sigma_{xx} - \sigma_{yy})/2]^2 + \tau_{xy}^2}$. The circle is called **Mohr's circle** for stress and the coordinates of each point on the circle are the stresses (σ_{nn}, τ_{nt}). These are the normal and shear stresses on an arbitrarily oriented plane that is passing through the point at which the stresses σ_{xx}, σ_{yy}, and τ_{xy} are specified. Thus:

- Each point on Mohr's circle represents a unique plane that passes through the point at which the stresses are specified.
- The coordinates of the point on Mohr's circle are the normal and shear stresses on the plane represented by the point.

8.3.1 Construction of Mohr's Circle

We can construct Mohr's circle in five steps:

Step 1: Show the stresses σ_{xx}, σ_{yy}, and τ_{xy} on a stress cube and label the vertical plane V and the horizontal plane H, as shown in Figure 8.17*a*.

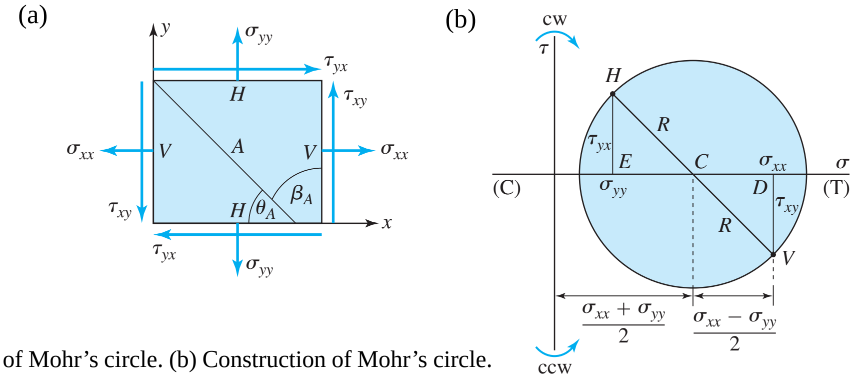


Figure 8.17 (a) Stress cube for construction of Mohr's circle. (b) Construction of Mohr's circle.

Step 2: Write the coordinates of points V and H as $V(\sigma_{xx},\ \tau_{xy})$ and $H(\sigma_{yy},\ \tau_{yx})$. The rotation arrow next to the shear stresses corresponds to the rotation of the cube caused by the set of shear stresses on planes V and H.

Step 3: Draw the horizontal axis with the tensile normal stress to the right and the compressive normal stress to the left, as shown in Figure 8.17*b*. Draw the vertical axis with the clockwise direction of shear stress up and the counterclockwise direction of rotation down.

Step 4: Locate points V and H and join the points by drawing a line. Label the point at which line VH intersects the horizontal axis as C.

Step 5: With C as the center and CV or CH as the radius, draw Mohr's circle.

To justify our construction, note the two triangles VCD and HCE in Figure 8.17*b* are identical, because

- angle VCD = angle HCE
- right angle CDV = right angle CEH
- side HE = side DV from the symmetry of shear stresses.

Thus side CE = side CD. In other words, C is the midpoint of DE, and the coordinates of the center point C are the mean values of the coordinates of points D and E, $[(\sigma_{xx} + \sigma_{yy})/2, 0]$, which represents the center of the circle, as in Equation (8.14). Since the length of side CD is the difference between the coordinates of D and C, we obtain the radius of the circle from the Pythagorean theorem as $\sqrt{[(\sigma_{xx} - \sigma_{yy})/2]^2 + \tau_{xy}^2}$, which is consistent with Equation (8.14).

An important point to remember is the differentiation made in Step 2 between τ_{xy} and τ_{yx}. Equations (8.4) and (8.5) tell us that the stresses on different planes are related by twice the angle between the planes. The vertical plane V and the horizontal plane H are 90° apart on the stress cube. Thus these planes must be 180° apart on Mohr's circle, as each point on Mohr's circle represents a unique plane. This implies that if the vertical plane V is located above the σ axis, then the horizontal plane H should be is located below the σ axis to maintain the 180° difference on Mohr's circle. If we use the conventional method of using the upper half-plane for positive values of shear stress and the lower half-plane for negative values of shear stress, then V and H will both be either in the upper half or in the lower half because the shear stresses $\tau_{xy} = \tau_{yx}$. By associating the clockwise and counterclockwise rotation, we can satisfy the requirement that the horizontal plane and the vertical plane on Mohr's circle be 180° apart. In summary:

- Angles between planes on a stress cube are doubled when plotted on Mohr's circle.
- The sign of shear stress cannot be determined directly from Mohr's circle, as the only information from Mohr's circle is that the shear stress causes the plane to rotate clockwise or counterclockwise.

8.3.2 Principal Stresses from Mohr's Circle

Figure 8.18 shows that the shear stresses are zero at points P_1 and P_2, and hence these points represent planes that are the principal planes. The normal stresses at these points are principal stresses, and these principal stresses can be found by inspection as the average normal stress plus or minus the radius. The stress at point P_1 is principal stress 1. In other words, principal stress 1 is the rightmost normal stress on Mohr's circle.

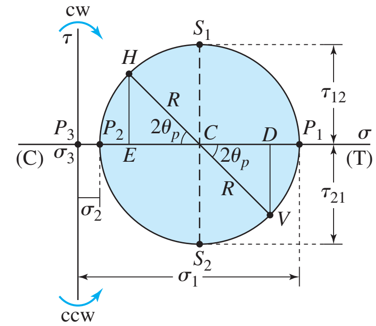


Figure 8.18 Principal stresses and in-plane maximum shear.

The angle between lines CV and CP_1 is $2\theta_p$, because all angles on Mohr's circle are double the actual angle between planes. The value of angle $2\theta_p$ can be found from the known dimensions of triangle VCD or triangle HCE, and we confirm the relationship given in Equation (8.6).

Is θ_p same as θ_1? In Figure 8.18 it would seem so. Figure 8.19 shows two examples in which the principal angle θ_p is different from principal angle θ_1. To clarify which angle we need consider that θ_1 is the angle of principal plane 1 (P_1) from the x direction. The outward normal to the vertical plane (V) is the x direction. On the Mohr's circle CV represents the x direction and CP_1 represents the principle direction 1. The angle between CV and CP_1 on the Mohr's circle is $2\theta_1$ in all cases, with counterclockwise rotation from CV as positive. As a final check, the value of θ_1 calculated from the Mohr's circle can be substituted into Equation (8.1), and the result should be principal stress one σ_1.

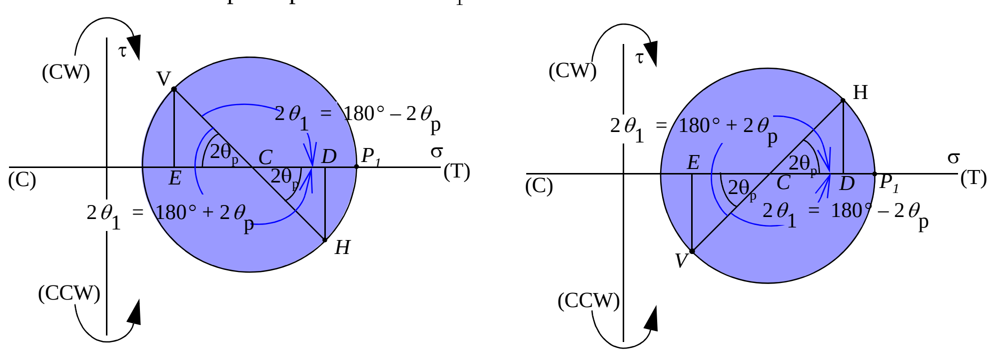


Figure 8.19 Examples of θ_1 different from θ_p.

Inspection of Figure 8.18 confirms that the maximum and the minimum normal stresses will be principal stresses. The observation that the principal planes are orthogonal is also obvious from Figure 8.18, as points P_1 and P_2, which represent principal planes, are at 180° on Mohr's circle. The coordinates of the center of the circle are the mean value of the normal stresses of any two points that are on a diameter of the circle. This confirms that the sum of normal stresses on orthogonal planes is invariant with respect to coordinate transformation.

8.3.3 Maximum In-Plane Shear Stress

The maximum in-plane shear stress will exist on the plane represented by points S_1 and S_2 in Figure 8.18, and its value is the radius of Mohr's circle, which is consistent with Equation (8.12). Points S_1 and S_2 are at 90° from points P_1 and P_2 on Mohr's circle, which is consistent with the earlier observation that the maximum in-plane shear stress exists on two planes which are at 45° to the principal planes.

8.3.4 Maximum Shear Stress

The circle in Figure 8.18 is the in-plane Mohr's circle, as the coordinate axes *n* and *t* are always in the *xy* plane. The in-plane circle represent all the planes that are obtained by rotating about principal axis 3. Let point P_3 represent principal plane 3. For plane stress problems, point P_3 coincides with the origin, as shown in Figure 8.20.

We can draw two more circles, one between P_1 and P_3 and the second between P_2 and P_3, as shown in Figure 8.20. These two circles represent rotation about principal axis 2 and principal axis 1, respectively, and are termed out-of-plane circles. The three circles together represent the complete state of stress at a point. *The maximum shear stress at a point is the radius of the biggest circle* [see Equation (8.13)]. This observation is also valid for plane strain. The difference is that the value of σ_3 will have to be found by using Equation (8.10), plotted on the horizontal axis, and labeled P_3.

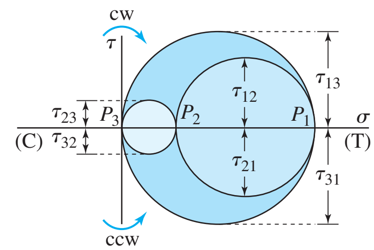


Figure 8.20 Maximum shear stress in plane stress.

8.3.5 Principal Stress Element

The principal stress element is a visualization aid used in the prediction of failure surfaces. Potential failure surfaces are the planes on which maximum normal or maximum shear stress acts—in other words, the principal planes and the plane of maximum shear. A **principal stress element** shows stresses on a wedge constructed from the principal planes and the plane of maximum shear stress.

We can describe our construction in terms of stress cubes, although they are not required once the method is understood:
Step 1: Draw a square and label the vertical side *V* and Horizontal side *H* as shown in Figure 8.21.
Step 2: Rotate the coordinate axis by an angle θ_1 and the cube along with it. The vertical plane (*V*) rotates to principal plane 1 (P_1). The horizontal plane (*H*) rotates to principal plane 2 (P_2) as shown in Figure 8.21.
Step 3: Draw the diagonals, that is planes 45° to the principal planes P_1 and P_2, representing the plane of maximum shear stress. Label the plane S_1 and S_2. Plane S_1 is 45° counterclockwise from plane P_1 in Figure 8.21*a*. On the Mohr's circle in Figure 8.18, if we rotate counterclockwise direction by 90° starting from point P_1, we reach point S_1. The stress wedge obtained is shown in Figure 8.21*b*. Similarly, starting from point P_1 on the Mohr's circle and rotating in the clockwise direction by 90°, we reach S_2 in Figure 8.18. The corresponding stress wedge is shown in Figure 8.21*c*.

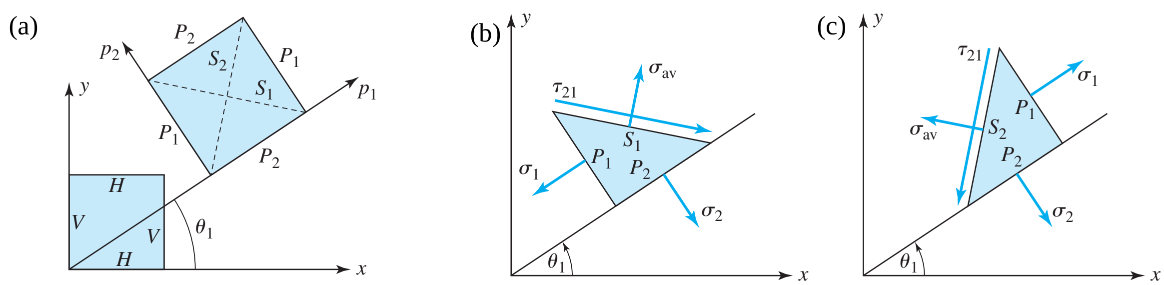


Figure 8.21 (a) Principal planes and planes of maximum in-plane shear stress. (b, c) Principal elements.

Step 4: Show principal stress 1 on plane P_1 and principal stress 2 on plane P_2. On the inclined plane show the maximum in-plane shear stress in the clockwise (or counterclockwise) direction if the inclined plane corresponds to the point in the upper (lower) half of Mohr's circle. Also show the average normal stress value on the inclined plane.

Figure 8.22 shows Mohr's circle and the principal element associated with the axial loading of a circular bar. Cast iron, a brittle material, fails from maximum tensile stress—that is, due to principal stress 1—and the failure surface is the principal plane 1. Aluminum, a ductile material, fails from maximum shear stress, and the failure surface is the plane of maximum shear S_2. Local imperfections dictated that the failure surface was S_2 rather than S_1, which from our theory is equally likely. An explanation of the failure surfaces due to torsion shown in Figure 8.1 is left to Problem 8.35.

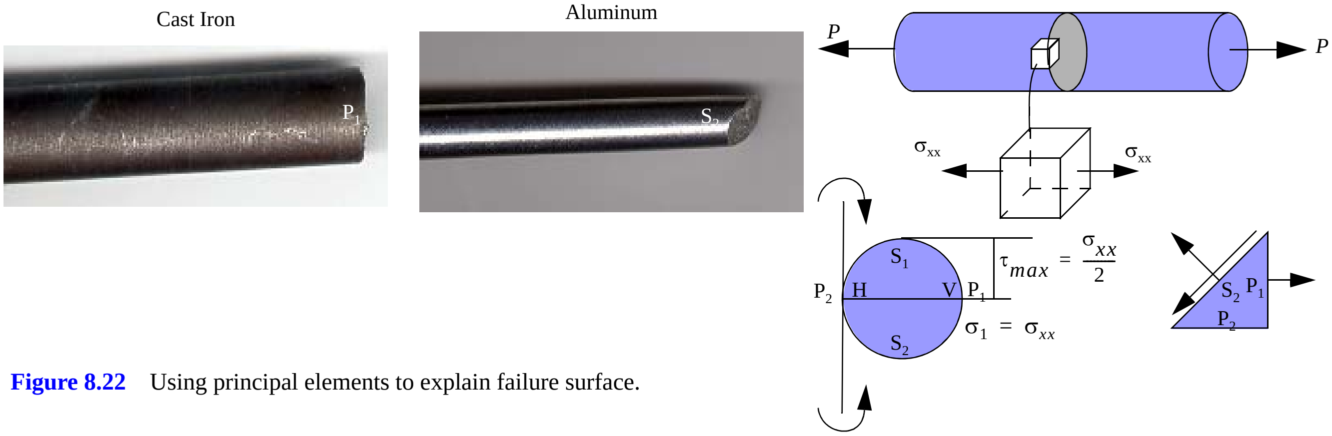


Figure 8.22 Using principal elements to explain failure surface.

8.3.6 Stresses on an Inclined Plane

The stresses on an inclined plane are found by first locating the plane on Mohr's circle and then determining the coordinates of the point representing the plane. This is achieved as follows:

Step 1: Draw the inclined plane on the stress cube and label it *A*, as shown in Figure 8.17.

Step 2: Locate the inclined plane on Mohr's circle as will be described later and label it *A*, as shown in Figure 8.23.

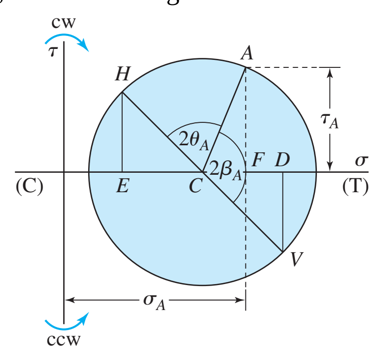


Figure 8.23 Stresses on inclined plane.

Step 3: Calculate the coordinates of point *A*.

Step 4: Determine the sign of shear stress.

There are two alternatives in Step 2.

1. On the stress cube in Figure 8.17, the inclined plane *A* is at an angle θ_A from the horizontal plane in the clockwise direction. Starting from line *CH* on Mohr's circle in Figure 8.23, we rotate by an angle $2\theta_A$ in the clockwise direction and then draw the line *CA*. Point *A* represents plane *A*.
2. On the stress cube, the inclined plane *A* is at an angle β_A from the vertical plane in the counterclockwise direction. Starting from line *CV* we rotate by an angle $2\beta_A$ in the counterclockwise direction and draw the line *CA*. Point *A* represents plane *A*.

We note from the stress cube that $\theta_A + \beta_A = 90°$, and from Mohr's circle we see that $2(\theta_A + \beta_A) = 180°$. This once more confirms that each point on Mohr's circle represents a unique plane, and it is immaterial how we locate that point on the circle.

Step 3 is the reverse of Step 2 in the construction of Mohr's circle and is a simple problem in geometry. Angle *FCA* can be found from the known angles. Radius *CA* of the circle is known, and lengths *FA* and *CF* can be found from triangle *FCA*. The coordinates of point *A* are (σ_A, τ_A). The direction of rotation is recorded as clockwise because point *A* is in the upper plane in Figure 8.23. If point *A* had been in the lower plane, we would have recorded a counterclockwise rotation with the shear stress.

To determine the sign of shear stress, we start by drawing the shear stress such that the inclined plane *A* rotates in the same direction as was recorded with the coordinates in Step 3. A local coordinate system is established, and if the shear stress is in the positive tangent direction, then it is positive. The two possibilities are shown in Figure 8.24. In both cases the shear stress is negative.

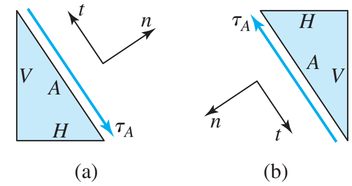


Figure 8.24 Sign of shear stress on an incline.

EXAMPLE 8.4

For each of the two states of stress below, plot the normal stress and the shear stress on a plane versus θ—the angle of the outward normal of the plane—draw Mohr's circle for each state of stress, on each diagram identify the planes at $\theta_A = 30^\circ$, $\theta_B = 75^\circ$, $\theta_D = 105^\circ$, and $\theta_E = 150^\circ$.

Case I: The uniaxial stress state is $\sigma_{xx} = \sigma_{0,}$ and all other stress components are zero.

Case II: The state of pure shear is $\tau_{xy} = \tau_0$, and all other stress components are zero.

PLAN

We can substitute the given states of stress into Equations (8.1) and (8.2) to obtain σ_{nn} and τ_{nt} as a function of θ and plot them. For each state of stress we can draw the stress cube, write the coordinates of planes V and H, and draw Mohr's circle. Starting from point V on Mohr's circle we can rotate by twice the angle in the counterclockwise direction to get the various points on the circle.

SOLUTION

Case I: Substituting the stress components for the uniaxial stress states into Equations (8.1) and (8.2), we obtain

$$\sigma_{nn} = \sigma_0 \cos^2\theta \qquad \text{or} \qquad \frac{\sigma_{nn}}{\sigma_0} = \cos^2\theta \tag{E1}$$

$$\tau_{nt} = -\sigma_0 \sin\theta\cos\theta \qquad \text{or} \qquad \frac{\tau_{nt}}{\sigma_0} = -\sin\theta\cos\theta \tag{E2}$$

Equations (E1) and (E2) can be plotted as shown in Figure 8.25*a*. We can also draw the stress cube showing uniaxial tension and record the coordinates of points V and H (Figure 8.25*b*). With no shear stress, the two points V and H are on the horizontal axis forming the diameter of Mohr's circle with the center at C and radius $R = \sigma_0/2$. Starting from point V on Mohr's circle we rotate counterclockwise by twice the angle θ to get the inclined planes, as shown in Figure 8.25*c*.

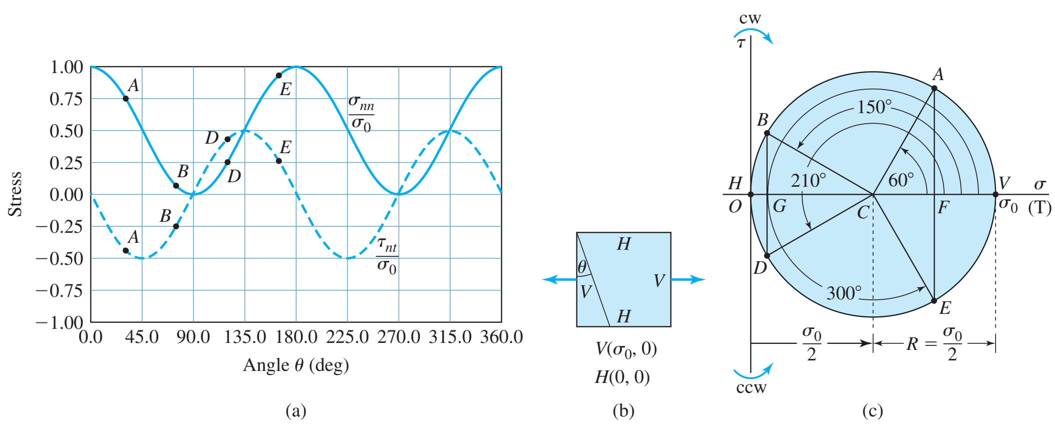


Figure 8.25 Stresses on inclined plane in uniaxial state of stress.

Case II: Substituting the stress components for state of pure shear in Equations (8.1) and (8.2), we obtain

$$\sigma_{nn} = 2\tau_0 \sin\theta\ \cos\theta \qquad \text{or} \qquad \frac{\sigma_{nn}}{\tau_0} = 2\sin\theta\ \cos\theta \tag{E3}$$

$$\tau_{nt} = \tau_0(\cos^2\theta - \sin^2\theta) \qquad \text{or} \qquad \frac{\tau_{nt}}{\tau_0} = \cos^2\theta - \sin^2\theta \tag{E4}$$

Equations (E3) and (E4) can be plotted as shown in Figure 8.26. We can also draw the stress cube showing pure shear and record the coordinates of points V and H. With normal stress, the two points V and H are on the vertical axis forming the diameter of Mohr's circle with center at C and radius $R = \tau_0$. Starting from point V on Mohr's circle we rotate counterclockwise by twice the angle θ to get the inclined planes, as shown in Figure 8.26.

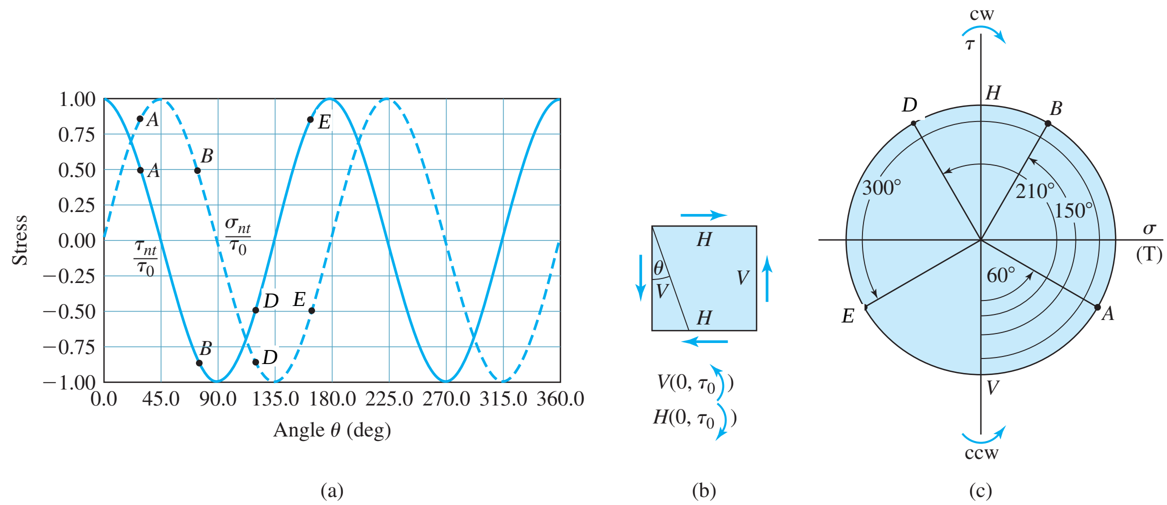


Figure 8.26 Stresses on inclined plane in state of pure shear stress.

COMMENTS

1. The example shows the relationship of planes on a graph and on a Mohr's circle, emphasizing that angles are double when plotted on a Mohr's circle.
2. Recall that in axial members and on top and bottom surface of beam, the only non-zero stress component is σ_{xx}. For both these cases, Figure 8.25 shows that the plane of maximum shear is 45° to the axis of the member and the value of maximum shear stress is half the value the normal stress at that point.
3. Recall that in torsion of circular shafts only non-zero stress component is $\tau_{x\theta}$. For a shaft in torsion, Figure 8.26 shows that the principal planes are 45° to the axis of the shaft and the magnitude of the principal stresses is same as the magnitude of torsional shear stress at that point.
4. Observations in comments 3 and 4 should be remembered as these can be used in strength based design of brittle and ductile materials.

EXAMPLE 8.5

(a) Solve Example 8.2 using Mohr's circle. Determine (b) the principal stresses, and principal angle 1; (c) the maximum shear stress at the point. (d) Show the results on a principal element.

PLAN

We can follow the steps outlined for the construction of Mohr's circle in Section 8.3.1 and calculate the various quantities from geometry.

SOLUTION

Step 1: We draw the stress cube and label the vertical and horizontal planes V and H, as shown in Figure 8.27*a*.

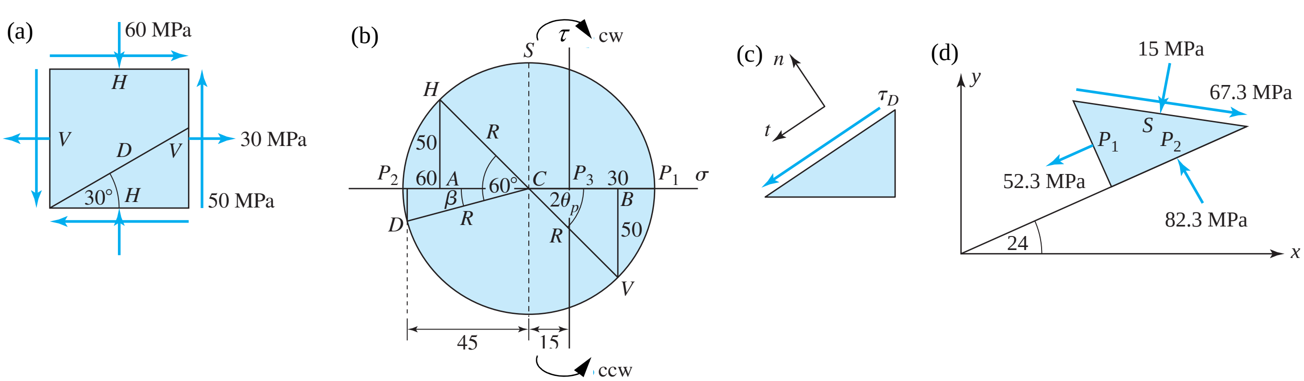


Figure 8.27 (a) Stress cube. (b) Mohr's circle. (c) Sign of shear stress. (d) Principal element

Step 2: From Figure 8.27*a* we note the coordinates of points V and H as

$$V(30, 50) \qquad H(-60, 50)$$

Step 3: We draw the axes for Mohr's circle, as shown in Figure 8.27*b*.

Step 4: We locate points *V* and *H* and join the two points. The coordinates of the center *C* are the mean value of the coordinates of points *A* and *B*, that is, $[30 + (-60)]/2 = -15$. The distance *AC* is $[30 - (-15)] = 45$, from which the radius as: $R = \sqrt{45^2 + 50^2} = 67.27$. The angle θ_p can then be calculated from triangle *VCB* as

$$\tan 2\theta_p = \frac{50}{45} = 1.1111 \qquad \text{or} \qquad 2\theta_p = 48.01° \tag{E1}$$

(a) Plane *D* is 30° counterclockwise from the horizontal plane in the stress cube in Figure 8.27*a*. We rotate by twice the angle (i.e., by 60°) from line *CH* on Mohr's circle in Figure 8.27*b* and draw the line *CD*. The coordinates (σ_D, τ_D) of point *D* are calculated from geometry,

$$\beta = 60° - 2\theta_p = 11.99° \qquad \sigma_D = -(15 + R\cos\beta) = -80.8 \qquad \tau_D = R\sin\beta = 13.97 \tag{E2}$$

We next draw the plane and determine the sign of τ_D. The shear stress must cause the plane to rotate counterclockwise, because point *D* on Mohr's circle was in the lower half of the plane. We establish a local coordinate system and determine that the shear stress has a positive sign (Figure 8.27*c*). The results are

ANS. $\sigma_D = 80.8 \text{ MPa (C)} \qquad \tau_D = 14.0 \text{ MPa}$

(b) The principal stresses are the coordinates of points P_1 and P_2,

$$\sigma_1 = -15 + R = 52.27 \qquad \sigma_2 = -15 - R = -82.27 \tag{E3}$$

$$\theta_p = \frac{48.02}{2} = 24.01° \tag{E4}$$

The principal stresses and principal angle 1 for the problem are

ANS. $\sigma_1 = 52.3 \text{ MPa (T)} \qquad \sigma_2 = 82.3 \text{ MPa (C)} \qquad \sigma_3 = 0 \qquad \theta_1 = 24.0° \text{ ccw}$

(c) The circle between P_1 and P_3 and the one between P_2 and P_3 are both inscribed within the in-plane circle shown in Figure 8.27*b*. Thus in this problem the in-plane maximum shear stress and the maximum shear stress at the given point are the same,

ANS. $\tau_{max} = 67.3 \text{ MPa}$

(d) Figure 8.27*d* shows the principal element which is drawn following the procedure in Section 8.3.5.

COMMENTS

1. The Mohr's circle method looks longer than the method of equations because of the explanation needed for the geometric constructions. However, computationally the difference between the two methods is small. The advantage of the method of equations is that the equations can be programmed and solved by computer. The advantage of using Mohr's circle is that it helps in the intuitive understanding of stress transformation.
2. The maximum shear stress shown in Figure 8.27 is negative, as can be deduced by establishing a local *n*, *t* coordinate system.

EXAMPLE 8.6

A 30-in.-long thin cylindrical tube is to transmit a torque of 25π in.·kips. The tube is to be fabricated by butt welding a $\frac{1}{16}$-in.-thick steel plate (G = 12,000 ksi) along a spiral seam, as shown in Figure 8.28. Buckling considerations limit the allowable stress in steel to 10 ksi in compression. The allowable shear stress in the weld is 12 ksi, and the allowable tensile stress in the weld is 20 ksi. Stiffness considerations limit the relative rotation of the two ends to 3°. Determine the minimum outer radius of the tube to the nearest $\frac{1}{16}$ in.

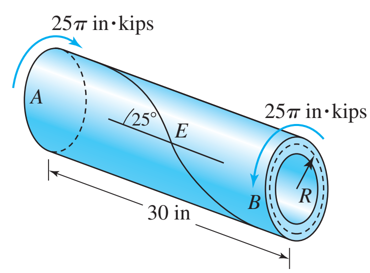


Figure 8.28 Geometry of shaft and loading in Example 8.6.

PLAN

We are required to find *R* to satisfy four limitations. By inspection we see that the weld material would be put into compression, and hence we can ignore the constraint on the maximum tensile stress in the weld. We can use the thin-tube approximation for computing *J* in terms of *R*, as given in Problem 5.28. For this simple loading we can determine the internal torque by inspection as $\boldsymbol{T} = +25\pi$ in.·kips. We can find $\phi_B - \phi_A$ in terms of *R* using Equation (5.12) and find one limit on *R*. Since the tube is thin, we can further assume that the torsional shear stress will not vary significantly from the inside to the outside. Hence we can evaluate it at the centerline radius. We can find the torsional shear stress in terms of R using Equation (5.10). We can then find the maximum compressive stress in steel and the shear stress in the seam using either Mohr's circle or the method of equations and find two other limits on R. We choose R that satisfies all limits and round it upward to the nearest $\frac{1}{16}$ in.

SOLUTION

For thin tubes, from Problem 5.28, we have

$$J = 2\pi R^3 t \tag{E1}$$

Substituting Equation (E1), $\boldsymbol{T} = 25\pi$ in.·kips, $G = 12{,}000$ ksi, $t = (1/16)$ in. , and $L = 30$ in. into Equation (5.12), we obtain the rotation of the section at B with respect to the section at A, which should be less than 3° = 0.0524 rad. We thus find one limit on R,

$$\phi_B - \phi_A = \frac{(25\pi\ \text{in.}\cdot\text{kips})(30\ \text{in.})}{(12{,}000\ \text{kips/in.}^2)[2\pi R^3(1/16)\ \text{in.}]} = \frac{0.5\ \text{in.}^3}{R^3} \le 0.0524 \qquad \text{or} \qquad R \ge 2.12\ \text{in.} \tag{E2}$$

Substituting Equation (E1), $\boldsymbol{T} = 25\pi$ in.·kips, and $\rho = R$ into Equation (5.10),

$$\tau_{x\theta} = \frac{(25\pi\ \text{in.}\cdot\text{kips})R}{[2\pi R^3(1/16)\ \text{in.}]} = \frac{200\ \text{kips}}{R^2} \tag{E3}$$

The direction of shear stress at point E can be determined using subscripts or intuitively, as shown in Figure 8.29. A two-dimensional representation of the stress cube is shown in Figure 8.29*c*. The directions of shear stress on the other surfaces are determined using the fact that pairs shear stresses either point toward the corner or away from the corner, as shown in Figure 8.29*c*

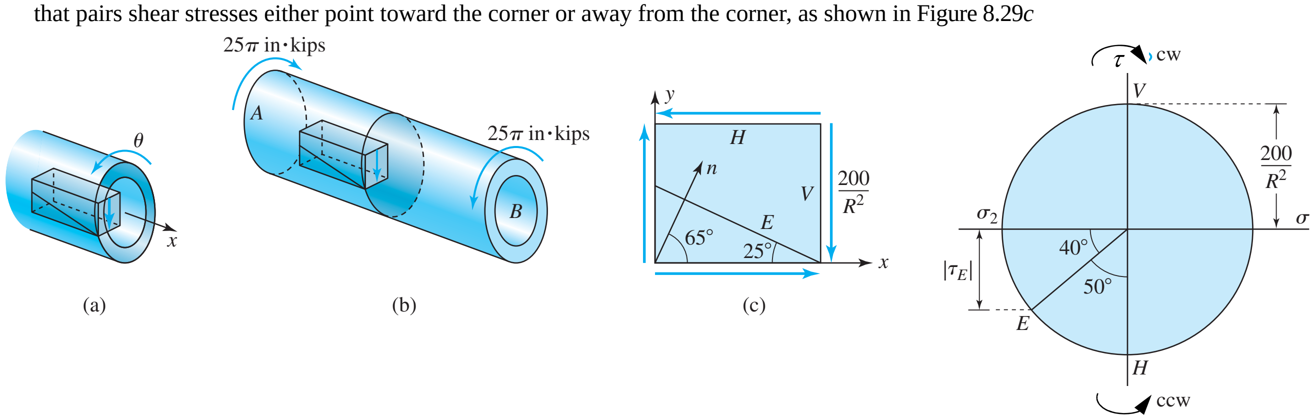


Figure 8.29 Direction of shear stress (a) by subscript.(b) by inspection. (c) Stress cube. (d) Stresses on the inclined plane using Mohr's circle.

We can determine the maximum compressive stress and the shear stress on the inclined plane using either Mohr's circle for stress or the method of equations.

Mohr's circle method: We record the coordinates of point V as $V(0, 200/R^2)$ and the coordinates of point H as $H(0, 200/R^2)$ and draw Mohr's circle as shown in Figure 8.29*d*. We determine principal stress 2,

$$\sigma_2 = \frac{(200\ \text{kips})}{R^2}\ (\text{C}) \tag{E4}$$

We then locate point E on Mohr's circle and determine the shear stress,

$$|\tau_E| = \frac{(200\ \text{kips})}{R^2}\sin 40° = \frac{(128.56\ \text{kips})}{R^2} \tag{E5}$$

Method of equations: We note from Figure 8.29*c* that $\tau_{xy} = -(200\ \text{kips})/R^2$ ksi, the normal stresses are zero, and the angle of the normal to the inclined plane is 65°. Substituting this information into Equations (8.7) and (8.2), we obtain principal stress 2,

$$\sigma_2 = 0 - \sqrt{0 + \left(\frac{200\ \text{kips}}{R^2}\right)^2} = \frac{(200\ \text{kips})}{R^2}\ (\text{C}) \tag{E6}$$

and the shear stress on the inclined plane,

$$\tau_E = \left(-\frac{200\ \text{kips}}{R^2}\right)(\cos^2 65° - \sin^2 65°) \qquad \text{or} \qquad |\tau_E| = \frac{(128.56\ \text{kips})}{R^2} \tag{E7}$$

We now consider the limitation on the compressive stress in steel and the shear stress in the weld and find two other limitations on R,

$$\sigma_2 = \frac{(200\ \text{kips})}{R^2} \le (10\ \text{kips/in.}^2) \qquad \text{or} \qquad R \ge 4.472\ \text{in.} \tag{E8}$$

$$|\tau_E| = \frac{(128.56\ \text{kips})}{R^2} \le (12\ \text{kips/in.}^2) \qquad \text{or} \qquad R \ge 3.273\ \text{in.} \tag{E9}$$

Comparing Equations (E2), (E8), and (E9), we see that the minimum value of R that will satisfy all three conditions is 4.472 in., given by Equation (E8). Rounding upward to the closest $\frac{1}{16}$ in., we obtain the value of the centerline circle radius.

ANS. $R = 4\frac{1}{2}$ in.

COMMENTS

1. Consider the error due to the thin-tube approximation for J. The outer radius for the tube is $R_o = R + t/2 = 4\frac{17}{32}$ in., and the inner radius of the tube is $R_i = R - t/2 = 4\frac{15}{32}$ in. . Thus the value of exact $J = \pi(R_o^4 - R_i^4)/2$ would be 35.786 in.^{4}. The value of approximate $J = 2\pi R^3 t$ for thin tubes is 35.785 in.^{4}, a percentage difference from exact J of 0.003%, which is negligible for any engineering calculation.
2. Consider the approximation of uniform shear stress in the tube. If we substitute $\rho = 4\frac{17}{32}$ in. into Equation (5.10), we obtain a value of 9.945 ksi at the outer surface. If we substitute $\rho = 4\frac{1}{2}$ in. into Equation (5.10), we obtain a value of 9.876 ksi at the centerline, for a difference of 0.69%, which is also negligible.
3. If we do not use the thin-tube approximation, then we have to find roots of nonlinear equations by numerical methods (see Problem 8.70). The thin-tube approximation can be used if $t < R/10$.
4. In this problem the direction (sign of τ_{xy}) of shear stress is important only to ensure that the weld is subjected to compressive stress and not tensile stress. The magnitude of the shear stress in the weld is unaffected by the direction (sign of τ_{xy}) of shear stress. (This will not be true in combined loading problems.)

EXAMPLE 8.7

A T-section beam is constructed by gluing two pieces of wood together, as shown in Figure 8.30. The maximum normal stress in the glue joint is to be limited to 2 MPa in tension and the maximum shear stress is to be limited to 1.7 MPa. Determine the maximum value for load *w*.

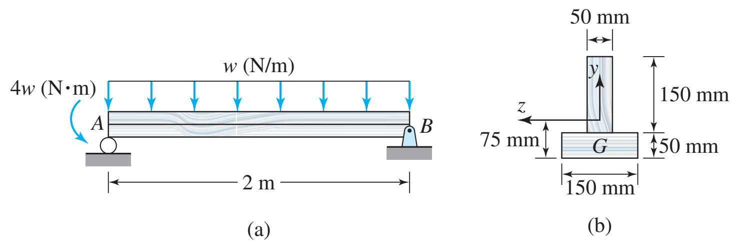


Figure 8.30 Beam and loading in Example 8.7.

PLAN

We are given that the principal stress 1 in the glue cannot exceed 2 MPa, and the maximum shear stress in the glue cannot exceed 1.7 MPa. We can draw the shear force and bending moment diagrams in terms of *w*. We then find the bending normal stress in glue and the bending shear stress in glue, considering in the sections where the moment M_z is maximum and the shear force V_y is maximum. We can draw stress cubes at the various sections and find principal stress 1 and the maximum shear stress in terms of *w*. Using the limiting values we can find the value of *w*.

SOLUTION

We can find the reaction forces at A and B by considering the free-body diagram of the entire beam. We then draw the shear force and bending moment diagrams, as shown in Figure 8.31*a*. The maximum shear force and bending moment are given by

$$(V_y)_{max} = -3w \text{ N} \qquad M_{max} = -4w \text{ N}\cdot\text{m} \tag{E1}$$

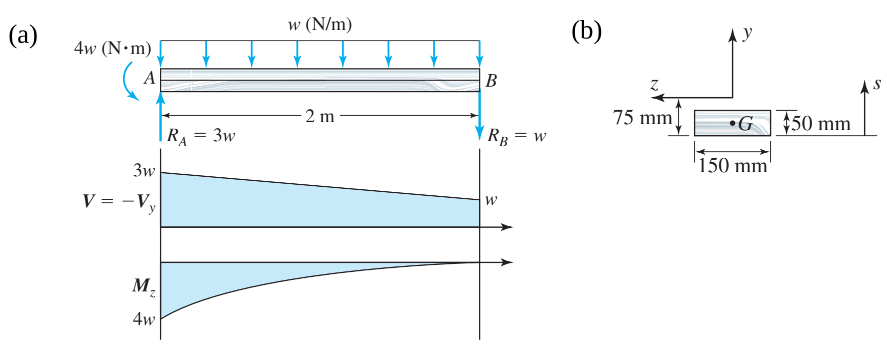


Figure 8.31 (a) Shear force and bending moment diagrams. (b) Calculation of Q_z.

Since the maximum bending moment and the maximum shear force exist in the section at A, the maximum principal flexural normal stress in glue and the maximum shear stress in glue will also exist in the section at A. The area moment of inertia of the cross section can be calculated as $I_{zz} = 53.125(10^6)\ \text{mm}^4$. Substituting $y_G = -25$ mm and Equation (E1) into Equation (6.12), we obtain the flexural normal stress at G as

$$\sigma_G = -\frac{-(4w\ \text{N}\cdot\text{m})[-25(10^{-3})\ \text{m}]}{53.125(10^{-6}\ \text{m}^4)} = -1882w\ \text{N/m}^2 \tag{E2}$$

We can draw the area A as the area between the bottom surface and point G, as shown in Figure 8.31*b*, and find Q_z,

$$Q_G = (150\ \text{mm})(50\ \text{mm})(-50\ \text{mm}) = -375(10^3)\ \text{mm}^3 \tag{E3}$$

Substituting Equations (E1) and (E3) into Equation (6.27), we obtain the shear stress,

$$\tau_{xs} = -\frac{(-3w\ \text{N})[-375(10^{-6})\ \text{m}^3]}{[53.125(10^{-6})\ \text{m}^4][50(10^{-3})\ \text{m}]} = -423.5w\ \text{N/m}^2 \tag{E4}$$

We can draw the stress cube and show on it the stresses in Equations (E2) and (E4). Note that with s positive upward, the shear stress τ_{xs} on the surface with the outward normal in the positive x direction will be downward to reflect the negative sign in Equation (4). Alternatively, the sign of the shear stress τ_{xy} is negative as the shear force $\mathbf{V}_y$ is negative. We can draw Mohr's circle as shown in Figure 8.32. The radius R is given by

$$R = \sqrt{(941w\ \text{N/m}^2)^2 + (423.5w\ \text{N/m}^2)^2} = 1032w\ \text{N/m}^2 \tag{E5}$$

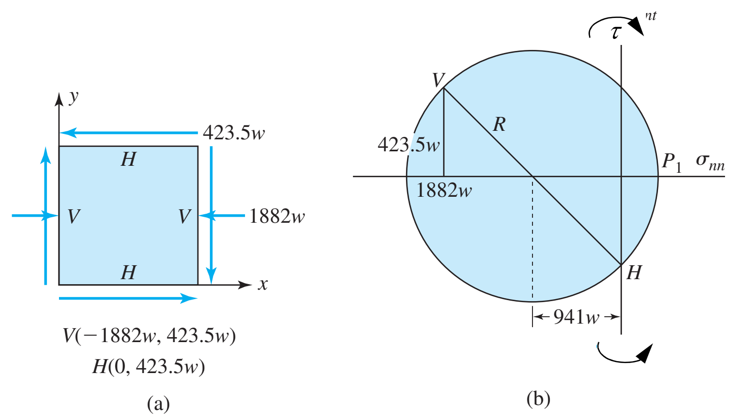


Figure 8.32 Mohr's circle in Example 8.7.

Principal stress 1 σ_1 can be found, and the limiting value on σ_1 yields one limit on w,

$$\sigma_1 = (-941w\ \text{N/m}^2) + (1032w\ \text{N/m}^2) = 91w\ \text{N/m}^2 \le 2(10^6)\ \text{N/m}^2 \quad \text{or} \quad w \le 22{,}000\ \text{N/m} \tag{E6}$$

The maximum shear stress τ_{max} is the radius of the circle, and the limiting value on τ_{max} yields the other limit on w,

$$\tau_{max} = 1032w\ \text{N/m}^2 = (1.7)10^6\ \text{N/m}^2 \quad \text{or} \quad w \le 1647\ \text{N/m} \tag{E7}$$

Comparing Equations (E6) and (E7), we conclude that the maximum permissible value of w is

ANS. $w_{max} = 1647\ \text{N/m}$

COMMENTS

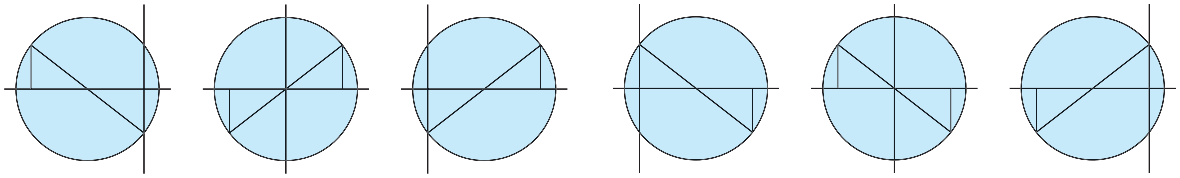
1. The maximum normal stress in glue is $\sigma_2 = -941w\ \text{N/m}^2 - 1032w\ \text{N/m}^2 = -1973w\ \text{N/m}^2$, which in magnitude is nearly 20 times greater than σ_1. However it is not even considered, because it is compressive and so does not affect the failure of glue.
2. The maximum bending normal stress in wood is at the top of the beam at section A and its value is $\sigma_{xx} = -9411.8w\ \text{N/m}^2$. The maximum bending shear stress is at the neutral axis and its value is $\tau_{xy} = -441.2w\ \text{N/m}^2$. At the top of the beam the only nonzero stress is σ_{xx}. Thus, from Figure 8.25, the maximum shear stress is $\tau_{max} = \sigma_{xx}/2 = -4705.9w\ \text{N/m}^2$, which is an order of magnitude greater than the maximum bending shear stress. If we had to consider shear strength failure of wood, then we would use the maximum value of $4705.9w\ \text{N/m}^2$ in our calculation.
3. Comment 2 emphasizes the difference between maximum stresses in a material and maximum bending stresses. Comment 1 emphasizes that it is not the magnitude of the maximum stress but the type of stress that causes failure in a material and is important in design.

QUICK TEST 8.2

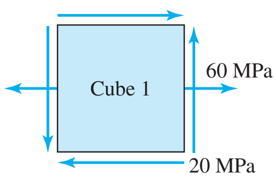
Time: 20 minutes/Total: 20 points

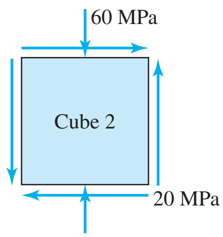
Each question is worth 2 points. Use the solutions given in Appendix E to grade yourself.

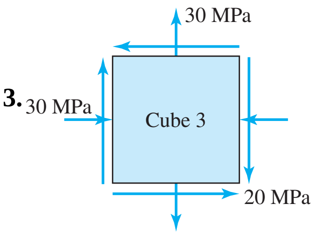
In Questions 1 through 3, associate the stress cubes with the appropriate Mohr's circle given:



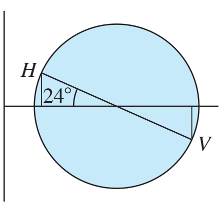
Circle *A* Circle *B* Circle *C* Circle *D* Circle *E* Circle *F*

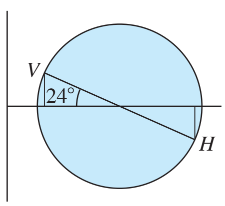
1. 

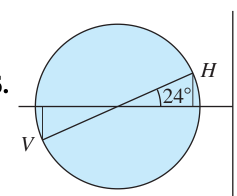
2. 

3. 

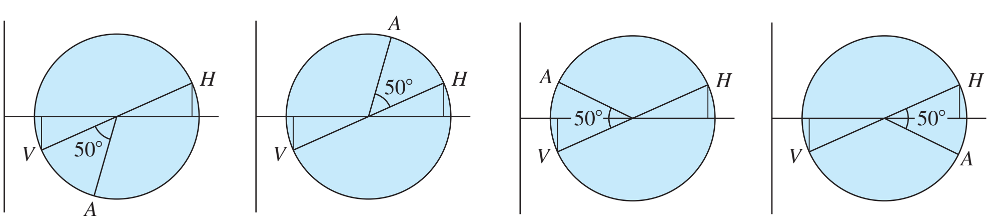
In Questions 4 through 6, Mohr's circles correspond to a plane state of stress. Determine the two possible values of principal angle 1 θ_1 in each question.

4. 

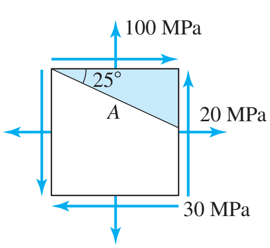
5. 

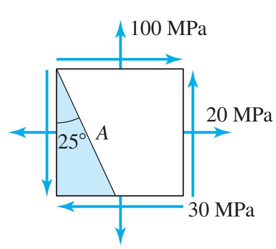
6. 

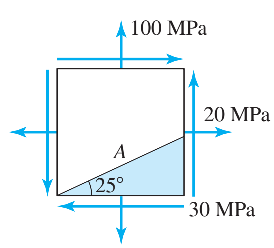
In Questions 7 through 9, Mohr's circle corresponds to the state of stress shown. Associate plane *A* on the stress cube with the corresponding Mohr's circles showing plane *A*, which are given:



Circle *A* Circle *B* Circle *C* Circle *D*

7. 

8. 

9. 

10. Plane *E* passes through a point that has the state of stress given in Question 7. The normal and shear stresses on plane *E* were found to be $\sigma_E = 90$ MPa (T) and $\tau_E = -40$ MPa . What are the normal stress and the shear stress on the plane that is 90° counterclockwise from plane *E*?

MoM in Action: Sinking of Titanic

On April 14, 1912, on the fourth day of her maiden voyage, the *RMS Titanic* (Figure 8.33*a*) struck an iceberg in Atlantic ocean. The ship dubbed “the unsinkable” sank in less than three hours (Figure 8.33*b*), with a loss of 1500 people, but it was not the last startling catastrophic failure. On January 15, 1919, a large molasses tank burst in Boston, killing 21 people and injuring another 150. On March 12, 1928, near Los Angeles, the St. Francis Dam failed suddenly, and the resulting flood killed more than 600 people. And on April 28, 1988, on Aloha flight 243 between Hilo and Honolulu in Hawaii, the fuselage of the aircraft blew open at 24,000 feet, killing a flight attendant and injuring another eight. The initiating cause of each tragedy was different. The final mechanism of catastrophic failure, however, was the same in each case—*brittle fracture*.

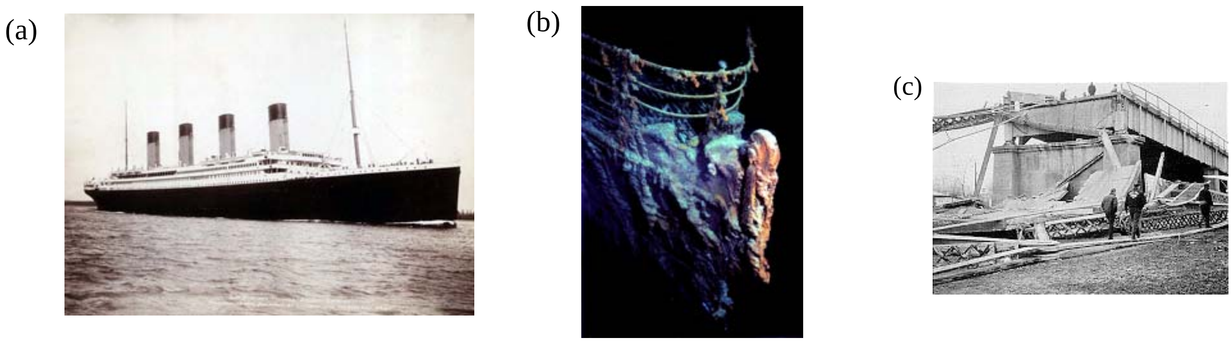


Figure 8.33 (a) RMS Titanic (b) Titanic bow at bottom of ocean. (c) Sliver Bridge.

To highlight some fracture issues, consider a crack in a windshield. It can stay that way for years, grow slowly, or grow suddenly into a crisscross pattern. A crack must reach critical length that is material and stress dependent before it starts growing. The slow growth is due to *ductile fracture,* while the rapid growth is due to *brittle fracture*.

In ductile materials, the high stresses at the crack tip causes plastic deformation, thus blunting the tip of the crack. Subsequent growth depends on there being sufficient energy in the deformed solid to create new crack surfaces, resulting in ductile fracture. On December 15, 1967, for example, Silver Bridge (Figure 8.33*b*), which spanned the Ohio River, collapsed owing to ductile fracture of a pin, killing 46 and injuring 9 other people. National Bridge Inspection Standards (NBIS) were established soon after and now require periodic inspection of all bridges.

In brittle materials, once a crack reaches critical length, fracture proceeds at extremely high speeds (in the neighborhood of 7000 ft/sec). The breaking in two of the tanker *S.S. Schenectady* (see Figure 1.1) is a vivid example. Brittle fracture can also occur in ductile materials. Polymer in a windshield, a ductile material, may become brittle due to sunlight and aging. The four days of sailing in icy cold water of Atlantic made the metal of *Titanic* more brittle, increasing its propensity to brittle fracture. Failure due to fatigue (see Section 3.10) always produces brittle fracture surfaces, even in ductile materials.

Tensile principal stress one (σ_1) is the dominant cause of crack growth. Though brittle fracture can cause catastrophe, it can also be used productively. By scoring glass with a diamond cutter, or a plasterboard with a utility knife, we introduce a sharp tipped crack in the material. By bending the glass and plasterboard we create tensile stress at the crack tip and thus produce a clean surface break. *Brittle coating method* is an experimental technique in which a brittle material is spray coated onto a machine component. As the component is loaded, cracks perpendicular to principal direction one (θ_1) start appearing in the coating.

A good design ensures that the critical crack length in a structure is never smaller then what can be detected by instruments. Regular inspections can then provide the warning needed to fix the crack. Drilling a hole to blunt the crack tip, putting obstacles like rivets in path of crack growth, or inserting glue as in windshield cracks are some of the ways of arresting crack growth from becoming catastrophic.

Brittle fracture is in itself neither good nor bad. It is, however, nature’s constant reminder of a fundamental engineering lesson: *to command nature we need to obey it first*.

PROBLEM SET 8.2

8.29 Show that Equations (8.7) and (8.8) are correct by substituting the values of sines and cosines following Figure 8.12 into Equations (8.4) and (8.5).

8.30 Show that Equation (8.12) is correct by substituting the appropriate sines and cosines following Figure 8.13 into Equations (8.4) and (8.5).

8.31 Derive Equation (8.7) by starting from Equation (8.3).

8.32 Draw the Mohr's circle and determine the normal and shear stresses on plane A in Figure P8.32.

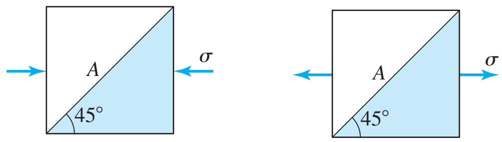


Figure P8.32 (a) (b)

8.33 Draw the Mohr's circle and determine the normal and shear stresses on plane A in Figure P8.33.

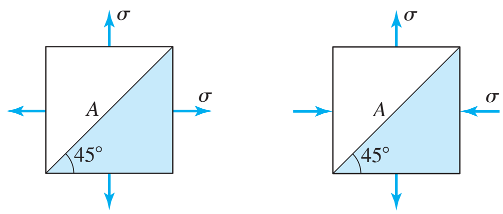


Figure P8.33 (a) (b)

8.34 Draw the Mohr's circle and determine the normal and shear stresses on plane A in Figure P8.34.

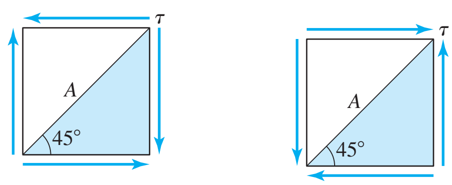


Figure P8.34 (a) (b)

8.35 Explain the failure surfaces due to torsion that are shown in Figure 8.1.

8.36 Solve Problem 8.19 by the method of equations.

8.37 Solve Problem 8.19 by Mohr's circle.

8.38 Solve Problem 8.20 by the method of equations.

8.39 Solve Problem 8.20 by Mohr's circle.

8.40 Solve Problem 8.21 by the method of equations.

8.41 Solve Problem 8.21 by Mohr's circle.

8.42 In a thin body (plane stress) the stress element is as shown in Figure P8.42. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

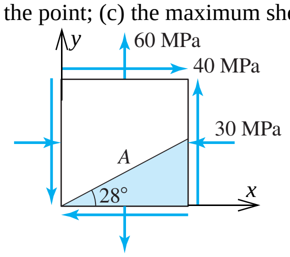


Figure P8.42

8.43 In a thin body (plane stress) the stress element is as shown in Figure P8.43. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

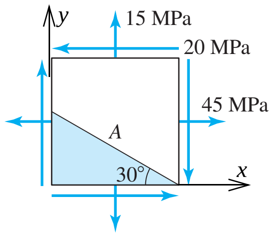


Figure P8.43

8.44 In a thin body (plane stress) the stress element is as shown in Figure P8.44. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

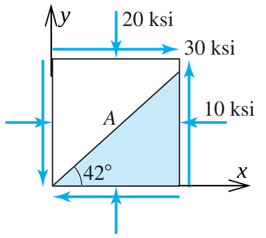


Figure P8.44

8.45 In a thick body (plane strain) the stress element is as shown in Figure P8.45. The Poisson's ratio of the material is $\nu = 0.3$. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

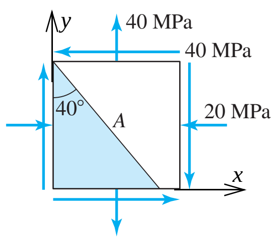


Figure P8.45

8.46 In a thick body (plane strain) the stress element is as shown in Figure P8.46. The Poisson's ratio of the material is $\nu = 0.3$. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

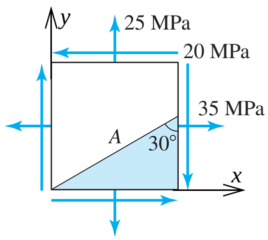


Figure P8.46

8.47 In a thick body (plane strain) the stress element is as shown in Figure P8.47. The Poisson's ratio of the material is $\nu = 0.3$. Determine (a) the normal and shear stresses on plane A; (b) the principal stresses at the point; (c) the maximum shear stress at the point. (d) Draw the principal element.

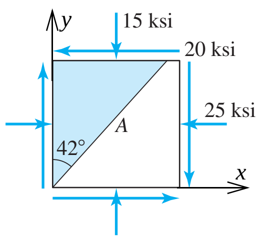


Figure P8.47

8.48 A thin plate (E=30,000 ksi, ν = 0.25) is subjected to a uniform stress σ = 10 ksi as shown in Figure P8.48. Assuming plane stress, determine the maximum shear stress in the plate.

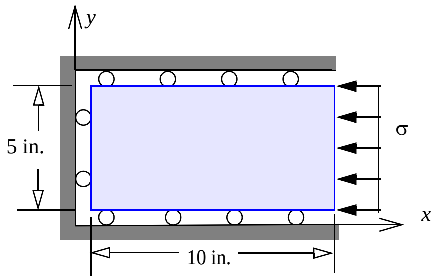


Figure P8.48

8.49 The strains at a point in plane stress and the material properties are as given below. Determine the principal stresses, principal angle 1, and the maximum shear stress at the point. (See Problem 3.79.)

$$\varepsilon_{xx} = 500\ \mu \qquad \varepsilon_{yy} = 400\ \mu \qquad \gamma_{xy} = -300\ \mu \qquad E = 200\ \text{GPa} \qquad \nu = 0.32$$

8.50 The strains at a point in plane stress and the material properties are as given below.Determine the principal stresses, principal angle 1, and the maximum shear stress at the point. (See Problem 3.80.)

$$\varepsilon_{xx} = -3000\ \mu \qquad \varepsilon_{yy} = 1500\ \mu \qquad \gamma_{xy} = 2000\ \mu \qquad E = 70\ \text{GPa} \qquad G = 28\ \text{GPa}$$

8.51 The strains at a point in plane stress and the material properties are as given below. Determine the principal stresses, principal angle 1, and the maximum shear stress at the point. (See Problem 3.81.)

$$\varepsilon_{xx} = -800\ \mu \qquad \varepsilon_{yy} = -1000\ \mu \qquad \gamma_{xy} = -500\ \mu \qquad E = 30{,}000\ \text{ksi} \qquad \nu = 0.3$$

8.52 A rectangle inscribed on an aluminum (10,000 ksi, ν = 0.25) plate is observed to deform into the colored shape shown in Figure P8.52. Determine the principal stresses, principal angle 1, and the maximum shear stress.

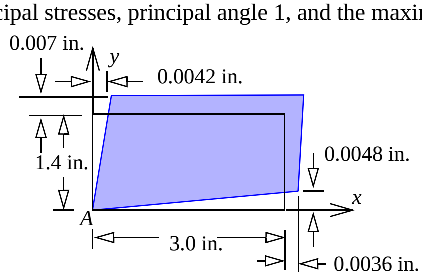


Figure P8.52

In Problems 8.53 through 8.55, the difference in the principal stresses $\sigma_1 - \sigma_2$ *and the principal direction 1* θ_1 *from the x axis were measured by photoelasticity (see* Section 8.4.1*) at several points and are given in each problem. The sum of the principal stresses* $\sigma_1 + \sigma_2$ *was found from elasticity*[5] *and is also given. Assuming plane stress, determine the stresses* σ_{xx}, σ_{yy}, *and* τ_{xy} *at the point.*

Problem	$\sigma_1 - \sigma_2$	θ_1	$\sigma_1 + \sigma_2$
	10 ksi	-15°	6 ksi
	3 ksi	+25°	-17 ksi.
	5 ksi	+35°	5 ksi.

8.56 A broken 2-in.× 6-in. wooden bar was glued together as shown in Figure P8.56. Determine the normal and shear stresses in the glue when F =12 kips.

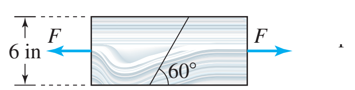


Figure P8.56

8.57 A 10-lb picture is hung using a wire of diameter of 1/8 in. as shown in Figure P8.57. What is the maximum shear stress in the wire?

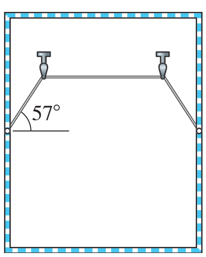
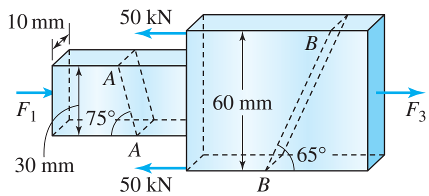


Figure P8.57

[5]Equations of elasticity show that $(\partial^2/\partial x^2)(\sigma_1 + \sigma_2) + (\partial^2/\partial y^2)(\sigma_1 + \sigma_2) = 0$. This differential equation can be solved numerically or analytically with the appropriate boundary conditions.

8.58 Two rectangular bars of thickness 10 mm are loaded as shown in Figure P8.58. For a force F_1 = 25 kN, determine the normal and shear stress on planes *AA* and *BB*.



FigureP8.58

8.59 An aluminum rod (*E*= 70 GPa) and a steel rod (*E*= 210 GPa) are securely fastened to a rigid plate that does not rotate during the application of the load *P* as shown in Figure P8.59. The diameter of aluminum and steel rods are 20 mm and 10 mm, respectively. If the applied force *P*=25 kN, determine the maximum shear stress in aluminum and steel.

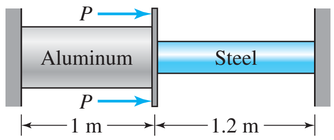


Figure P8.59

8.60 Determine the normal and shear stresses in the seam of the shaft passing through point *A* at an angle θ= 60º to the axis of a solid shaft of 2-in. diameter, as shown in Figure P8.60.

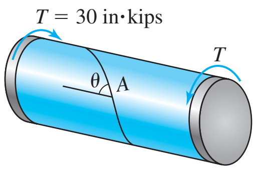


Figure P8.60

8.61 Two circular steel shafts (G =12,000 ksi) of diameter 2 in. are securely connected to an aluminum shaft (G =4,000 ksi) of diameter 1.5 in. as shown in Figure P8.61. Determine (a) the maximum normal stress in the shaft; (b) the normal and shear stress on a weld running through point E.

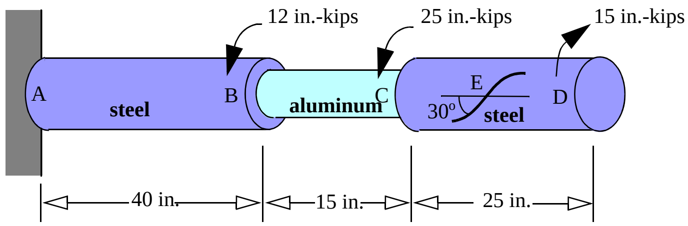


Figure P8.61

8.62 Two pieces of solid shafts of diameter 75 mm are securely connected to a rigid wheel as shown in Figure P8.62. The shaft material has a modulus of rigidity *G* = 80 GPa. If the applied torque *T* = 8 kN-m, determine the maximum normal stress in the shaft.

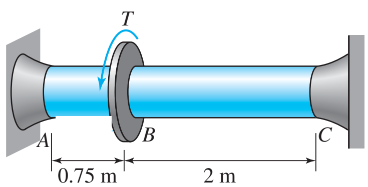


Figure P8.62

8.63 If the applied force in Figure P8.63 is P = 1.8 kN, determine the maximum normal and shear stress at points *A*, *B*, and *C* which are on the surface of the beam.

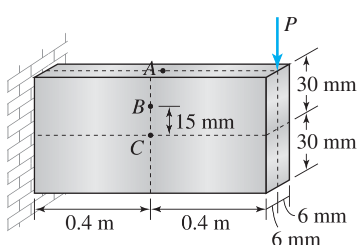


Figure P8.63

8.64 If the applied force in Figure P8.64 is P = 1.8 kN, determine the maximum normal and shear stress at points at points *A*, *B*, and *C* which are on the surface of the beam.

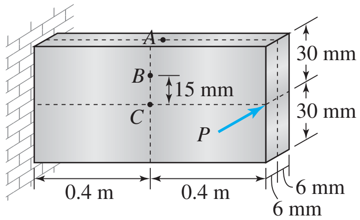


Figure P8.64

8.65 Two pieces of lumber are glued together to form the beam shown in Figure P8.65. The intensity of the distributed load is *w* = 25 lb/in. Determine (a) the maximum shear stress in the beam; (b) the maximum normal stress in the glue.

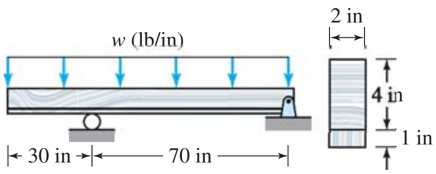


Figure P8.65

8.66 Two pieces of wood are glued together to form a beam, as shown in Figure P8.66. The applied moment M_{ext}= 9 in.-kips. Determine (a) the maximum shear stress in the beam; (b) the maximum normal stress in the glue.

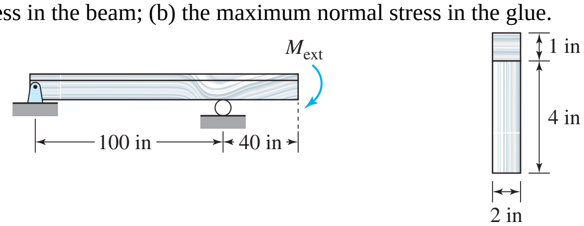


Figure P8.66

Design problems

8.67 A broken 2 in. x 6 in. wooden bar is glued together as shown in Figure P8.56. The allowable normal and shear stress in the glue are 600 psi (T) and 400 psi, respectively. Determine the maximum force F to the nearest pound that can be transmitted by the bar.

8.68 A rigid bar *ABC* is supported by two aluminum cables (E = 10,000 ksi) as shown in Figure P8.68. The allowable shear stress in aluminum is 20 ksi. If the applied force *P* = 10 kips, determine the minimum diameter of cables *CE* and *BD* to the nearest 1/16 in. Both cables have the same diameter.

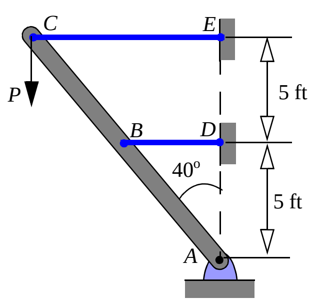


Figure P8.68

8.69 A thin tube of $\frac{1}{8}$-in. thickness has a mean diameter of 6 in. What is the maximum torque the tube can transmit if the allowable normal stress in compression is 10 ksi?

8.70 Solve Example 8.6 again, but without the thin-tube approximation.

8.71 An aluminum rod (E_{al} = 70 GPa) and a steel rod (E_s = 210 GPa) are securely fastened to a rigid plate that does not rotate during the application of load *P*, as shown in Figure P8.59. The diameters of the aluminum and steel rods are 20 mm and 10 mm, respectively. The allowable shear stresses in aluminum and steel are 120 MPa and 150 MPa. Determine the maximum force *P* that can be applied to the rigid plate.

8.72 A shaft is welded along the seam that makes an angle of θ = 60° to the axis of a 2 in. diameter shaft as shown in Figure P8.72. The allowable normal and shear stresses in the weld are 15 ksi (T) and 10 ksi, respectively. Determine the maximum torque T_{ext} that can be applied.

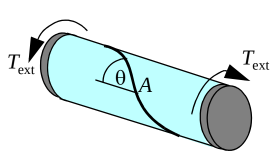
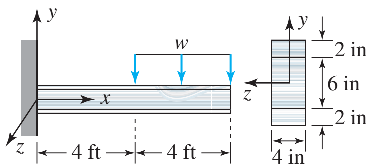


Figure P8.72

8.73 Two pieces of solid shaft of 75-mm diameter are securely connected to a rigid wheel, as shown in Figure P8.62. The shaft material has a modulus of shear rigidity $G = 80$ GPa and an allowable normal stress in tension or compression of 90 MPa. Determine the maximum torque T that can act on the wheel.

8.74 A cantilever beam is constructed by gluing three pieces of timber, as shown in Figure P8.74. The allowable shear stress in the glue is 300 psi and the allowable tensile stress is 200 psi. The allowable tensile or compressive stress in wood is 2000 psi. Determine the maximum intensity of the distributed load w.

Figure P8.74

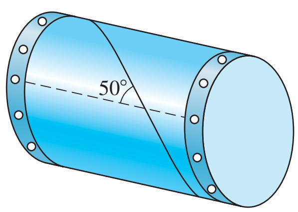


8.75 Two pieces of lumber are glued together to form the beam shown in Figure P8.66. The allowable shear stress in the in the wood is 600 psi and the allowable tensile stress in the glue is 650 psi (T). Determine the maximum moment M_{ext} that can be applied.

8.76 Determine the thickness of a steel plate required for a thin cylindrical boiler with a centerline diameter of 2.5 m, if the maximum tensile stress is not to exceed 100 MPa and the maximum shear stress is not to exceed 60 MPa, when the pressure in the boiler is 1800 kPa.

8.77 A thin cylindrical tank is fabricated by butt welding a $\frac{1}{2}$-in.-thick plate, as shown in Figure P8.77. The centerline diameter of the tank is 4 ft. The maximum tensile stress of the plate cannot exceed 30 ksi. The normal and shear stresses in the weld are limited to 25 ksi and 18 ksi, respectively. What is the maximum pressure the tank can hold?

Figure P8.77



Stretch yourself

8.78 By multiplying the matrices, show that the following matrix equation is the same as Equations (8.1), (8.2), and (8.3):

$$[\sigma]_{nt} = [T]^T[\sigma][T] \qquad \text{where} \qquad [\sigma]_{nt} = \begin{bmatrix} \sigma_{nn} & \tau_{nt} \\ \tau_{tn} & \sigma_{tt} \end{bmatrix} \qquad [T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad [\sigma] = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{yx} & \sigma_{yy} \end{bmatrix}$$

$[T]^T$ represents the transpose of the matrix $[T]$. The matrix $[T]$ is the transformation matrix that relates the x and y coordinates to the n and t coordinates.

8.79 Show that the eigenvalues of the matrix $[\sigma]$ are the principal stresses given by Equation (8.7).

8.80 Using the wedge shown in Figure P8.26, show that the normal stress on an inclined plane is related to the stresses in Cartesian coordinates by the equation

$$\sigma_{nn} = \sigma_{xx}n_x^2 + \sigma_{yy}n_y^2 + \sigma_{zz}n_z^2 + 2\tau_{xy}n_xn_y + 2\tau_{yz}n_yn_z + 2\tau_{zx}n_zn_x \tag{8.15}$$

8.81 Figure P8.81 show eight (octal) planes that make equal angles with the principal planes. These planes are called *octahedral planes*. Though the signs of the direction cosines change with each plane, the magnitude of the direction cosines is the same for all eight planes; that is, $|n_x| = 1/\sqrt{3}$, $|n_y| = 1/\sqrt{3}$, and $|n_z| = 1/\sqrt{3}$. The normal stress and the shear stress on the octahedral planes σ_{oct} and τ_{oct} are given by equations below. Using Equation (8.15), obtain Equation (8.16).

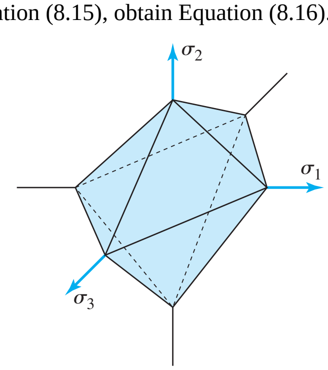


Figure P8.81

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{8.16}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{8.17}$$

Computer problems

8.82 On a machine component made of steel ($E = 30{,}000$ ksi, $G = 11{,}600$ ksi) the following strains were found: $\varepsilon_{xx} = [100(2x + y) + 50]\ \mu$, $\varepsilon_{yy} = -100(2x + y)\ \mu$, and $\gamma_{xy} = 200(x - 2y)\ \mu$. Assuming plane stress, determine the principal stresses, principal angle 1, and the maximum shear stress every 30° on a circle of radius 1 around the origin. Use a spreadsheet or write a computer program for the calculations.

8.83 The stresses in polar coordinates around a hole in a very large plate subject to a uniform stress σ (Figure P8.83) are given by equations below. On a ship deck with a manhole having a diameter of 2 ft, it was estimated that $\sigma = 10$ ksi. Calculate the principal stresses every 15° at a radius of 18 in. Use a spreadsheet or write a computer program for the calculations.

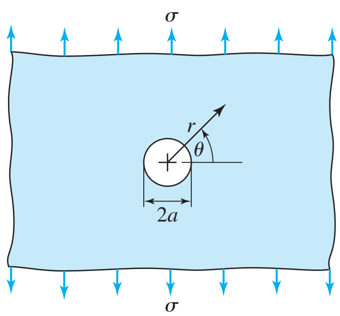


Figure P8.83

$$\sigma_{rr} = \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 - \frac{4a^2}{r^2} + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{8.18a}$$

$$\sigma_{\theta\theta} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \tag{8.18b}$$

$$\tau_{r\theta} = \frac{\sigma}{2}\left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4}\right)\sin 2\theta \tag{8.18c}$$

QUICK TEST 8.3

Time: 15 minutes/Total: 20 points

Answer true or false. If false, give the correct explanation. Each question is worth 2 points. Use the solution given in Appendix E to grade yourself.

1. In plane stress there are two principal stresses and in plane strain there are three principal stresses.

2. Principal planes are always orthogonal.

3. For a given state of stress at a point, the principal stresses depend on the material.

4. Depending on the coordinate system used for finding stresses at a point, the values of the stress components differ. Hence the principal stress at that point will depend on the coordinate system in which the stresses were found.

5. Planes of maximum shear stress are always at 90° to principal planes.

6. The sum of the normal stresses in an orthogonal coordinate system is independent of the orientation of the coordinate system.

7. If principal stress 1 is tensile and principal stress 2 is compressive, then the in-plane maximum shear stress and the maximum shear stress are the same for plane *stress* problems.

8. If principal stress 1 is tensile and principal stress 2 is compressive, then the in-plane maximum shear stress and the maximum shear stress are the same for plane *strain* problems.

9. Two planes passing through a point can be represented by the same point on Mohr's circle.

10. Two points on Mohr's circle can represent the same plane.

*8.4 CONCEPT CONNECTOR

Photoelasticity is an experimental method for deducing stress information from observing the effects on light as it passes through a transparent material that is stressed. The analysis is complex, and you will study it in advanced courses, but the principles behind this remarkable visual representation of stress are simpler. To explain photoelasticity, we must first understand the transmission of light.

8.4.1 Photoelasticity

The color of light depends on the frequency of the wave, and it may be affected by transmission through different-colored materials. White light is a mixture of different frequencies. However, even waves of the same frequency can travel differently in different materials. That is because light waves may lie in different planes at a point. (In quantum mechanics, we learn to think of light also as particles, called *photons*, which can then vibrate in different planes). Light of a single frequency is *monochromatic*, and light with one plane of vibrations is *polarized*.

A *polarizer* is a material that permits light rays to pass only if their plane of vibration lies parallel to one axis, called the *polarizing* axis. If two polarizers are used, then the second polarizer is called an *analyzer.* If the polarizer and the analyzer have polarizing axes arranged perpendicular to each other, then no light will pass through, and a *dark field* will be produced. If the polarizer and the analyzer have polarizing axes parallel to each other, then all the light that passes through the first polarizer will pass through the analyzer, too, and a *light field* will be produced.

The velocity of light depends on the material through which it is passing. Usually the velocity of light is the same in all directions, and the material is said to be *isotropic*. However, when some transparent materials are stressed, light travels through the material at different speeds along different planes of vibration. These materials have *two* polarizing axes, at right angles to each other. And, unlike in a polarizer, neither axis simply selects the light. Rather, the velocities of light along the two polarizing axes are different. This behavior of light is called *birefringence*, and materials in which light velocities are different in the two polarizing axes are called *birefringent*.

When light passes through birefringent material, a ray along one axis takes longer to pass through it than a ray through the other axis. In other words, the ray along the slows axis reaches the same amplitude as the faster ray after a time Δt. The time difference Δt is called *retardation* time.

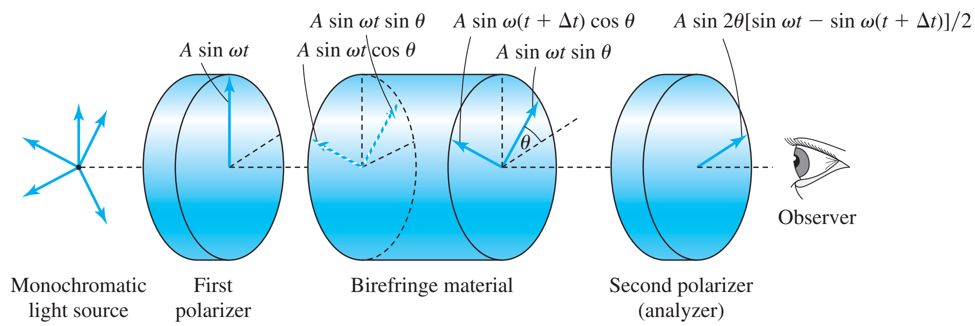


Figure 8.34 Transmission of light in photoelasticity.

Figure 8.34 shows light originating from a monochromatic source. It then encounters not just a polarizer and an analyzer, but also a birefringent material placed between them. As light passes first through the polarizer, it emerges with its plane of vibration parallel to the vertical axis. Now suppose that the two axes of the birefringent material are set at an angle θ with respect to the vertical. Light passing through the polarizer will have components along each of these axes, and the component of light along the slow axis is retarded by time Δt. In other words, two rays emerge from the birefringent material. The observer sees only the horizontal components of these two rays, because the second polarizer will pass only light that is parallel to the horizontal axis. In sum, the light that the observer sees depends on two variables—the angle θ of the birefringent material's fast axis with respect to the analyzer axis and the retardation time Δt.

What makes *photoelasticity* possible is that birefringence is directly related to stress, as observed by James Clerk Maxwell, the Scottish mathematician and physicist, in 1857:

1. The principal axes of stresses in a birefringent material are the fast and slow axes, with the fast axis corresponding to principal stress 1.

2. The retardation time is proportional to the difference of principal stresses.

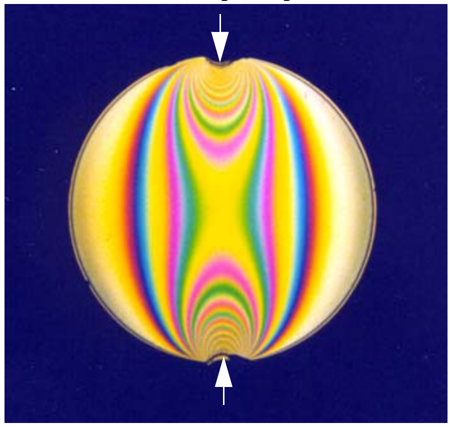


Figure 8.35 Photoelastic fringes showing principal stress difference. (Courtesy Professor I. Miskioglu.)

Suppose we start with a light field, so that the polarizer and analyzer axes are parallel. Note that $\sin(n\pi)$ is zero. We conclude that the observer will see dark spots where $\theta = n\pi/2$. Lines that connect these dark spots, called *isoclinic lines*, thus give us the direction of principal stress 1 at different points. If we start instead with a dark field, then the isoclinic lines join points of maximum transmission of light, where $\sin 2\theta = 1$.

If we start with a light field, the observer will also not see any light if the term in brackets in Figure 8.34 equals zero. At these points $\Delta t = 2n\pi/\omega$, and lines connecting these points are called *fringes*. Because Δt is related to the difference in principal stresses, the fringes thus yield the values of $\sigma_1 - \sigma_2$. Figure 8.35 shows these fringes in a disc subjected to diametrically opposite compressive forces.

By choosing different orientations for the polarizing axes of the polarizer, the analyzer, and the birefringe material, we can obtain different isoclinic lines and different fringes. Through a succession of such combinations of axes, we can then obtain a visual representation of stress in a material. Photographs are taken for each isoclinic line and fringe, and a composite photograph that shows all isoclinic lines and fringes is made.

Actually, to describe plane stress completely, we need three pieces of information. Photoelasticity yields only two—the orientation of principal axis 1 and the difference of principal stresses. However, on a free surface we know that one of the principal stresses is zero. Thus photoelasticity will give a complete state of stress for a point on a free surface. In the interior, if we know the sum of the principal stresses, then we can obtain the complete state of stress at that point. To obtain the sum of the principal stresses may require a mix of analytical, numerical, and other experimental methods.

8.5 CHAPTER CONNECTOR

In this chapter we studied the relationship of stresses in different coordinate systems and methods to determine the maximum tensile normal stress, maximum compressive normal stress, and maximum shear stress. In Chapter 10 we shall study various failure theories, including maximum-normal-stress and maximum-shear-stress theories. We also apply these theories to the design and failure analysis of simple structures and machines.

POINTS AND FORMULAS TO REMEMBER

- Stress transformation equations relate stresses *at a point* in different coordinate systems:
- $$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad (8.1)$$
- $$\tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta) \qquad (8.2)$$
- where σ_{xx}, σ_{yy}, and τ_{xy} are the stresses in *x*, *y*, *z* coordinate system, σ_{nn}, σ_{tt}, and τ_{nt} are the stresses in *n*, *t*, *z* coordinate system and θ is measured from the *x* axis in the counterclockwise direction to the *n* direction.
- The values of stresses on a plane through a point are unique and depend on the orientation of the plane only and not on how its orientation is described or measured.
- Planes on which the shear stresses are zero are called *principal planes*.
- Principal planes are orthogonal.
- The normal direction to the principal planes is referred to as the principal direction or *principal axis*.
- The angles the principal axis makes with the global coordinate system are called *principal angles*.
- Normal stress on a principal plane is called *principal stress*.
- The greatest principal stress is called *principal stress 1*.
- Principal stresses are the maximum and minimum normal stresses at a point.
- The maximum shear stress on a plane that can be obtained by rotating about the *z* axis is called *in-plane maximum shear stress*.
- The maximum shear stress at a point is the *absolute* maximum shear stress that is on any plane passing through the point.
- Maximum in-plane shear stress exists on two planes which are at 45° to the principal planes.
- $$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}} \quad (8.6) \qquad \sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \quad (8.7)$$
- $$|\tau_p| = \left|\frac{\sigma_1 - \sigma_2}{2}\right| \qquad (8.12)$$
- where θ_p is the angle to either principal plane 1 or 2, σ_1 and σ_2 are the principal stresses, τ_p is the in-plane maximum shear stress.
- $$\sigma_{nn} + \sigma_{tt} = \sigma_{xx} + \sigma_{yy} = \sigma_1 + \sigma_2 \qquad (8.9)$$
- $$\sigma_3 = \sigma_{zz} = \begin{cases} 0, & \text{plane stress} \\ \nu(\sigma_{xx} + \sigma_{yy}), & \text{plane strain} \end{cases} \qquad (8.10)$$
- $$\tau_{max} = \left|\max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right)\right| \qquad (8.13)$$
- Each point on Mohr's circle represents a unique plane that passes through the point at which the stresses are specified.
- The coordinates of the point on Mohr's circle are the normal and shear stresses on the plane represented by the point.
- Angles between planes on a stress cube are doubled when plotted on Mohr's circle.
- The sign of shear stress cannot be determined directly from Mohr's circle, which tells only that the shear stress causes the plane to rotate clockwise or counterclockwise.
- The maximum shear stress at a point is the radius of the biggest circle.
- A principal stress element shows stresses on a wedge constructed from principal planes and the plane of maximum shear stress.

CHAPTER NINE

STRAIN TRANSFORMATION

Learning Objectives

1. Learn the equations and procedures of relating strains at a point in different coordinate systems.
2. Learn the analysis in using strain gages.

To minimize the likelihood of occupants being thrown from the vehicle as a result of impact, Federal Motor Vehicle Safety Standard specifies requirements for attaching of doors. Automobile companies conduct and sponsor research to understand the stresses near bolts attaching doors to the car body. The door's own weight subjects the bolts to bending loads.

In the experiment shown in Figure 9.1, a composite plate is subjected to bending loads like those at the attachment points of the car door to a fiber glass body. Strain gages—the most popular strain-measuring devices—are used to determine the strains (and hence the stresses) in the critical region. In previous chapters, we obtained formulas for predicting strain. How do we relate the experimentally measured strains to those obtained from theory in Cartesian or polar coordinates? This chapter discusses *strain transformation,* which relates strains in different coordinate systems.

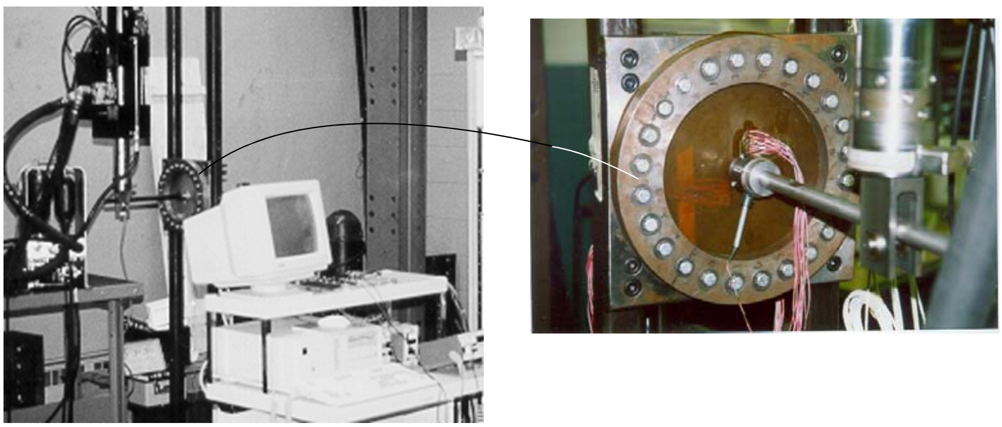


Figure 9.1 Measurement of strains using strain gages. (Courtesy Professor I. Miskioglu.)

The idea of strain transformation is very similar to stress transformation. We shall rely on this similarity to develop the key definitions and equations for strain transformation. But there are also differences, and they are critical to a successful understanding of the methods.

9.1 PRELUDE TO THEORY: THE LINE METHOD

In the wedge method of stress transformation (see Section 8.1), the key idea was to convert *stresses* into *forces*—that is, to convert a second-order tensor into a vector. We adopt a similar strategy for strain transformation. By multiplying a strain component by the length of a line, we obtain the *deformation*, which is a vector quantity. Using the small-strain approximation, we can then find the component of deformation in a given direction (see Problems 2.40-2.47). Section 9.1.1 elaborates this strategy.

We restrict ourselves to plane strain problems (see Section 2.5.1), where all strains with subscript z are zero. We further assume that the strains in the global Cartesian coordinate system (ε_{xx}, ε_{yy}, and γ_{xy}) at a point are known. We define a right-handed local coordinate system n, t, z, as shown in Figure 9.2. As in stress transformation, only those coordinate systems that can be obtained by rotation about the z axis are considered. Our objective is to find ε_{nn}, ε_{tt}, and γ_{nt}.

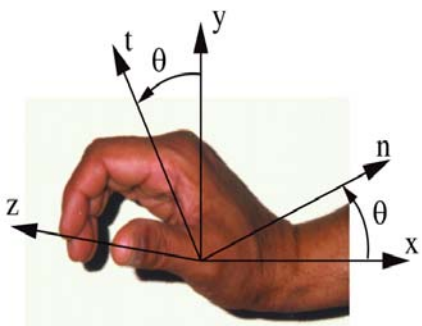


Figure 9.2 Global and local coordinate system.

Recall that normal strains are a measure of change in the *length* of a line, whereas shear strains are a measure of change in the *angle* between two lines. By finding the change in length and the rotation each axes we can find the strains in that coordinate system. This process is formally described in next section and elaborated in Example 9.1.

9.1.1 Line Method Procedure

The normal and shear strains in the local coordinate system can be found by the following steps:

Step 1: Consider each strain component one at a time and view the *n* and *t* directions as two separate lines.

Step 2: Construct a rectangle with a diagonal in the direction of the line. Relate the length of the diagonal to the lengths of the rectangle's sides.

Step 3: Draw the deformed shape assuming one side is fixed and apply the deformation on the opposite side. Find the deformation and rotation of the diagonal using small-strain approximations.

Step 4: Calculate the normal and shear strains in the *n* and *t* directions.

Step 5: Repeat steps *2* through *4* for each strain component. Superpose the results to obtain the strains in the *n* and *t* directions.

Because the line method is repetitive and tedious, we will consider problems with only one nonzero strain component. However, the same principle will be used to develop the equations of strain transformation.

EXAMPLE 9.1

At a point, the only nonzero strain component is $\varepsilon_{xx} = 200\ \mu$. Determine the strain components in the *n*, *t* coordinate system that is rotated 25° counterclockwise to the *x* axis.

PLAN

We follow the procedure described in Section 9.1.1.

SOLUTION

Step 1: View the axes of the *n*, *t* coordinate system as two lines, as shown in Figure 9.3*a*. Due to the normal strain in the *x* direction, the lines in the *n* and *t* directions deform to n_1 and t_1, as shown in Figure 9.3*a*.

Calculations in the n direction

Step 2: We can draw a rectangle with a diagonal in the *n* direction, as shown in Figure 9.3*a*. The diagonal length be Δn can be related to Δx as shown.

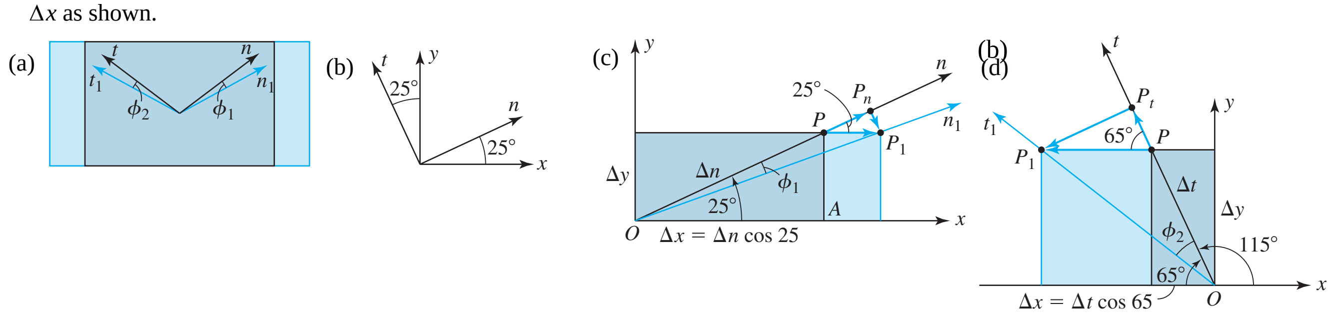


Figure 9.3 (a) (b) Movement of *n* and *t* lines. Deformation calculations (c) in the *n* direction; (d) in the *t* direction in Example 9.1.

Step 3: Let point *P* move to point P_1 due to strain ε_{xx}. The deformed shape can be drawn as shown in Figure 9.3*c*.

Step 4: A perpendicular line from P_1 to the n direction is drawn. The sides of the triangle P_nPP_1 are

$$PP_1 = \varepsilon_{xx}\Delta x = 200(10^{-6})\Delta n\cos 25° = 181.3\Delta n(10^{-6}) \tag{E1}$$

$$PP_n = PP_1\cos 25° = 164.3\Delta n(10^{-6}) \qquad P_nP_1 = PP_1\sin 25° = 76.6\Delta n(10^{-6}) \tag{E2}$$

The angle ϕ_1 can be found from triangle P_nOP_1 as

$$OP_n = OP + PP_n = \Delta n[1 + 164.3(10^{-6})] \approx \Delta n \qquad \tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} = 76.6(10^{-6})\ \text{rad} \tag{E3}$$

Calculations in the t direction

Step 2: We can draw a rectangle with a diagonal representing the t direction, as shown in Figure 9.3*d*. The diagonal length Δt can be related to Δx as shown.

Step 3: Let point P move to point P_1 due to strain ε_{xx}. The deformed shape can be drawn as shown in Figure 9.3*d*.

Step 4: A perpendicular line from P_1 to the t direction is drawn. The sides *of* triangle P_tPP_1 are

$$PP_1 = \varepsilon_{xx}\Delta x = 200(10^{-6})\Delta t\cos 65° = 84.5\Delta t(10^{-6}) \tag{E4}$$

$$P_tP_1 = PP_1\sin 65° = 76.58\Delta t(10^{-6}) \qquad PP_t = PP_1\cos 65° = 35.7\Delta t(10^{-6}) \tag{E5}$$

We can calculate the angle ϕ_2 from triangle P_tOP_1 as

$$OP_t = OP + PP_t = \Delta t[1 + 35.7(10^{-6})] \approx \Delta t \qquad \tan\phi_2 \approx \phi_2 = \frac{P_tP_1}{OP_t} = 76.6(10^{-6}) \tag{E6}$$

Step 5: The normal strain in the n and t directions are

$$\varepsilon_{nn} = \frac{PP_n}{\Delta n} = \frac{164.3\Delta n(10^{-6})}{\Delta n} = 164.3(10^{-6}) \qquad \varepsilon_{tt} = \frac{PP_t}{\Delta t} = \frac{35.7\Delta t(10^{-6})}{\Delta t} = 35.7(10^{-6}) \tag{E7}$$

$$\gamma_{nt} = -(\phi_1 + \phi_2) = -[76.6(10^{-6}) + 76.6(10^{-6})] = -153.2(10^{-6}) \tag{E8}$$

ANS. $\varepsilon_{nn} = 164.3\ \mu \qquad \varepsilon_{tt} = 35.7\ \mu \qquad \gamma_{nt} = -153.2\ \mu$

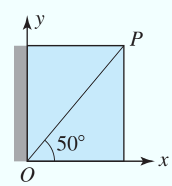
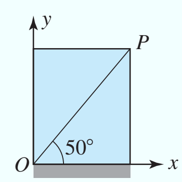
COMMENTS

1. In Figure 9.3*a* the displacement of point P to P_1 is in the positive x direction, whereas in Figure 9.3*d* the displacement is in the negative x direction. But notice that both rectangles show elongation to reflect positive ε_{xx}. Both rectangles represent the same point. This emphasizes once more the difference between displacements and deformations.
2. The negative sign in Equation (E8) reflects the increase in angle between n and t as shown in Figure 9.3.
3. We repeated the calculations for n and t directions for one strain component. For all three components we would do similar calculations six times, making the line method a repetitive, tedious process. We will develop formulas using this method to overcome the tedium.

PROBLEM SET 9.1

Line method

In Problems 9.1 through 9.4, determine the rotation of line OP and the normal strain in the direction OP due to the strain given in each problem.

Problem	*Strain*	Use	
9.1	$\varepsilon_{xx} = 500\ \mu$	Figure P9.1	**Figure P9.1**
9.2	$\gamma_{xy} = 300\ \mu$	Figure P9.1	
9.3	$\varepsilon_{yy} = -400\ \mu$	Figure P9.3	**Figure P9.3**
9.4	$\gamma_{yx} = 300\ \mu$	Figure P9.3	

In Problems 9.5 through 9.13, at a point, the nonzero strain component are as given in each problem. Determine the strain components in the n, t coordinate system shown.

Problem	*Strain*	Use
9.5	$\varepsilon_{xx} = -400\ \mu$	Figure P9.5
9.6	$\varepsilon_{yy} = 600\ \mu$	Figure P9.5
9.7	$\gamma_{xy} = -500\ \mu$	Figure P9.5
9.8	$\varepsilon_{xx} = -600\ \mu$	Figure P9.8
9.9	$\varepsilon_{yy} = -1000\ \mu$	Figure P9.8
9.10	$\gamma_{xy} = 500\ \mu$	Figure P9.8
9.11	$\varepsilon_{xx} = 600\ \mu$	Figure P9.11
9.12	$\varepsilon_{yy} = 600\ \mu$	Figure P9.11
9.13	$\gamma_{xy} = 600\ \mu$	Figure P9.11

Figure P9.5

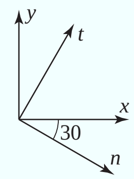


Figure P9.8

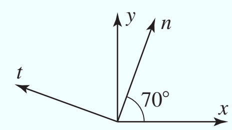
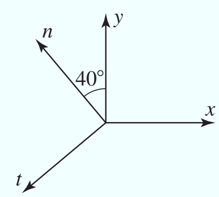


Figure P9.11



9.2 METHOD OF EQUATIONS

In this section we develop strain transformation equations using the line method.[1] We assume the point is in plane strain and the strains ε_{xx}, ε_{yy}, and γ_{xy} are known. As in stress transformation, we consider only those coordinate systems that can be obtained by rotation about the z axis shown in Figure 9.2. Our objective is to find the strains ε_{nn}, ε_{tt}, and γ_{nt}.

Sign Convention: The angle θ describing the orientation of the local coordinate system is positive in counterclockwise direction from the x axis.

We will follow the procedure outlined in Section 9.1.1 to determine the deformation and rotation of a line in the n direction. By substituting $90^\circ + \theta$ in place of θ in the expressions obtained in the n direction, we will obtain the expressions in the t direction.

Calculations for ε_{xx} acting alone

Step 1: View the axes of the n, t coordinate system as two lines, as shown in Figure 9.4*a*. Due to the normal strain in the x direction, the lines in the n and t directions deform to n_1 and t_1, as shown.

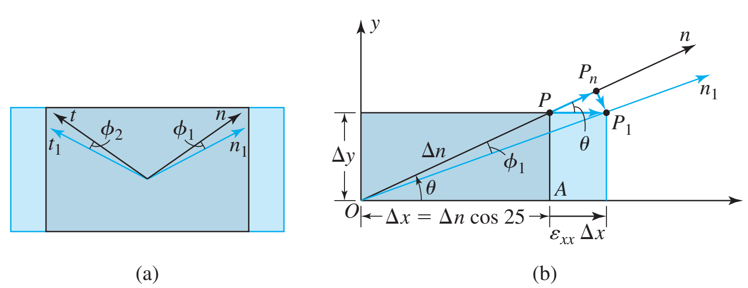


Figure 9.4 Strain transformation with ε_{xx} only.

Step 2: Draw a rectangle with a diagonal at an angle θ as shown in Figure 9.4*b*. The diagonal length Δn can be related to Δx as shown.

[1]See Problems 9.78 through 9.80 for an alternative derivation of the strain transformation equations.

Step 3: Let point P move to point P_1 due to strain ε_{xx}. Draw an exaggerated deformed shape as shown in Figure 9.4*b*. A perpendicular line from P_1 to the n direction is drawn. The sides of the triangle P_nPP_1 are

$$PP_1 = \varepsilon_{xx}\Delta x = \varepsilon_{xx}\Delta n\cos\theta \tag{9.1.a}$$

$$PP_n = PP_1\cos\theta = (\varepsilon_{xx}\cos^2\theta)(\Delta n) \tag{9.1.b}$$

$$P_nP_1 = PP_1\sin\theta = (\varepsilon_{xx}\sin\theta\ \cos\theta)\ \Delta n \tag{9.1.c}$$

Now $OP_n = OP + PP_n = OP(1 + PP_n/OP) = OP(1 + \varepsilon_{nn})$. For small strain, $\varepsilon_{nn} \ll 1$ and hence can be neglected, giving $OP_n = OP = \Delta_n$. For small strain the tangent function can be approximated by its argument. With these two approximations, we obtain the rotation ϕ_1 from triangle P_nOP_1,

$$\tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} \approx \frac{P_nP_1}{OP} = \frac{(\varepsilon_{xx}\sin\theta\ \cos\theta)\ \Delta n}{\Delta n} = \varepsilon_{xx}\sin\theta\cos\theta$$

Step 4: The normal strain in the n direction can be found as

$$\varepsilon_{nn}^{(1)} = PP_n/\Delta n = \varepsilon_{xx}\cos^2\theta \tag{9.1.d}$$

The superscript 1 differentiates the strains calculated from ε_{xx} from those calculated from ε_{yy} and γ_{xy}. We note that the t axis is a line like the n axis, but at an angle of 90º + θ instead of θ. We can obtain the normal strain in the t direction and the rotation ϕ_2 of the t axis by substituting 90º + θ in place of θ.

$$\varepsilon_{tt}^{(1)} = \varepsilon_{xx}\cos^2(90° + \theta) = \varepsilon_{xx}\sin^2\theta \tag{9.1.e}$$

$$\phi_2 = |\varepsilon_{xx}\sin(90° + \theta)\cos(90° + \theta)| = \varepsilon_{xx}\sin\theta\ \cos\theta \tag{9.1.f}$$

The angle between the n and t directions increases, as seen from the rectangle in Figure 9.4*a*. This implies that the shear strain will be negative and is given as

$$\gamma_{nt}^{(1)} = -(\phi_1 + \phi_2) = -2\varepsilon_{xx}\sin\theta\ \cos\theta \tag{9.1.g}$$

Calculations for ε_{yy} acting alone

The preceding calculations can be repeated for ε_{yy}. The calculations for Steps 1–3 are shown in Figure 9.5. Based on small strain, we once more approximate $OP_n \cong OP$ and $\tan\phi_1 \approx \phi_1$ to obtain

$$\tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} \approx \frac{P_nP_1}{OP} = \varepsilon_{yy}\sin\theta\ \cos\theta \tag{9.2.a}$$

Step 4: The normal strain in the n direction can be found as

$$\varepsilon_{nn}^{(2)} = PP_n/\Delta n = \varepsilon_{yy}\sin^2\theta \tag{9.2.b}$$

We can obtain the normal strain in the t direction and the rotation ϕ_2 of the t axis by substituting 90º + θ in place of θ.

$$\varepsilon_{tt}^{(2)} = \varepsilon_{yy}\sin^2(90° + \theta) = \varepsilon_{yy}\cos^2\theta \tag{9.2.c}$$

$$\phi_2 = |\varepsilon_{yy}\sin(90° + \theta)\cos(90° + \theta)| = \varepsilon_{yy}\sin\theta\ \cos(\tag{9.2.d}$$

The angle between the n and t directions decreases, as seen from the rectangle in Figure 9.5*a*. This implies that the shear strain will be positive:

$$\gamma_{nt}^{(2)} = \phi_1 + \phi_2 = 2\varepsilon_{yy}\sin\theta\ \cos\theta \tag{9.2.e}$$

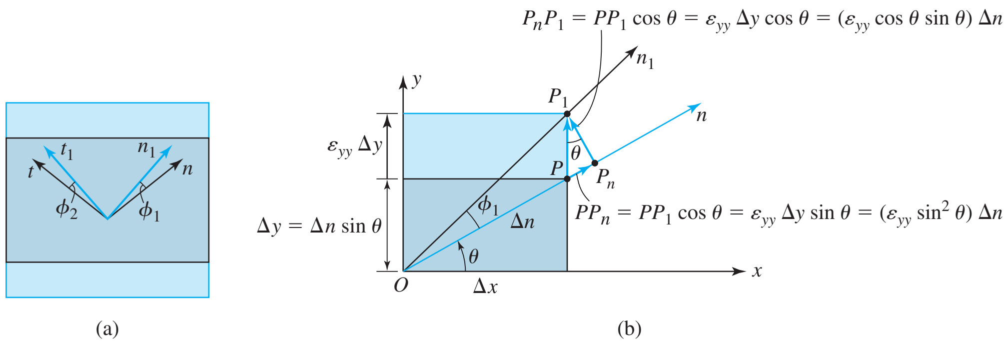


Figure 9.5 Strain transformation with ε_{yy} only.

Calculations for γ_{xy} acting alone

The preceding calculations can be repeated for γ_{xy}. The calculations for Steps 1–3 are shown in Figure 9.6. Based on small strain, we once more approximate $OP_n \cong OP$ and $\tan\phi_1 \approx \phi_1$ to obtain

$$\tan\phi_1 \approx \phi_1 = \frac{P_nP_1}{OP_n} \approx \frac{P_nP_1}{OP} = \gamma_{xy}\sin^2\theta \tag{9.3.a}$$

Step 4: The normal strain in the n direction can be found as

$$\varepsilon_{nn}^{(3)} = PP_n/\Delta n = \gamma_{xy}\sin\theta\cos\theta \tag{9.3.b}$$

We can obtain the normal strain in the t direction and the rotation ϕ_2 of the t axis by substituting $90° + \theta$ in place of θ.

$$\varepsilon_{tt}^{(3)} = \gamma_{xy}\sin(90° + \theta)\cos(90° + \theta) = -\gamma_{xy}\sin\theta\cos\theta \tag{9.3.c}$$

$$\phi_2 = \left|\gamma_{xy}\cos^2(90° + \theta)\right| = \gamma_{xy}\cos^2\theta \tag{9.3.d}$$

From Figure 9.6*a* it is seen that the movement of the line in the n direction to n_1 increases the initial angle, and the movement of the line in the t direction to t_1 decreases the initial angle. The final angle between the n_1 and t_1 directions is $\pi/2 + \phi_1 - \phi_2$. Thus from the definition of shear strain in Chapter 2, we obtain

$$\gamma_{nt}^{(3)} = \phi_1 - \phi_2 = \gamma_{xy}(\cos^2\theta - \sin^2\theta) \tag{9.3.e}$$

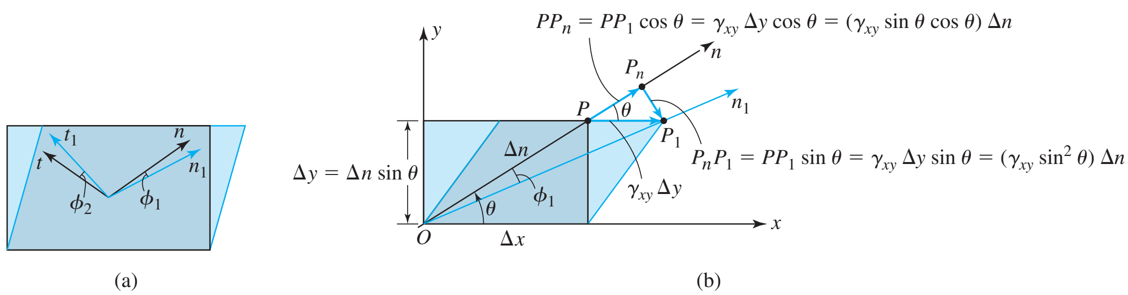


Figure 9.6 Strain transformation with γ_{xy} only.

Total strains

Step 5: As we are working with small strains, we have a linear system, and the total strain in the n and t directions is the superposition of the strains due to the individual components. That is,

$$\varepsilon_{nn} = \varepsilon_{nn}^{(1)} + \varepsilon_{nn}^{(2)} + \varepsilon_{nn}^{(3)} \qquad \varepsilon_{tt} = \varepsilon_{tt}^{(1)} + \varepsilon_{tt}^{(2)} + \varepsilon_{tt}^{(3)} \qquad \gamma_{nt} = \gamma_{nt}^{(1)} + \gamma_{nt}^{(2)} + \gamma_{nt}^{(3)}$$

We obtain the following equations:

$$\boxed{\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta} \tag{9.4}$$

$$\boxed{\varepsilon_{tt} = \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta} \tag{9.5}$$

$$\boxed{\gamma_{nt} = -2\varepsilon_{xx}\sin\theta\ \cos\theta + 2\varepsilon_{yy}\ \sin\theta\ \cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)} \tag{9.6}$$

Equations (9.4), (9.5), and (9.6) are similar to the stress transformation equations, Equations (8.1), (8.2), and (8.3). However, the coefficient of the shear strain term is *half* the coefficient of the shear stress term. This is because we are using engineering strain instead of tensor strain.[2] With this difference accounted for, we can rewrite the results from stress transformation for strain transformation, as described next.

9.2.1 Principal Strains

Analogous to the case of stress transformation, we have the following definitions.

- The **principal directions** are the coordinate axes in which the shear strain is zero.
- The angles the principal axes make with the global coordinate system are called **principal angles**.
- Normal strains in the principal directions are called **principal strains**.
- The greatest principal strain is called **principal strain 1** (ε_1). By greatest principal strain we refer to the magnitude and the sign of the principal strain. Thus a strain of –600 μ is greater than one of –1000 μ

Note that the coefficient of the shear strain in strain transformation equations is half the coefficient of shear stress in the stress transformation equations. We therefore obtain the principal angle θ_p and principal strains [see Equations (8.6) and (8.7)],

$$\boxed{\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}}} \tag{9.7}$$

$$\boxed{\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}} \tag{9.8}$$

Here $\varepsilon_{1,2}$ represents the two strains ε_1 and ε_2. The plus sign is to be taken with ε_1 and the minus sign with ε_2. Like principal stresses, the principal strains correspond to the maximum and minimum normal strains at a point.

Adding Equations (9.4) and (9.5) and the principal strains in Equation (9.8), we obtain

$$\varepsilon_{nn} + \varepsilon_{tt} = \varepsilon_{xx} + \varepsilon_{yy} = \varepsilon_1 + \varepsilon_2 \tag{9.9}$$

Equation (9.9) shows that the sum of the normal strains at any point in an orthogonal coordinate system does not depend on the orientation of the coordinate system.

The angle of principal axis 1 from the *x* axis is reported only in describing the principal coordinate system in two-dimensional problems. Counterclockwise rotation from the *x* axis is defined as positive.

Two values of θ_p satisfy Equation (9.7), separated by 90°. The direction θ_1 corresponding to ε_1 is 90° from the direction θ_2 corresponding to ε_2. In other words, the *principal directions are orthogonal*. It is not clear whether the principal angle found from Equation (9.7) is associated with ε_1 or ε_2. We will resolve this problem as we did in stress transformation, as elaborated in Example 9.4.

In plane strain, the shear strains with subscript z are zero. Therefore the *z* direction is a principal direction and the normal strain ε_{zz} is a principal strain of zero value. In plain stress the shear strains with subscript *z* are again zero, but ε_{zz} is not zero; as shown in Figure 3.27, it is equal to $\nu(\sigma_{xx} + \sigma_{yy})$. If we add Equations (3.12a) and (3.12b) for plane stress problems, we obtain $\sigma_{xx} + \sigma_{yy} = E[(\varepsilon_{xx} + \varepsilon_{yy})/(1-\nu)]$. Thus [See Equation (3.18)],

$$\varepsilon_{zz} = -\frac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy})$$

[2]An alternative is to let $\gamma_{xy} = 2\varepsilon_{xy}$ and $\gamma_{nt} = 2\varepsilon_{nt}$ in Equations (9.4) through (9.6), where it is understood that ε_{xy} is the tensor shear strain and γ_{xy} is the engineering shear strain. In such a case the equations of stress and strain transformation have identical forms.

and we can write the third principal strain as

$$\varepsilon_3 = \begin{cases} 0, & \text{plane strain} \\ -\dfrac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}) = -\dfrac{\nu}{1-\nu}(\varepsilon_1 + \varepsilon_2), & \text{plane stress} \end{cases} \tag{9.10}$$

9.2.2 Visualizing Principal Strain Directions

A circle at a given point will deform into an ellipse with the major axis in the direction of maximum normal strain (principal strain 1) and the minor axis in the direction of minimum normal strain (principal strain 2). We make use of this observation to estimate the direction of the principal strains within a 45° quadrant.

Step 1: Visualize or draw a square with a circle inside.

Step 2: Visualize or draw the deformed shape of the square due to *only* normal strains.

The deformed shape will be a rectangle. The circle within the square has now become an ellipse with the major axis either along the *x* direction or along the *y* direction, depending which normal strain is greater.

Step 3: Visualize or draw the deformed shape of the rectangle due to the shear strain.

The rectangle will deform into a rhombus, and the ellipse inside would have rotated such that the major axis is in the direction of the longer diagonal of the rhombus. The major axis can rotate at most 45° from its orientation in Step 2. The major axis represents principal direction 1 and the minor axis represents principal direction 2.

Step 4: Using the eight 45° sectors shown in Figure 9.7, report the orientation of principal direction 1. Also report principal direction 2 as two sectors counterclockwise from the sector reported for principal direction 1.

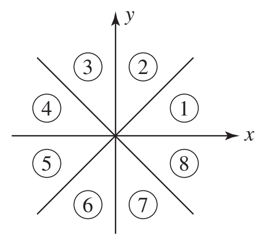


Figure 9.7 The eight sectors in which the principal axis will lie.

As in stress transformation, principal directions 1, 2, and 3 form a right-handed coordinate system. The *z* direction is the third principal direction. Once principal direction 1 is determined, the right-hand rule places principal direction 2 at two sectors (90°) counterclockwise from it.

EXAMPLE 9.2

At a point in plane strain, the strain components are $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = 500\ \mu$, and $\gamma_{xy} = 600\ \mu$. Estimate the orientation of the principal directions and report your results using the sectors shown in Figure 9.7.

PLAN

We will follow the steps outlined in section Section 9.2.2.

SOLUTION

Step 1: We draw a circle inside a square, as shown in Figure 9.8*a*.

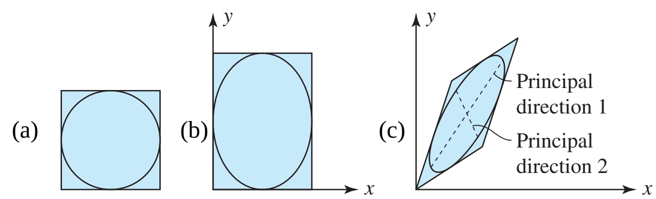


Figure 9.8 (*a*) Undeformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

Step 2: As $\varepsilon_{yy} > \varepsilon_{xx}$, the extension in the *y* direction is greater than that in the *x* direction. The square becomes a rectangle and the circle becomes an ellipse, as shown Figure 9.8*b*.
Step 3: As γ_{xy} is positive, the angle between the *x* and *y* axes must decrease and we obtain the rhombus shown in Figure 9.8*c*.
Step 4: The two solutions follow by inspection.

ANS. Principal axis 1 is in sector 2 and principal axis 2 is in sector 4.
or
Principal axis 1 is in sector 6 and principal axis 2 is in sector 8.

COMMENTS

1. In Figure 9.8*b* the major axis is along the *y* axis. This major axis can rotate at most 45° clockwise or counterclockwise, as dictated by the shear strain. Thus principal axis 1 will be either in sector 2 or in sector 6, according to Figure 9.8*c*.
2. We will always obtain two answers for principal angle 1 as we did in stress transformation. Both answers are correct, and either can be reported.

EXAMPLE 9.3

At a point in plane strain, the strain components are $\varepsilon_{xx} = -200\ \mu$, $\varepsilon_{yy} = -400\ \mu$, and $\gamma_{xy} = -300\ \mu$. Estimate the orientation of the principal directions and report your results using the sectors shown in Figure 9.7.

PLAN

This time we will visualize but not draw any deformed shapes.

SOLUTION

Step 1: We visualize a square with a circle.
Step 2: Due to normal strains, the contraction in the *y* direction is greater than that in the *x* direction. Hence the rectangle will have a longer side in the *x* direction, that is, the major axis is along the *x* axis.
Step 3: As the shear strain is negative, the angle will increase. The major axis will rotate clockwise, and it will lie either in sector 8 or in sector 4.
Step 4: Principal axis 1 is either in sector 8 or in sector 4, giving the solution.

ANS. Principal axis 1 is in sector 8 and principal axis 2 is in sector 2.
or
Principal axis 1 is in sector 4 and principal axis 2 is in sector 6.

9.2.3 Maximum Shear Strain

As in stress transformation, we differentiate between in-plane maximum shear strain and maximum shear strain. The maximum shear strain in coordinate systems that can be obtained by rotating about the *z* axis is called **in-plane maximum shear strain. Since** the coefficient of shear strain in strain transformation equations is half the coefficient of shear stress in stress transformation equations, we obtain the in-plane maximum shear strain as

$$\left|\frac{\gamma_p}{2}\right| = \left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right| \tag{9.11}$$

The **maximum shear strain** at a point is the maximum shear strain in any coordinate system given by

$$\boxed{\frac{\gamma_{max}}{2} = \max\left(\left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right|, \left|\frac{\varepsilon_2 - \varepsilon_3}{2}\right|, \left|\frac{\varepsilon_3 - \varepsilon_1}{2}\right|\right)} \tag{9.12}$$

Equation (9.12) shows that the value of the maximum shear strain depends on the value of principal strain 3.Equation (9.10) shows that the value of principal strain 3 depends on the plane stress or plane strain problem. As in stress transformation, the maximum shear strain exists in two coordinate systems that are at 45° to the principal coordinate system.

EXAMPLE 9.4

At a point in plane strain, the strain components are $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = 1000\ \mu$, and $\gamma_{xy} = -600\ \mu$. Determine (a) the principal strains and principal angle 1; (b) the maximum shear strain; (c) the strain components in a coordinate system that is rotated 25° counterclockwise, as shown in Figure 9.9.

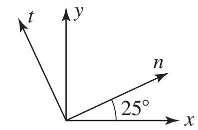


Figure 9.9

PLAN

(a) Using Equation (9.7), we can find θ_p. We can substitute θ_p into Equation (9.4) and find one of the principal strains. Using Equation (9.9), we find the other principal strain and decide which is principal strain 1. (b) We can find the maximum shear strain using Equation (9.12). (c) We can find the strains in the *n* and *t* coordinates by substituting $\theta = 25^{\circ}$ in Equations (9.4), (9.5), and (9.6).

SOLUTION

(a) From Equation (9.7) we obtain the principal angle,

$$\tan 2\theta_p = \frac{-600\ \mu}{200\ \mu - 1000\ \mu} = 0.75 = \tan 36.87^{\circ} \qquad \text{or} \qquad \theta_p = 18.43^{\circ}\text{ccw} \tag{E1}$$

Substituting θ_p into Equation (9.4), we obtain one of the principal strains,

$$\varepsilon_p = (200\ \mu)\cos^2 18.43^{\circ} + (1000\ \mu)\sin^2 18.43^{\circ} + (-600\ \mu)\sin 18.43^{\circ}\cos 18.43^{\circ} = 100\ \mu \tag{E2}$$

Now $\varepsilon_{xx} + \varepsilon_{yy} = 1200\ \mu$. From Equation (9.9) we obtain the other principal strain as $1200 - 100 = 1100\ \mu$, which is greater than the principal strain in Equation (E2). Thus $1100\ \mu$ is principal strain 1, and principal angle 1 is obtained by adding (or subtracting) 90° from Equation (E1). As the point is in plane strain, the third principal strain is zero. We report our results as

ANS. $\varepsilon_1 = 1100\ \mu \qquad \varepsilon_2 = 100\ \mu \qquad \varepsilon_3 = 0 \qquad \theta_1 = 108.4^{\circ}\text{ ccw or } 71.6^{\circ}\text{ cw}$

We can check the principal strain values using Equation (9.8),

$$\varepsilon_{1,2} = \frac{(200\ \mu) + (1000\ \mu)}{2} \pm \sqrt{\left(\frac{200\ \mu - 1000\ \mu}{2}\right)^2 + \left(\frac{-600\ \mu}{2}\right)^2} = 600\ \mu \pm 500\ \mu\text{----Checks}$$

Intuitive check orientation of principal axis 1: We visualize a circle in a square, as shown in Figure 9.10*a*. As $\varepsilon_{yy} > \varepsilon_{xx}$, the rectangle will become longer in the *y* direction than in the *x* direction and the circle will become an ellipse with major axis along the *y* direction, as shown in Figure 9.10*b*. As $\gamma_{xy} < 0$, the angle between the *x* and *y* directions will increase. The rectangle will become a rhombus and the major axis of the ellipse will rotate counterclockwise from the *y* axis. Hence we expect principal axis 1 to be in either the third sector or the seventh sector, confirming our result.

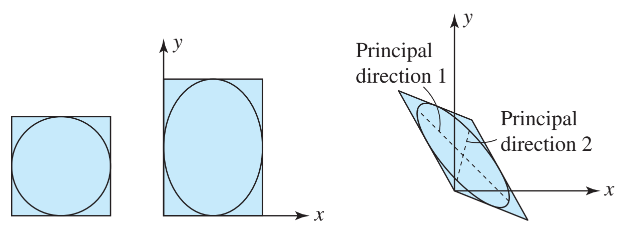


Figure 9.10 (*a*) Undeformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

(b) We can find the maximum shear strain from Equation (9.12), that is, the maximum difference is between ε_1 and ε_3, thus the maximum shear strain is

ANS. $\gamma_{max} = 1100\ \mu$

(c) Substituting $\theta = 25^{\circ}$ in Equations (9.4), (9.5), and (9.6), we obtain

$$\varepsilon_{nn} = (200\ \mu)\cos^2 25^{\circ} + (1000\ \mu)\sin^2 25^{\circ} + (-600\ \mu)\sin 25^{\circ}\cos 25^{\circ} = 113.1\ \mu \tag{E3}$$

$$\varepsilon_{tt} = (200\ \mu)\sin^2 25^{\circ} + (1000\ \mu)\cos^2 25^{\circ} - (-600\ \mu)\sin 25^{\circ}\cos 25^{\circ} = 1086.9\ \mu \tag{E4}$$

$$\gamma_{nt} = -2(200\ \mu)\sin 25^{\circ}\cos 25^{\circ} + 2(1000\ \mu)\sin 25^{\circ}\cos 25^{\circ} + (-600\ \mu)(\cos^2 25^{\circ} - \sin^2 25^{\circ}) = 227.2\ \mu \tag{E5}$$

ANS. $\varepsilon_{nn} = 113.1\ \mu \qquad \varepsilon_{tt} = 1086.9\ \mu \qquad \gamma_{nt} = 227.2\ \mu$

We can use Equation (9.9) to check our results. We note that $\varepsilon_{nn} + \varepsilon_{tt} = 1200\ \mu$, which is the same value as for $\varepsilon_{xx} + \varepsilon_{yy}$, confirming the accuracy of our results.

COMMENTS

1. It can be checked that if we substitute $\theta = 25° + 180° = 205°$ or $\theta = 25° - 180° = -155°$ in Equations (9.4), (9.5), and (9.6), we will obtain the same values for ε_{nn}, ε_{tt}, and γ_{nt} as in part (c). In other words, adding or subtracting 180° from the angle θ in Equations (9.4), (9.5), and (9.6) does not affect the results. This emphasizes that the strain at a point in a given direction (coordinate system) is unique and does not depend on how we describe or measure the orientation of the line.
2. If the point were in plane stress on a material with a Poisson's ratio of $\frac{1}{3}$, then the third principle strain would be $\varepsilon_3 = -[\nu/(1-\nu)](\varepsilon_{xx} + \varepsilon_{yy}) = -600\ \mu$ and the maximum shear strain would be $\gamma_{max} = 1700\ \mu$ which is different than the value we obtained in part (b) for plane strain.

EXAMPLE 9.5

For the wooden cantilever beam shown in Figure 9.11 determine at point A (a) the principal strains and the angle of first principal direction θ_1; (b) the maximum shear strain. Use the modulus of elasticity $E = 12.6$ GPa and Poisson's ratio $\nu = 0.3$.

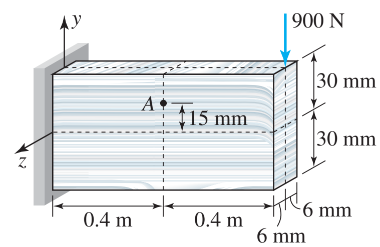


Figure 9.11 Beam and loading in Example 9.5.

PLAN

The bending stresses σ_{xx} and τ_{xy} at point A can be found using Equations (6.12) and (6.27), respectively. Using Hooke's law, the strains ε_{xx}, ε_{yy}, and γ_{xy} can be found. Using Equation (9.7), θ_p can be found and substituted into Equation (9.4) to obtain one of the principal strains. Using Equation (9.9), we find the other principal strain and decide which is principal strain 1. The maximum shear strain can be found using Equation (9.12).

SOLUTION

Bending stress calculations: Recall that A_s is the area between the free surface and the parallel line passing through point A, where shear stress is to be found. The area moment of inertia I_{zz} and the first moment Q_z of the area A_s shown in Figure 9.12*a* are

$$I_{zz} = \frac{(12\text{ mm})(60\text{ mm})^3}{12} = 0.216(10^6)\text{ mm}^4 = 0.216(10^{-6})\text{ m}^4 \quad \text{(E1)}$$

$$Q_z = (12\text{ mm})(15\text{ mm})(15\text{ mm} + 7.5\text{ mm}) = 4.050(10^3)\text{ mm}^3 = 4.050(10^{-6})\text{ m}^3 \quad \text{(E2)}$$

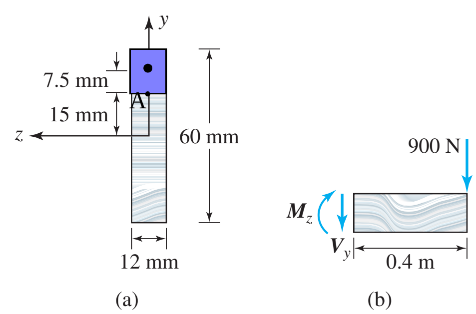


Figure 9.12 Calculation of geometric and internal quantities.

Figure 9.12*b* shows the free-body diagram of the right part of the beam after making the imaginary cut through point A in Figure 9.11. The shear force $\boldsymbol{V}_y$ and the bending moment $\boldsymbol{M}_z$ are drawn according to our sign convention. By balancing forces and moment we obtain

$$\boldsymbol{V}_y = -900\text{ N} \quad \text{(E3)}$$

$$\boldsymbol{M}_z = -(0.4\text{ m})(900\text{ N}) = -360\text{ N}\cdot\text{m} \quad \text{(E4)}$$

Substituting Equations (E1), (E4), and $y_A = 0.015$ m into Equation (6.12), we obtain the bending normal stress,

$$\sigma_{xx} = -\frac{\boldsymbol{M}_z y}{I_{zz}} = -\frac{(-360\text{ N}\cdot\text{m})(0.015\text{ m})}{0.216(10^{-6})\text{ m}^4} = 25(10^6)\text{ N/m}^2 \quad \text{(E5)}$$

By visualizing the beam deformation, we expect σ_{xx} to be tensile consistent with the calculations above.

Substituting Equations (E1), (E2), and (E3) and $t = 0.012$ m into Equation (6.27), we obtain the magnitude of τ_{xy}. Noting that τ_{xy} must have the same sign as V_y, we obtain the sign of τ_{xy} (see Section 6.6.6) as given by

$$|\tau_{xy}| = \left|\frac{V_y Q_z}{I_{zz} t}\right| = \left|\frac{(-900\ N)[4.050(10^{-6})\ \text{m}^3]}{[0.216(10^{-6})\ \text{m}^4](0.012\ \text{m})}\right| = 1.41(10^6)\ \text{N/m}^2 \quad \text{or} \quad \tau_{xy} = -1.41(10^6)\ \text{N/m}^2 \tag{E6}$$

Bending strain calculations: The shear modulus of elasticity can be found from $G = E/2(1 + \nu)$. Substituting $E = 12.6$ GPa and $\nu = 0.3$, we obtain $G = 4.85$ GPa. The only two nonzero stress components are given by Equations (E5) and (E6). Using the generalized Hooke's law [or Equation (6.29)], we obtain the bending strains,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{25(10^6)\ \text{N/m}^2}{12.6(10^9)\ \text{N/m}^2} = 1.984(10^{-3}) = 1984\ \mu \tag{E7}$$

$$\varepsilon_{yy} = -\nu\sigma_{xx}/E = -\nu\varepsilon_{xx} = -0.3(1984\ \mu) = -595.2\ \mu \tag{E8}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{-1.41(10^6)\ \text{N/m}^2}{4.85(10^9)\ \text{N/m}^2} = -0.2907(10^{-3}) = -290.7\ \mu \tag{E9}$$

(a) *Stress transformation calculations:* From Equation (9.7) we obtain the principal angle,

$$\tan 2\theta_p = \frac{-290.7\ \mu}{1984\ \mu - (-595.2\ \mu)} = -0.1124 = -\tan 6.41° \quad \text{or} \quad \theta_p = -3.21° \tag{E10}$$

Substituting θ_p into Equation (9.4) we obtain one of the principal strains,

$$\varepsilon_p = (1984\ \mu)\cos^2(-3.21°) + (-595.2\ \mu)\sin^2(-3.21°) + (-290.7\ \mu)\sin(-3.21°)\cos(-3.21°) = 1992\ \mu \tag{E11}$$

Now $\varepsilon_{xx} + \varepsilon_{yy} = 1389\ \mu$. From Equation (9.9) we obtain the other principal strain as $1389\ \mu - 1992\ \mu = -603\ \mu$, which is less than the principal strain in Equation (E11). Thus $1992\ \mu$ is principal strain 1, and principal angle 1 is obtained from Equation (E10). The third principal strain is . $\varepsilon_3 = \varepsilon_{zz} = -[\nu/(1-\nu)](\varepsilon_{xx} + \varepsilon_{yy}) = -595.2\ \mu$. We report our results as

ANS. $\varepsilon_1 = 1992\ \mu \quad \varepsilon_2 = -603\ \mu \quad \varepsilon_3 = -595.2\ \mu \quad \theta_1 = 3.21^o\text{cw}$

Check on principal strains: We can check the principal strain values using Equation (9.8),

$$\varepsilon_{1,2} = \frac{1984\ \mu + (-595.2\ \mu)}{2} \pm \sqrt{\left(\frac{1984\ \mu - (-595.2\ \mu)}{2}\right)^2 + \left(\frac{-290.7\ \mu}{2}\right)^2} = 694.4\ \mu \pm 1297.7\ \mu \quad \text{or}$$

$$\varepsilon_1 = 1992.1\ \mu \qquad \varepsilon_2 = -603.3\ \mu\text{----checks}$$

Intuitive check orientation of principal axis 1: We visualize a circle in a square. As $\varepsilon_{xx} > \varepsilon_{yy}$ the rectangle will become longer in the x direction. The circle will become an ellipse with its major axis along the x direction. As the shear strain is negative, the angle will increase. The major axis will rotate clockwise, and it will lie either in sector 8 or in sector 4, confirming our result.

(b) We can find the maximum shear strain from Equation (9.12), as the difference between ε_1 and ε_2 (or ε_3).

ANS. $\gamma_{max} = 2595\ \mu$

COMMENT

1. The example demonstrates the synthesis of the theory of symmetric bending of beams and the theory of strain transformation. A similar synthesis can be elaborated for axial and torsion members.

9.3 MOHR'S CIRCLE

As for stress transformation, Mohr's circle is graphical technique for solving problems in strain transformation. We eliminate θ from Equations (9.4) and (9.6) written in terms of double angles, to obtain

$$\left(\varepsilon_{nn} - \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{nt}}{2}\right)^2 = \left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2 \tag{9.13}$$

Comparing Equation (9.13) with the equation of a circle, $(x - a)^2 + y^2 = R^2$, we see that Equation (9.13) it represents a circle with a center that has coordinates $(a, 0)$ and radius R, where

$$a = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \qquad R = \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \tag{9.14}$$

The circle is called Mohr's circle for strain. *Each point on Mohr's circle represents a unique direction* passing through the point at which the strains are specified. The coordinates of each point on the circle are the strains $(\varepsilon_{nn}, \gamma_{nt}/2)$. These represent the normal strain of a line in the n direction and half the shear strain, which represents the rotation of the line passing through the point at which strains ε_{xx}, ε_{yy}, and γ_{xy} are specified.

9.3.1 Construction of Mohr's Circle for Strains

The construction of Mohr's circle for strain is very similar to that for stress. However, there are two important differences. (i) In stress transformation we talked about planes, while here we talk about directions. The directions are the outward normals of the planes. (ii) The vertical axis is shear strain divided by 2. All values of shear strain that are plotted on Mohr's circle or calculated from Mohr's circle must take into account that the vertical coordinate is shear strain divided by 2.

The steps in the construction of Mohr's circle for strain are as follows.

Step 1: Draw a square with a shape deformed due to shear strain γ_{xy}. Label the intersection of the vertical plane and the x axis as V and the intersection of the horizontal plane and the y axis as H, as shown in Figure 9.13.

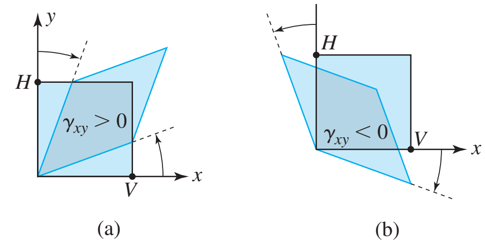


Figure 9.13 Deformed cube for construction of Mohr's circle.

Unlike in stress transformation, where V and H represented planes, here V and H refer to directions. The outward normal to the vertical plane is the x axis, and V is the label associated with it. Similarly, the outward normal to the horizontal plane is the y axis, which is represented by point H.

Step 2: Write the coordinates of points V and H,

$$V(\varepsilon_{xx}, \gamma_{xy}/2) \qquad H(\varepsilon_{yy}, \gamma_{xy}/2), \qquad \text{for } \gamma_{xy} > 0$$

The arrow of rotation along side the shear strains corresponds to the rotation of the line on which the point lies, as shown in Figure 9.13.

Step 3: Draw the horizontal axis to represent the normal strain, with extensions (E) to the right and contractions (C) to the left, as shown in Figure 9.14*a*. Draw the vertical axis to represent half the shear strain, with clockwise rotation of a line in the upper plane and counterclockwise rotation of a line in the lower plane.

As this step emphasizes, the value of shear strain read from Mohr's circle does not tell us whether shear strain is positive or negative. Rather, it shows that the shear strain will cause a line in a given direction to rotate clockwise or counterclockwise. This point is further elaborated in Section 9.3.2.

Step 4: Locate points V and H and join the points by drawing a line. Label the point at which line VH intersects the horizontal axis as C.

Step 5: The horizontal coordinate of point C is the average normal strain. Distance CE can be found from the coordinates of points E and C and the radius R calculated using the Pythagorean theorem. With C as the center and CV or CH as the radius, draw Mohr's circle.

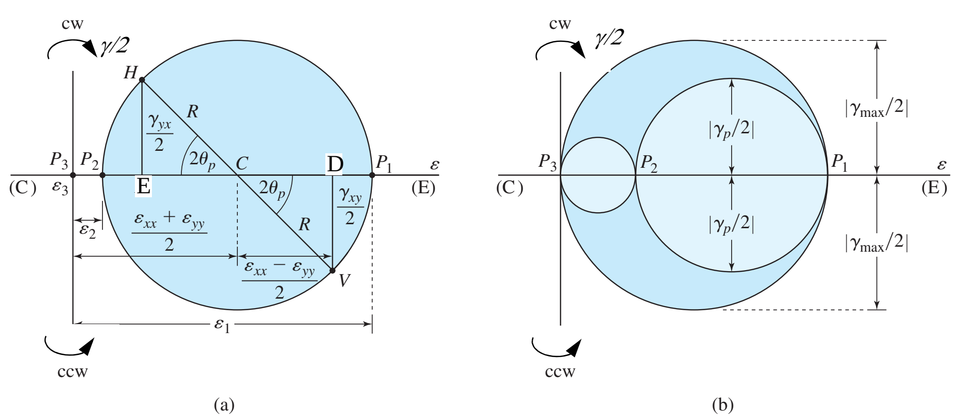


Figure 9.14 Mohr's circle for

Step 6: Calculate the principal strains by finding the coordinates of points P_1 and P_2 in Figure 9.14*a*.

Step 7: Calculate principal angle θ_p from either triangle *VCD* or triangle *ECH*. Find the angle between lines *CV* and CP_1 if θ_1 is different from θ_p.

In Figure 9.14*a*, θ_p and θ_1 have the same value, but this may not always be the case, as elaborated in Example 9.6. θ_1 is the angle measured from the *x* axis, which is represented by point *V* on Mohr's circle, and principal direction 1 is represented by point P_1.

Step 8: Check your answer for θ_1 intuitively using the visualization technique of Section 9.2.2.

Step 9: The in-plane maximum shear strain $\gamma_p/2$ equals *R*, the radius of the in-plane circle shown in Figure 9.14*a*. To find the absolute maximum shear strain, locate point P_3 at the value of the third principal strain. Then draw two more circles between P_1 and P_3 and between P_2 and P_3, as shown in Figure 9.14*b*. The maximum shear strain at a point is found from the radius of the largest circle.

For plane strain P_3 is at the origin, as shown in Figure 9.14*b*. But for plane stress, the third principal strain must be found from Equation (9.10) and located before drawing the remaining two circles. Notice that the radii of the circles yield half the value of the maximum shear strain.

9.3.2 Strains in a Specified Coordinate System

The strains in a specified coordinate system are found by first locating the coordinate directions on Mohr's circle and then determining the coordinates of the point representing the directions. This is achieved as follows.

Step 10: Draw the Cartesian coordinate system and the specified coordinate system along with a square in each coordinate system, representing the undeformed state. Label points *V*, *H*, *N*, and *T* to represent the four directions, as shown in Figure 9.15*a*.

Step 11: Points *V* and *H* on Mohr's circle are known. Point *N* on Mohr's circle is located by starting from point *V* and rotating by $2\theta_V$ in the same direction, as shown in Figure 9.15*a*. Similarly, starting from point *H* on Mohr's circle, point *T* is located as shown in Figure 9.15*b*.

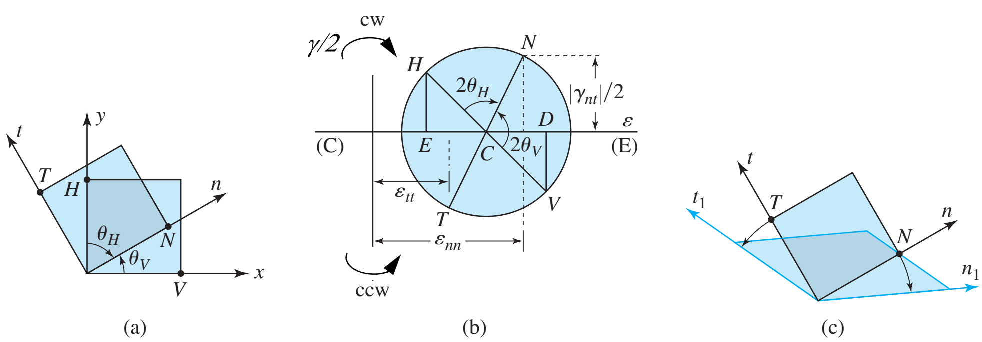


Figure 9.15 Strains in specified coordinate system.

It should be emphasized that we could start from point *H* on Mohr's circle and reach point *N* by rotating $2\theta_H$, as shown in Figure 9.15*b*. In Figure 9.15*a*, $\theta_H + \theta_V = 90^\circ$, and in Figure 9.15*b* we see that $2\theta_H + 2\theta_V$ is 180°, which once more emphasizes that each point on Mohr's circle represents a unique direction, and it is immaterial how one reaches it.

Step 12: Calculate the coordinates of points *N* and *T*.

This is the reverse of Step 2 in the construction of Mohr's circle and is a problem in geometry. As seen in Figure 9.15*b*, the coordinates of points *N* and *T* are

$$N(\varepsilon_{nn}, \gamma_{nt}/2 \circlearrowright) \qquad T(\varepsilon_{tt}, \gamma_{nt}/2 \circlearrowleft)$$

The rotation of the line at point *N* is clockwise, as it is in the upper plane, whereas the rotation of the line at point *T* is counterclockwise, as it is in the lower plane in Figure 9.15*b*.

Step 13: Determine the sign of the shear strain.

To draw the deformed shape we rotate the *n* coordinate in the direction shown for point *N* in Step 3. Similarly, we rotate the *t* coordinate in the direction shown for point *T* in Step 3, as illustrated in Figure 9.15*c*. The angle between the *n* and *t* directions increases, and hence the shear strain γ_{nt} is negative.

EXAMPLE 9.6

At a point in plane strain, the strain components are $\varepsilon_{xx} = 200\ \mu$, $\varepsilon_{yy} = 1000\ \mu$, and $\gamma_{xy} = -600\ \mu$. Using Mohr's circle, determine (a) the principal strains and principal angle 1; (b) the maximum shear strain; (c) the strain components in a coordinate system that is rotated 25° counterclockwise, as shown in Figure 9.16.

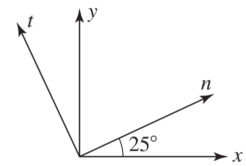


Figure 9.16

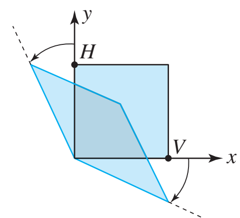
PLAN

We can follow the steps outlined for the construction of Mohr's circle and for the calculation of the various quantities as outlined in this section.

SOLUTION

Step 1: The shear strain is negative, and hence the angle between the x and y axes should increase. We draw the deformed shape of a square due to shear strain γ_{xy}. We label the intersection of the vertical plane and the x axis as V and the intersection of the horizontal plane and the y axis as H, as shown in Figure 9.17.

(a)



(b)

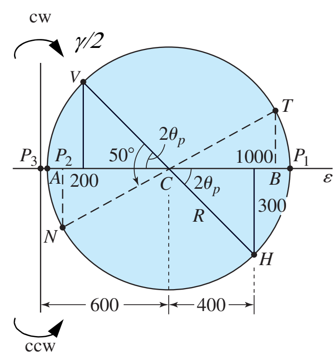


Figure 9.17 (a) Deformed cube. (b) Mohr's circle.

Step 2: Using Figure 9.17*a*, we can write the coordinates of points V and H,

$$V(200, 300 \circlearrowright) \qquad H(1000, 300 \circlearrowleft) \tag{E1}$$

Step 3: We draw the axes for Mohr's circle as shown in Figure 9.17*b*.
Step 4: Locate points V and H and join the points by drawing a line.
Step 5: Point C, the center of Mohr's circle, is midway between points A and B—that is, at 600 μ. The distance BC can thus be found as 400 μ, as shown in Figure 9.17*b*. From the Pythagorean theorem we can find the radius R,

$$R = \sqrt{CB^2 + BH^2} = \sqrt{400^2 + 300^2} = 500 \tag{E2}$$

Step 6: The principal strains are the coordinates of points P_1 and P_2 in Figure 9.17*b*. By adding the radius CP_1 to the coordinate of point C, we can obtain the principal strains, $\varepsilon_1 = 600 + 500 = 1100$ and $\varepsilon_2 = 600 - 500 = 100$. Note that for plane strain the third principal strain is zero.

ANS. $\varepsilon_1 = 1100\ \mu \qquad \varepsilon_2 = 100\ \mu \qquad \varepsilon_3 = 0$

Step 7: Using triangle BCH we can find the principal angle θ_p,

$$\tan 2\theta_p = \frac{BH}{BC} = \frac{300}{400} \qquad \text{or} \qquad 2\theta_p = 36.87° \tag{E3}$$

Principal angle 1 can be found from θ_p as shown in Figure 9.18.

ANS. $\theta_1 = 71.6°\ \text{cw} \qquad \text{or} \qquad \theta_1 = 108.4°\ \text{ccw}$

Step 8: Intuitive check: We visualize a circle in a square, as shown in Figure 9.18*b*. As $\varepsilon_{yy} > \varepsilon_{xx}$ the rectangle will become longer in the y direction than in the x direction, and the circle will become an ellipse with the major axis along the y direction, as shown in Figure 9.18*c*. As $\gamma_{xy} < 0$, the angle between the x and y directions will increase. The rectangle will become a rhombus, and the major axis of the ellipse will rotate counterclockwise from the y axis, as shown in Figure 9.18*d*. Hence we expect principal axis 1 to be either in the third sector or in the seventh sector, confirming the result.

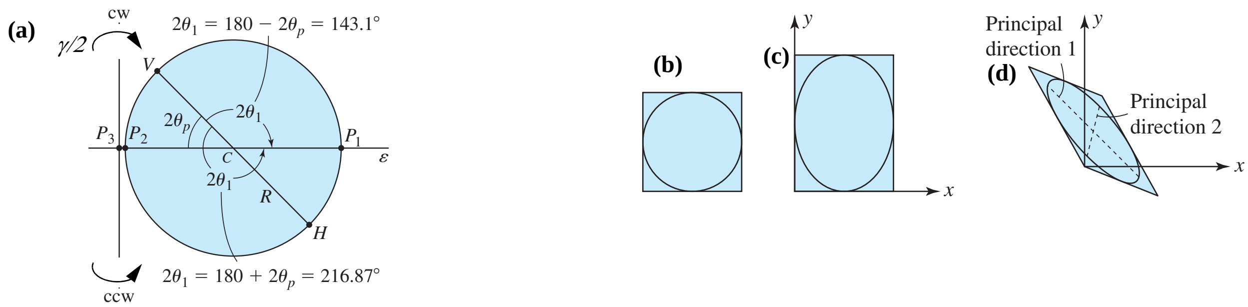


Figure 9.18 (a) Two values of principal angle 1. (b) Un-deformed shape. (c) Deformation due to normal strains. (d) Additional deformation due to shear strain.

Step 9: The circles between P_1 and P_2 and between P_2 and P_3 will be inscribed in the circle between P_1 and P_3. Thus the maximum shear strain at the point can be determined from the circle between P_1 and P_3,

$$\frac{\gamma_{max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} = \frac{1100}{2} \quad \text{or} \quad \textbf{ANS.} \quad \gamma_{max} = 1100\ \mu \tag{E4}$$

Step 10: We can draw the Cartesian coordinate system and the specified coordinate system with a square representing the undeformed state. Label points *V*, *H*, *N*, and *T* to represent the four directions, as shown in Figure 9.19.

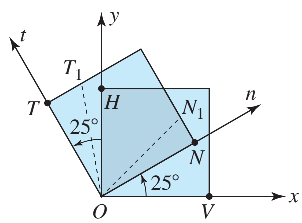


Figure 9.19 *n*, *t* coordinate system in Example 9.6

Step 11: Starting from point *V* on Mohr's circle, we rotate by 50° counterclockwise and obtain point *N* on Mohr's circle in Figure 9.17. Similarly, by starting from point *H* and rotating by 50° counterclockwise, we obtain point *T* on Mohr's circle in Figure 9.17.

Step 12: Angle *ACN* and angle *BCT* can be found as $50 - 2\theta_p = 13.13°$. From triangle *ACN* in Figure 9.21, the coordinates of point *N* are

$$\varepsilon_{nn} = 600 - 500\cos 13.13° = 113.1 \qquad \gamma_{nt}/2 = 500\sin 13.13° = 113.58 \tag{E5}$$

From triangle *BCT*, the coordinates of point *T* are

$$\varepsilon_{tt} = 600 + 500\cos 13.13° = 1086.9 \qquad \gamma_{nt}/2 = 500\sin 13.13° = 113.58 \tag{E6}$$

Step 13: In Figure 9.19 line *ON* rotates in the counterclockwise direction to ON_1, as seen in Equation (E5), and line *OT* rotates in the clockwise direction to OT_1, as seen in Equation (E6). Angle N_1OT_1 is less than angle *NOT*, and hence the shear strain in the *n*, *t* coordinate system is positive.

ANS. $\varepsilon_{nn} = 113.1\ \mu \qquad \varepsilon_{tt} = 1086.9\ \mu \qquad \gamma_{nt} = 227.2\ \mu$

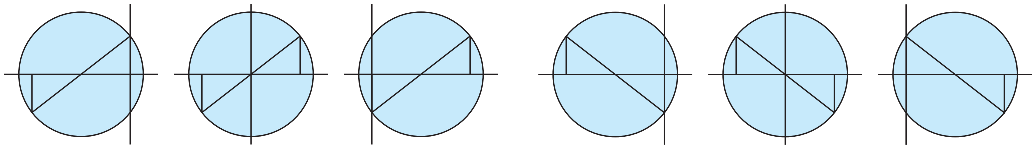
COMMENT

1. Example 9.4 and this example solve the same problem. But unlike with the method of equations used in Example 9.4, this example shows that we do not need an equation to solve the problem by Mohr's circle. Once Mohr's circle is constructed, the problem of strain transformation becomes a problem of geometry.

QUICK TEST 9.1

Time: 15 minutes/Total: 20 points

Grade yourself with the answers given in Appendix E. Each question is worth two points.
In Questions 1 through 3, associate the strain states with the appropriate Mohr's circle given.



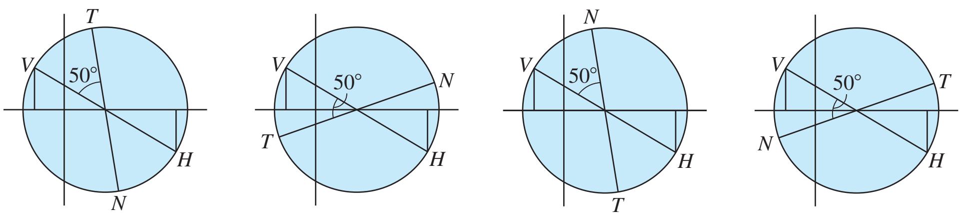
Circle *A* Circle *B* Circle *C* Circle *D* Circle *E* Circle *F*

1. $\varepsilon_{xx} = -600\ \mu$, $\varepsilon_{yy} = 0$, and $\gamma_{xy} = -600\ \mu$.

2. $\varepsilon_{xx} = 0$, $\varepsilon_{yy} = 600\ \mu$, and $\gamma_{xy} = 600\ \mu$.

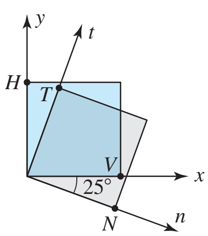
3. $\varepsilon_{xx} = 300\ \mu$, $\varepsilon_{yy} = -300\ \mu$, and $\gamma_{xy} = -600\ \mu$.

In Questions 4 and 5, the Mohr's circles corresponding to the states of strain $\varepsilon_{xx} = -500\ \mu$, $\varepsilon_{yy} = 1100\ \mu$, and $\gamma_{xy} = -1200\ \mu$ are shown. Identify the circle you would use to find the strains in the *n*, *t* coordinate system in each question.

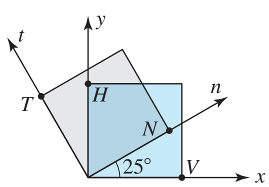


Circle *A* Circle *B* Circle *C* Circle *D*

4.

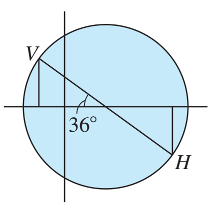


5.

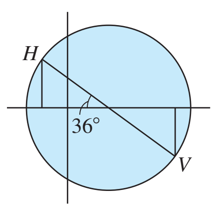


In Questions 6 and 7, the Mohr's circles for a state of strain are given. Determine the two possible values of principal angle 1 (θ_1) in each question.

6.

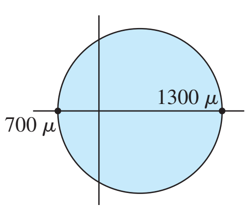


7.

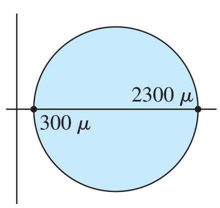


In Questions 8 through 10, the Mohr's circles for points in plane strain are given. Report principal strain 1 and maximum shear strain in each question.

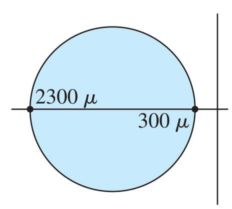
8.



9.



10.



9.4 GENERALIZED HOOKE'S LAW IN PRINCIPAL COORDINATES

In Section 3.5 it was observed that the generalized Hooke's law is valid for any orthogonal coordinate system. We have seen that the principal coordinates for stresses and strains are orthogonal.

It has been shown mathematically and confirmed experimentally that for isotropic materials the principal directions for strains are the same as the principal directions for stresses. In Example 9.9, we will see that the principal directions for stresses and strains are different when the material is orthotropic. For isotropic materials, we can write the generalized Hooke's law relating principal stresses to principal strains as follows:

$$\varepsilon_1 = \frac{\sigma_1 - \nu(\sigma_2 + \sigma_3)}{E} \tag{9.14.a}$$

$$\varepsilon_2 = \frac{\sigma_2 - \nu(\sigma_3 + \sigma_1)}{E} \tag{9.14.b}$$

$$\varepsilon_3 = \frac{\sigma_3 - \nu(\sigma_1 + \sigma_2)}{E} \tag{9.14.c}$$

Note that there are no equations for shear stresses and shear strains, as both these quantities are zero in the principal coordinate system. Now that we know that, at a point, principal axis 1 for stresses and strains is the same for isotropic material, we can extend our intuitive check to stress transformation. This can be done by viewing σ_{xx}, σ_{yy}, and τ_{xy} as analogous to ε_{xx}, ε_{yy}, and γ_{xy} in the visualization procedure outlined in Section 9.2.2.

EXAMPLE 9.7

The stresses $\sigma_{xx} = 4\text{ ksi}$ (T), $\sigma_{yy} = 10\text{ ksi}$ (C), and $\tau_{xy} = 4\text{ ksi}$ were calculated at a point on a free surface of an isotropic material. Determine (a) the orientation of principal axis 1 for stresses, using Mohr's circle for stress; (b) the orientation of principal axis 1 for strains, using Mohr's circle for strain. Use the following material constants: $E = 7500\text{ ksi}$, $G = 3000\text{ ksi}$, and $\nu = 0.25$.

PLAN

By substituting the stresses and material constants into the generalized Hooke's law in Cartesian coordinates, we can find the strains ε_{xx}, ε_{yy}, and γ_{xy}. We can draw Mohr's circle for stress to find principal direction 1 for stress, and we can draw Mohr's circle for strain to find principal direction 1 for strain.

SOLUTION

As the point is on a free surface, the state of stress is plane stress; hence $\sigma_{zz} = 0$. Substituting the stresses and the material constants into Equations (3.12a), (3.12b), and (3.12d), we obtain

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu}{E}\sigma_{yy} = \frac{4\text{ ksi}}{7500\text{ ksi}} - \frac{0.25}{7500\text{ ksi}}(-10\text{ ksi}) = 0.867(10^{-3}) = 867\ \mu \tag{E1}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E} - \frac{\nu}{E}\sigma_{xx} = \frac{-10\text{ ksi}}{7500\text{ ksi}} - \frac{0.25}{7500\text{ ksi}}(4\text{ ksi}) = -1.467(10^{-3}) = -1467\ \mu \tag{E2}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{4\text{ ksi}}{3000\text{ ksi}} = 1.333(10^{-3}) = 1333\ \mu \tag{E3}$$

(a) We draw the stress cube and record the coordinates of planes V and H,

$$V(4, 4) \qquad H(-10, 4) \tag{E4}$$

We then draw Mohr's circle for stress, as shown in Figure 9.20*a*. The angle θ_p can be found from triangle BCH (or ACV) and is given by

$$\tan 2\theta_p = \frac{4}{7} \qquad \text{or} \qquad \theta_p = 14.87° \tag{E5}$$

For this example $\theta_1 = \theta_p$ and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 14.87°\text{ ccw}$

(b) Since γ_{xy} is positive, the angle between the x and y coordinates decreases, as shown by the deformed shape in Figure 9.20*b*. Noting that the vertical coordinate is $\gamma/2$, we record the coordinates of points V and H,

$$V(867, 666.7) \qquad H(-1467, 666.7) \tag{E6}$$

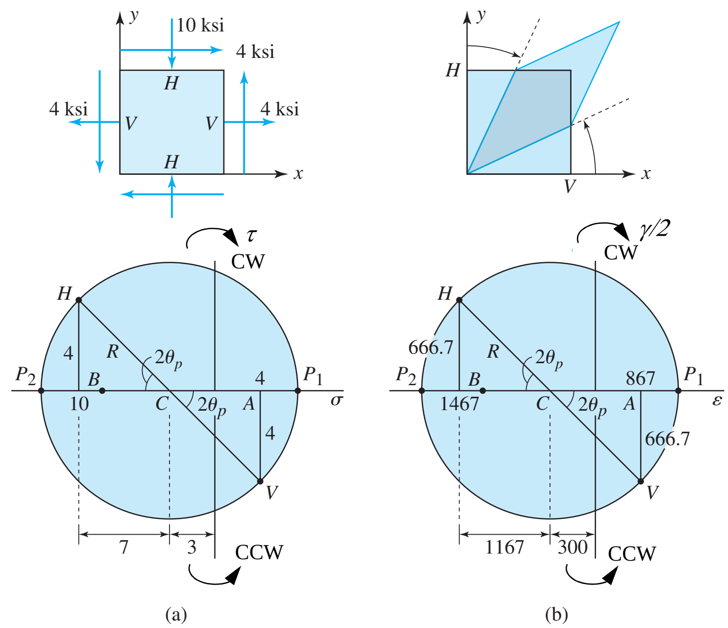


Figure 9.20 Mohr's circles in Example 9.7. (*a*) Stress. (*b*)

We then draw Mohr's circle for strain, as shown in Figure 9.20*b*. The angle θ_p can be found from triangle *BCH* (or *ACV*) and is given by

$$\tan 2\theta_p = \frac{666.7}{1167} \qquad \text{or} \qquad \theta_p = 14.87° \tag{E7}$$

For this example $\theta_1 = \theta_p$ and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 14.87°\text{ ccw}$

COMMENTS

1. The example highlights that for isotropic materials the principal axes for stresses and strains are the same.
2. The principal stresses can be found from Mohr's circle for stress as

$$\sigma_1 = -3\text{ ksi} + 8.06\text{ ksi} = 5.06\text{ ksi} \qquad \sigma_2 = -3\text{ ksi} - 8.06\text{ ksi} = -11.06\text{ ksi}$$

Noting that $\sigma_3 = 0$ because of the plane stress state, we obtain the principal strains from Equations (9.21a) and (9.21b),

$$\varepsilon_1 = \frac{5.06\text{ ksi} - 0.25(-11.06\text{ ksi})}{7500\text{ ksi}} = 1044\ \mu \qquad \varepsilon_2 = \frac{(-11.06\text{ ksi}) - 0.25(5.06\text{ ksi})}{7500\text{ ksi}} = -1644\ \mu \tag{E8}$$

3. From Mohr's circle for strain we obtain the same values,

$$\varepsilon_1 = -300\ \mu + 1344\ \mu = 1044\ \mu \qquad \varepsilon_2 = -300\ \mu - 1344\ \mu = -1644\ \mu \tag{E9}$$

The preceding highlights that the sequence of using the generalized Hooke's law and Mohr's circle does not affect the calculation of the principal strains.

4. We can conduct an intuitive check on the orientation of principal axis 1 for strain. We visualize a circle in a square, as shown in this example (Figure 9.21). Since $\varepsilon_{xx} > \varepsilon_{yy}$, the rectangle will become longer in the *x* direction than in the *y* direction, and the circle will become an ellipse with its major axis along the *x* direction. Since $\gamma_{xy} > 0$, the angle between the *x* and *y* directions will decrease. The rectangle will become a rhombus, and the major axis of the ellipse will rotate counterclockwise from the *x* axis. Hence we expect principal axis 1 to be either in the first sector or in the fifth sector of Figure 9.6. The result given in Equation (E7) puts principal axis 1 in sector 1, which is one of our intuitive answers.

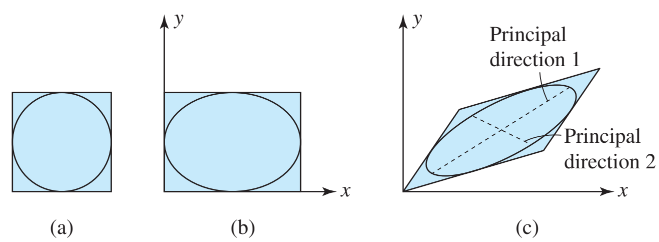


Figure 9.21 Estimating principal directions in Example 9.7 (*a*) Undeformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

EXAMPLE 9.8

For an isotropic materials show that $G = E/2(1+\nu)$.

PLAN

We can start with a state of pure shear and find principal stresses in terms of shear stress τ and principal strains in terms of shear strain γ. Using Equation (9.14.a) we can relate principal strain 1 to the principal stresses and then obtain a relationship between shear stress τ and shear strain γ. This relationship will have only E and ν in it. Comparing this to the relationship $\tau = G\gamma$, we can obtain the relationship between E, ν, and G.

SOLUTION

We start by assuming that all stress components except $\tau_{xy} = \tau$ are zero in the Cartesian coordinate system. We draw the stress cube and Mohr's circle in Figure 9.22*a* and find the principal stresses in terms of τ,

$$\sigma_1 = +\tau \qquad \sigma_2 = -\tau \tag{E1}$$

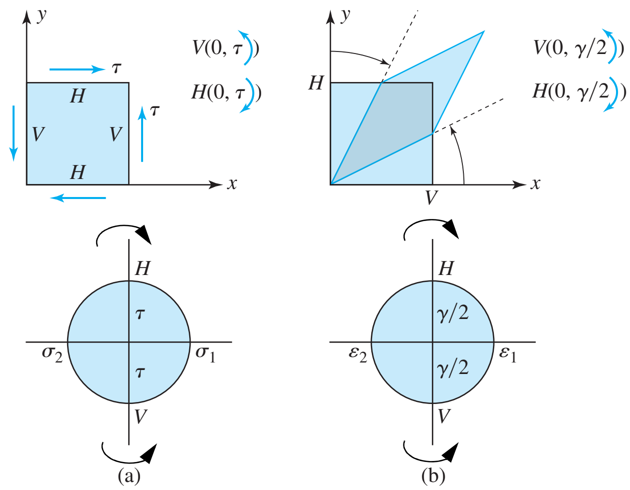


Figure 9.22 Mohr's circles for pure shear in Example 9.8. (*a*) Stress. (*b*) Strain.

We then start with all strains except $\gamma_{xy} = \gamma$ as zero. Using Mohr's circle in Figure 9.22*b*, we find the principal strains,

$$\varepsilon_1 = +\gamma/2 \qquad \varepsilon_2 = -\gamma/2 \tag{E2}$$

Noting that $\sigma_3 = 0$, we substitute Equations (E1), (E2), and (E3) into Equation (9.14.a) to obtain

$$\frac{\gamma}{2} = \frac{\tau - \nu(-\tau + 0)}{E} = \frac{1+\nu}{E}\tau \qquad \text{or} \qquad \tau = \frac{E}{2(1+\nu)}\gamma \tag{E3}$$

Comparing Equation (E5) to $\tau = G\gamma$, we obtain $G = E/2(1+\nu)$.

COMMENTS

1. Principal axes 1 in Mohr's circles for stress and for strain are seen to be at 90° counterclockwise from plane *V*. This implies that for isotropic materials the principal direction for stresses is the same as the principal direction for strains.
2. The state of pure shear can be produced by applying tensile stress in one direction (σ_1) and a compressive stress of equal magnitude in a perpendicular direction (σ_2). Then on a 45° plane a state of pure shear will be seen.

EXAMPLE 9.9

The stresses $\sigma_{xx} = 4$ ksi (T), $\sigma_{yy} = 10$ ksi (C), and $\tau_{xy} = 4$ ksi were calculated at a point on a free surface of an orthotropic composite material. An orthotropic material has the following stress–strain relationship at a point in plane stress:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} \qquad \varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} \qquad \frac{\nu_{yx}}{E_y} = \frac{\nu_{xy}}{E_x} \tag{9.15}$$

Determine (a) the orientation of principal axis 1 for stresses using Mohr's circle for stress; (b) the orientation of principal axis 1 for strains using Mohr's circle for strain. Use the following values for the material constants: $E_x = 7500$ ksi, $E_y = 2500$ ksi, $G_{xy} = 1250$ ksi, and $\nu_{xy} = 0.3$.

PLAN

By substituting the stresses and material constants into Equation (9.15), we can find the strains ε_{xx}, ε_{yy}, and γ_{xy}. We can draw Mohr's circle for stress to find principal direction 1 for stress, and we can draw Mohr's circle for strain to find principal direction 1 for strain.

SOLUTION

From $\nu_{yx}/E_y = \nu_{xy}/E_x$, we obtain

$$\nu_{yx} = \frac{E_y \nu_{xy}}{E_x} = \frac{(2500 \text{ ksi})(0.3)}{(7500 \text{ ksi})} = 0.1 \tag{E1}$$

Substituting the stresses and the material constants into Equation (9.15), we obtain

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} - \frac{\nu_{yx}}{E_y}\sigma_{yy} = \frac{4 \text{ ksi}}{7500 \text{ ksi}} - \frac{0.1}{2500 \text{ ksi}}(-10 \text{ ksi}) = 0.933(10^{-3}) = 933\ \mu \tag{E2}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy}}{E_y} - \frac{\nu_{xy}}{E_x}\sigma_{xx} = \frac{(-10 \text{ ksi})}{2500 \text{ ksi}} - \frac{0.3}{(7500 \text{ ksi})}(4 \text{ ksi}) = -4.160(10^{-3}) = -4160\ \mu \tag{E3}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G_{xy}} = \frac{4 \text{ ksi}}{1250 \text{ ksi}} = 3.200(10^{-3}) = 3200\ \mu \tag{E4}$$

(a) We draw the stress cube and record the coordinates of points *V* and *H*. We then draw Mohr's circle for stress, as shown in Figure 9.23*a*, and obtain

$$\tan 2\theta_p = \frac{4}{7} \quad \text{or} \quad \theta_p = 14.87° \text{ ccw} \tag{E5}$$

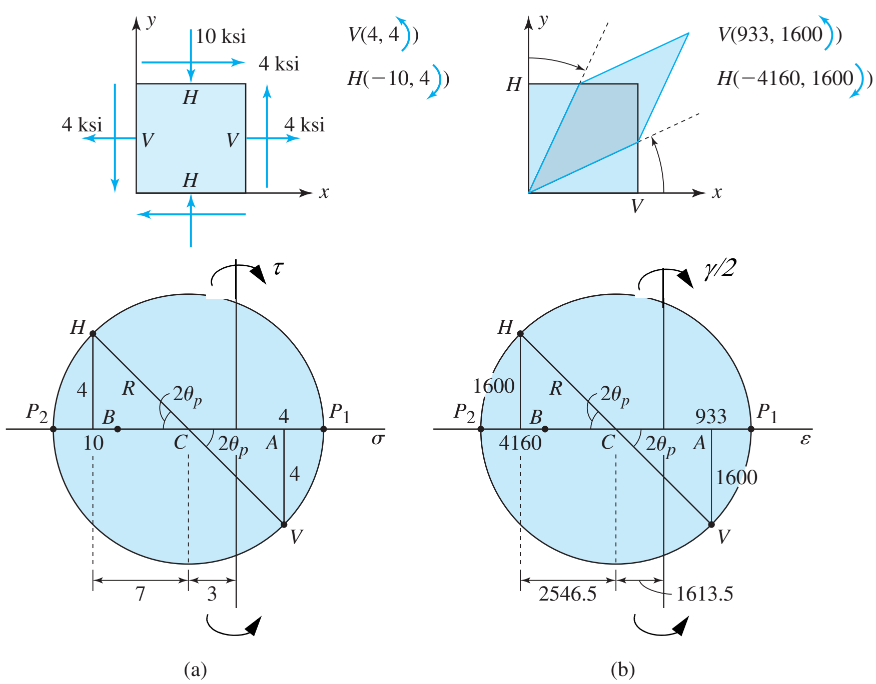


Figure 9.23 Mohr's circles in Example 9.9. (*a*) Stress. (*b*) Strain.

For this example $\theta_1 = \theta_p$, and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 14.87°$ ccw

(b) Since γ_{xy} is positive, the angle between the *x* and *y* coordinates decreases, as shown by the deformed shape in Figure 9.23*b*. Noting that the vertical coordinate is $\gamma/2$, we record the coordinates of points *V* and *H*. We then draw Mohr's circle for strain, as shown in Figure 9.23*b*. The angle θ_p can be found from triangle *BCH* (or *ACV*):

$$\tan 2\theta_p = \frac{1600}{2546.5} \quad or \quad \theta_p = 16.1° \text{ ccw} \tag{E6}$$

For this example $\theta_1 = \theta_p$, and we obtain the result for the orientation of principal axis 1.

ANS. $\theta_1 = 16.1°$ ccw

COMMENTS

1. The stress state in this example is the same as in Example 9.7. In Example 9.7 we concluded that for isotropic materials the principal directions for stresses and strains are the same. Equations (E5) and (E6) show that for orthotropic materials the principal directions for stresses and strains are different.
2. In Example 9.7, if we change the material constants for the isotropic material, then the stress values will be different, but the result for the principal angle for stress will not change. If we change the material constants for orthotropic materials, then we not only change the stress values but we may also change the principal angle for stress. This is because we may change the degree of orthotropicness—that is, the degree of difference in the material constants in the *x* and *y* directions.
3. The preceding two comments highlight some of the reasons why intuition based on isotropic materials can be misleading when working with composite materials. In such cases mathematical rigor can provide answers that once confirmed by experiment, can form a new knowledge base for the development of intuitive understanding.

PROBLEM SET 9.2

Visualization of principal axis

In Problems 9.14 through 9.18, the state of strain at a point in plane strain is as given in each problem. Estimate the orientation of the principal directions and report your results using the sectors shown in Figure 9.7.

	Strains		
Problem	ε_{xx}	ε_{yy}	γ_{xy}
9.14	–400 μ	600 μ	–500 μ
9.15	–600 μ	–800 μ	500 μ.
9.16	800 μ	600 μ	–1000 μ
9.17	0	600 μ,	–500 μ
9.18	–1000 μ	–500 μ	700 μ

Method of Equations and Mohr's circle

9.19 Starting from Equation (9.4), show that maximum or minimum normal strain will exist in the direction of θ_p, as given by Equation (9.7). (*Hint:* See the similar derivation in stress transformation.)

9.20 Show that the values of the maximum and minimum normal strains are given by Equation (9.8). (*Hint:* See the similar derivation in stress transformation.)

9.21 Show that angle θ_p as given by Equation (9.7) is the principal angle, that is, shear strain is zero in a coordinate system that is at an angle θ_p to the Cartesian coordinate system. (*Hint:* See the similar derivation in stress transformation.)

9.22 Show that the coordinate system of maximum in-plane shear strain is 45° to the principal coordinate system. (*Hint:* See the similar derivation in stress transformation.)

9.23 Show that the maximum in-plane shear strain is given by Equation (9.11). (*Hint:* See the similar derivation in stress transformation.)

9.24 Starting from Equations (9.4) and (9.6), obtain the expression of Mohr's circle given by Equation (9.13). (*Hint:* See the similar derivation in stress transformation.)

9.25 Solve Problem 9.5 by the method of equations.

9.26 Solve Problem 9.5 by Mohr's circle.

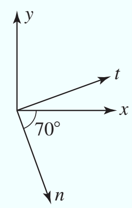
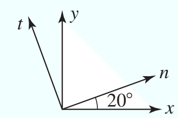
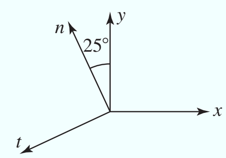
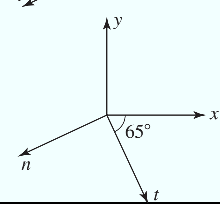
9.27 Solve Problem 9.6 by the method of equations.

9.28 Solve Problem 9.6 by Mohr's circle.

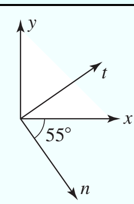
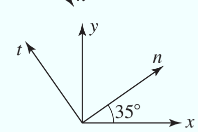
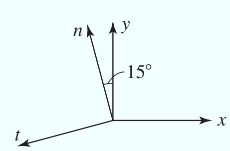
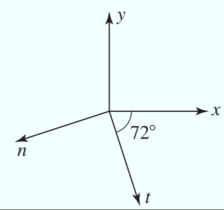
9.29 Solve Problem 9.7 by the method of equations.

9.30 Solve Problem 9.7 by Mohr's circle.

In Problems 9.31 through 9.34, at a point in plane strain, the strain components in the x, y coordinate system are as given. Using the associated figure, determine (a) the principal strains and principal angle 1; (b) the maximum shear strain; (c) the strain components in the n, t coordinate system.

	Strains				
Problem	ε_{xx}	ε_{yy}	γ_{xy}		
9.31	–400 μ	600 μ	–500 μ	**Figure P9.31**	
9.32	–600 μ	–800 μ,	500 μ	**Figure P9.32**	
9.33	250 μ	850 μ,	1600 μ	**Figure P9.33**	
9.34	–1800μ	–3600 μ	–1500 μ	**Figure P9.34**	

In Problems 9.35 through 9.38, at a point in plane strain, the strain components in the n, t coordinate system are as given. Using the associated figure, determine (a) the principal strains; (b) the maximum shear strain; (c) the strain components in the x, y coordinate system.

	Strains				
Problem	ε_{nn}	ε_{tt}	γ_{nt}		
9.35	2000 μ	–800 μ	750 μ	**Figure P9.35**	
9.36	–2000 μ	–800 μ	–600 μ	**Figure P9.36**	
9.37	350 μ	700 μ	1400 μ	**Figure P9.37**	
9.38	–3600 μ	2500 μ	–1000 μ	**Figure P9.38**	

In Problems 9.39 through 9.42, the principal strains ε_1 and ε_2 and the direction of principal direction 1 θ_1 from the x axis are given. Determine strains ε_{xx}, ε_{yy}, and γ_{xy} at the point.

	Principal Strains		*Principal Angle 1*
Problem	ε_1	ε_2	θ_1
9.39	1200 μ	300 μ	27.5°
9.40	900 μ	–600 μ	–20°
9.41	–200 μ	–2000 μ	105°
9.42	1400 μ	–600 μ	–75°

Generalized Hooke's law in principal coordinates

In Problems 9.43 through 9.45, the stresses in a thin body (plane stress) in the xy plane are as shown on each stress element. The modulus of elasticity E and Poisson's ratio ν are given in each problem. Using the associated figure, determine (a) the principal strains and principal angle 1 at the point; (b) the maximum shear strain at the point.

Problem	***E***	ν	
9.43	70 GPa	ν = 0.25	**Figure P9.43**
9.44	70 GPa	ν = 0.25	**Figure P9.44**
9.45	30,000 ksi	0.28	**Figure P9.45**

Figure P9.43

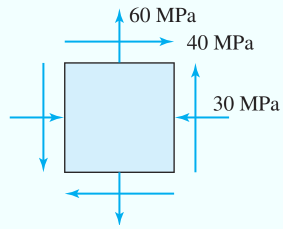


Figure P9.44

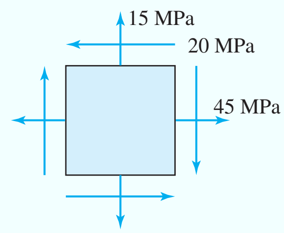
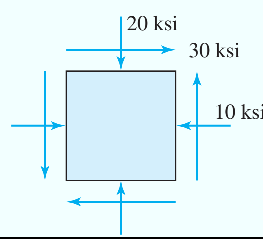


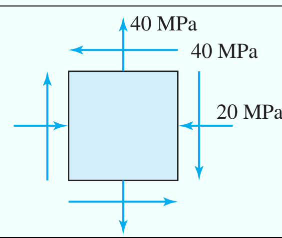
Figure P9.45



In Problems 9.46 through 9.48, the stresses in a thick body (plane strain) in the xy plane are as shown on each stress element. The modulus of elasticity E and Poisson's ratio ν are given in each problem. Using the associated figure, determine (a) the principal strains and principal angle 1 at the point; (b) The maximum shear strain at the point.

Problem	***E***	ν	
9.46	105 GPa	ν = 0.35	**Figure P9.46**

Figure P9.46



Problem	E	ν	
9.47	70 GPa	$\nu = 0.25$	Figure P9.47
9.48	30,000 ksi	0.28	Figure P9.48

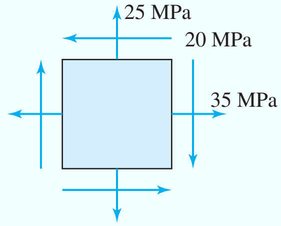


Figure P9.47

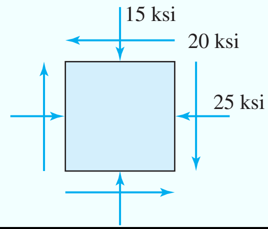


Figure P9.48

Orthotropic materials

In Problems 9.49 through 9.52, the properties of an orthotropic material and the stresses or strain are given at a point on a free surface. Using Equation (9.15), determine the principal direction for stresses and strains.

Problem	E_x	E_y	G_{xy}	ν_{xy}	Stresses / Strains
9.49	7500 ksi	2500 ksi	1250 ksi	0.25	$\varepsilon_{xx} = -400\ \mu$, $\varepsilon_{yy} = 600\ \mu$, and $\gamma_{xy} = -500\ \mu$
9.50	7500 ksi	2500 ksi	1250 ksi	0.25	$\sigma_{xx} = 10$ ksi (T), $\sigma_{yy} = 7$ ksi (C), and $\tau_{xy} = 5$ ksi.
9.51	50 GPa	18 GPa	9 GPa	0.25	$\varepsilon_{xx} = 800\ \mu$, $\varepsilon_{yy} = 200\ \mu$, and $\gamma_{xy} = 300\ \mu$.
9.52	50 GPa	18 GPa	9 GPa	0.25	$\sigma_{xx} = 70$ MPa (C), $\sigma_{yy} = 49$ MPa (C), and $\tau_{xy} = -30$ MPa

9.5 STRAIN GAGES

Strain gages are strain-measuring devices based on the changes in resistance in a wire with changes in its length. Since strain causes a length change, the change in resistance can be correlated to the strain in the wire by conducting an experiment. By bonding a wire to a stressed part, we can *assume* that the deformation of the wire is the same as that of the material. Hence, by measuring changes in the resistance of a wire, we can get the strains in the material. Strain gages are a sophisticated application of this technique.

Strain gages are usually manufactured by etching a thin foil of material, as shown in Figure 9.24. The back-and-forth pattern increases the sensitivity of the gage by providing a long length of wire in a very small area. Strain gages can be as small in length as $\frac{8}{1000}$ in., which for many engineering calculations is equivalent to measuring strain at a point.

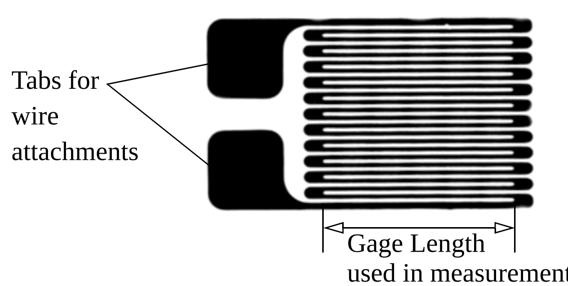


Figure 9.24 Typical strain gage.

Since we are measuring changes in the length of a wire, a strain gage measures only normal strains directly and not shear strains. In this section it will be shown how shear strains are calculated from the measured normal strains. Because of the finite sizes of strain gages, strain gages give an average value of strain at a point. To protect the strain gage from damage, no force is applied on its top. Hence strain gages are bonded to a free surface; that is, measurements take place in plane stress. We record the following observations:

1. Strain gages measure only normal strains directly.
2. Strain gages are bonded to a free surface. That is, the strains are in a state of plane stress and not plane strain.
3. Strain gages measure average strain at a point

In plane stress there are three *independent* strain components ε_{xx}, ε_{yy}, and γ_{xy}. To determine these, we need three observations at a point. In other words, we need to find normal strains in three directions. Figure 9.25 shows an assembly of three strain gages called a *strain rosette*. The strain gage readings ε_a, ε_b, and ε_c can be related to ε_{xx}, ε_{yy}, and γ_{xy} by Equation (9.4) as

$$\varepsilon_a = \varepsilon_{xx}\cos^2\theta_a + \varepsilon_{yy}\sin^2\theta_a + \gamma_{xy}\sin\theta_a\cos\theta_a \tag{9.16.a}$$

$$\varepsilon_b = \varepsilon_{xx}\cos^2\theta_b + \varepsilon_{yy}\sin^2\theta_b + \gamma_{xy}\sin\theta_b\cos\theta_b \tag{9.16.b}$$

$$\varepsilon_c = \varepsilon_{xx}\cos^2\theta_c + \varepsilon_{yy}\sin^2\theta_c + \gamma_{xy}\sin\theta_c\cos\theta_c \tag{9.16.c}$$

The three equations can be solved for the three unknowns ε_{xx}, ε_{yy}, and γ_{xy} since θ_a, θ_b, and θ_c are known.

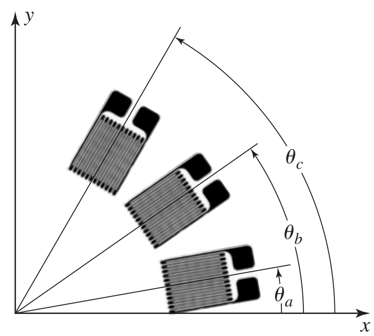


Figure 9.25 Strain rosette.

The angles at which strain gages are attached are chosen to reduce the algebra in the calculation of ε_{xx}, ε_{yy}, and γ_{xy}. Figure 9.26 shows two popular choices of angles in a strain rosette. Notice in Figure 9.26*b* that angle θ_c can be 120° or –60° (or 300° or –240°). This emphasizes that Equation (9.4) does not change if 180° is added to or subtracted from angle θ. (See Problem 9.53.) An alternative explanation is that normal strain is a measure of the deformation of a line and deformation is the relative movement of two points on a line. Hence the value does not depend on whether the two points on the line have positive or negative coordinates. We can summarize our observation simply:

- A change in strain gage orientation by ±180° makes no difference in the strain values.

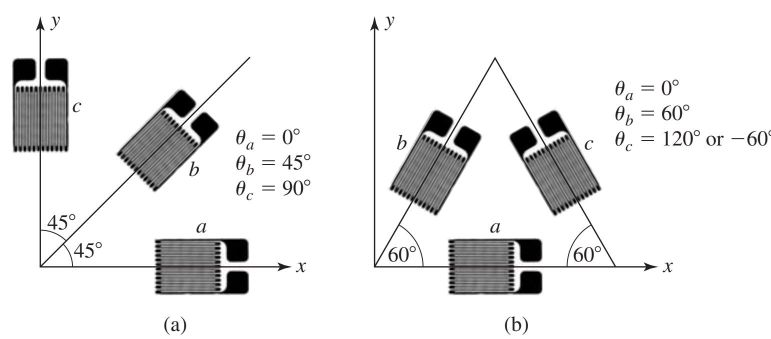


Figure 9.26 Strain rosettes. (*a*) 45°. (*b*) 60°.

Once strains ε_{xx}, ε_{yy}, and γ_{xy} are found, then the principal strains can be found. The principal stresses can be found next, if needed, from the generalized Hooke's law in principal coordinates. Alternatively, the stresses σ_{xx}, σ_{yy}, and τ_{xy} may be found first from the generalized Hooke's law, and then the principal stresses can be found. But it is important to remember that the point where strains are being measured is in plane stress, and hence $\sigma_{zz} = 0$. The strain in the *z* direction is the third principal strain and can be found from Equation (9.10).

EXAMPLE 9.10

Strains $\varepsilon_a = 900\ \mu$in./in., $\varepsilon_b = 200\ \mu$in./in., and $\varepsilon_c = 700\ \mu$in./in. were recorded by the three strain gages shown in Figure 9.27 at a point on the free surface of a material that has a modulus of elasticity $E = 30{,}000$ ksi and a Poisson ratio $\nu = 0.3$. Determine the principal stresses, principal angle 1, and the maximum shear stress at the point.

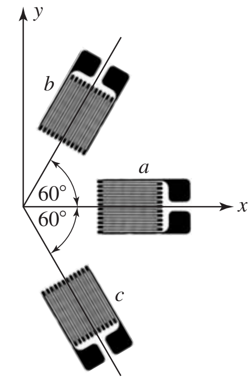


Figure 9.27 Strain rosette in Example 9.10.

PLAN: METHOD 1

We note that $\varepsilon_a = \varepsilon_{xx}$. We can find strains ε_{yy} and γ_{xy} from the two equations obtained by substituting $\theta_b = +60°$ and $\theta_c = -60°$ into Equation (9.4). We can then find principal strains 1 and 2 and principal angle 1 by using either Mohr's circle or the method of equations. Principal strain 3 can be found from Equation (9.10), and the maximum shear strain from the radius of the biggest circle. Using the generalized Hooke's law in principal coordinates we can find the principal stresses.

SOLUTION

Strain gages: The strain in the *x* direction is given by the strain gage *a* reading. Thus

$$\varepsilon_{xx} = 900\ \mu \tag{E1}$$

Substituting $\theta_b = +60°$ and $\theta_c = -60°$ into Equation (9.4), we obtain

$$\varepsilon_b = (900)\cos^2 60 + \varepsilon_{yy}\sin^2 60 + \gamma_{xy}\sin 60\cos 60 = 200 \quad \text{or} \quad 0.75\varepsilon_{yy} + 0.433\gamma_{xy} = -25 \tag{E2}$$

$$\varepsilon_c = (900)\cos^2(-60) + \varepsilon_{yy}\sin^2(-60) + \gamma_{xy}\sin(-60)\cos(-60) = 700 \quad \text{or} \quad 0.75\varepsilon_{yy} - 0.433\gamma_{xy} = 475 \tag{E3}$$

Solving Equations (E2) and (E3), we obtain

$$\varepsilon_{yy} = 300\ \mu \qquad \gamma_{xy} = -577.4\ \mu \tag{E4}$$

Mohr's circle for strain: We draw the deformed shape as shown in Figure 9.28*a* and write the coordinates of points *V* and *H* as

$$V(900, 288.7) \qquad H(300, 288.7) \tag{E5}$$

We then draw Mohr's circle for strain shown in Figure 9.28b and calculate the principal strains. From the Pythagorean theorem we can find the radius *R*,

$$R = \sqrt{CB^2 + BV^2} = \sqrt{300^2 + 288.7^2} = 416.4 \tag{E6}$$

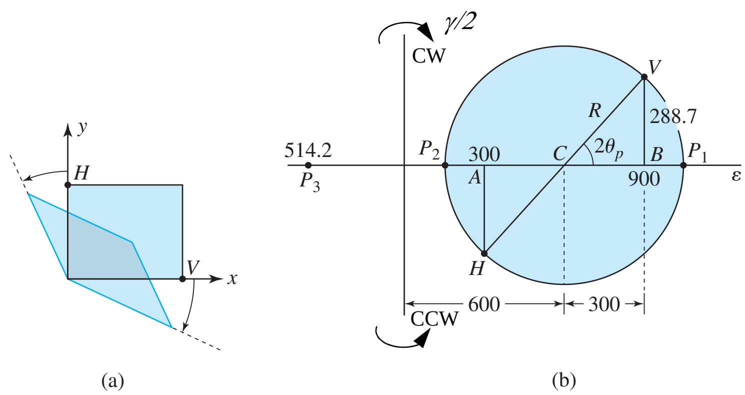


Figure 9.28 Mohr's circle in Example 9.4.

The principal strains are the coordinates of points P_1 and P_2 in Figure 9.28*b*,

$$\varepsilon_1 = 600 + 416.4 = 1016.4 \qquad \varepsilon_2 = 600 - 416.4 = 183.6 \tag{E7}$$

As the point is on a free surface, the state is in plane stress. Hence the third principal strain from Equation (9.10) is

$$\varepsilon_3 = \varepsilon_{zz} = -\frac{0.3}{1-0.3}(900+300) = -514.2\ \mu \tag{E8}$$

Using triangle *BCV* in Figure 9.28*b*, we can find the principal angle θ_p,

$$\cos 2\theta_p = \frac{CB}{CV} = \frac{300}{416.4} \tag{E9}$$

From Figure 9.28*b* we see that $\theta_1 = \theta_p$, and the direction is clockwise:

$$2\theta_p = 43.9° \qquad \theta_1 = \theta_p = 21.9°\ \text{cw} \tag{E10}$$

Intuitive check: We visualize a circle in a square, as in Figure 9.29*a*. Since $\varepsilon_{xx} > \varepsilon_{yy}$, the rectangle will become longer in the *x* direction than in the *y* direction, and the circle will become an ellipse with its major axis along the *x* direction, as shown in Figure 9.29*b*. Since $\gamma_{xy} < 0$, the angle between the *x* and *y* directions will increase. The rectangle will become a rhombus, and the major axis of the ellipse will rotate clockwise from the *x* axis, as shown in Figure 9.29*c*. Hence we expect principal axis 1 to be either in the eighth sector or in the fourth sector, confirming the result given in Equation (E10).

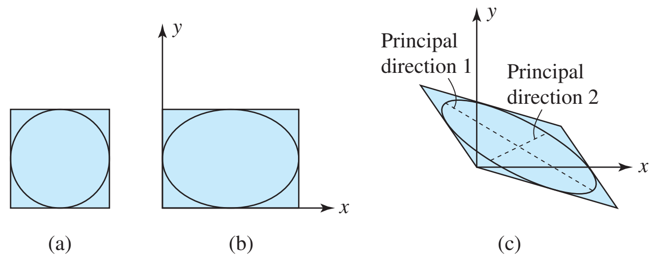


Figure 9.29 Estimating principal directions in Example 9.10. (*a*) Un-deformed shape. (*b*) Deformation due to normal strains. (*c*) Additional deformation due to shear strain.

Locating point P_3, which corresponds to the third principal strain in Figure 9.28*b*, we note that the circle between P_1 and P_3 will be a bigger circle than between P_2 and P_3, or between P_1 and P_2. Thus the maximum shear strain at the point can be determined from the circle between P_1 and P_3,

$$\frac{\gamma_{\max}}{2} = \frac{\varepsilon_1 - \varepsilon_3}{2} = 765.3 \qquad \gamma_{\max} = 1531\ \mu \tag{E11}$$

Hooke's law: For plane stress $\sigma_3 = 0$. From Equations (9.14.a) and (9.14.b) we obtain

$$\varepsilon_1 = \frac{\sigma_1 - \nu\sigma_2}{30{,}000\ \text{ksi}} = 1016(10^{-6}) \qquad \text{or} \qquad \sigma_1 - 0.3\sigma_2 = 30.48\ \text{ksi} \tag{E12}$$

$$\varepsilon_2 = \frac{\sigma_2 - \nu\sigma_1}{30{,}000\ \text{ksi}} = 184(10^{-6}) \qquad \text{or} \qquad \sigma_2 - 0.3\sigma_1 = 5.52\ \text{ksi} \tag{E13}$$

Solving Equations (E12) and (E13), we obtain $\sigma_1 = 35.31$ ksi and $\sigma_2 = 16.11$ ksi. For isotropic materials the principal direction for stresses and strains is the same.

ANS. $\sigma_1 = 35.3\ \text{ksi(T)} \qquad \sigma_2 = 16.1\ \text{ksi(T)} \qquad \sigma_3 = 0 \qquad \theta_1 = 21.9^o\ \text{CW}$

The shear modulus of elasticity is

$$G = \frac{E}{2(1+\nu)} = 11{,}538\ \text{ksi} \tag{E14}$$

The maximum shear stress can be found from Hooke's law as

$$\tau_{max} = G\gamma_{max} = (11,538)(1531)(10^{-6}) = 17.65\ \text{ksi} \tag{E15}$$

Check: We can also find the maximum shear stress as half the maximum difference between principal stresses. That is, from Equation (8.13), $\tau_{\max} = (35.3 - 0)/2 = 17.65\ \text{ksi}$, confirming Equation (E15).

ANS. $\tau_{\max} = 17.65\ \text{ksi}$

COMMENT

This example combines three concepts: the use of strain gages to find strain components in Cartesian coordinates, the use of Mohr's circle for finding principal strains, and the use of Hooke's law in principal coordinates for finding principal stresses.

PLAN: METHOD 2

We can find ε_{xx}, ε_{yy}, and γ_{xy} from the values of the strains recorded by the strain gages, as we did in Method 1. We can use Hooke's law in Cartesian coordinates to find σ_{xx}, σ_{yy}, and τ_{xy}. Using Mohr's circle for stress (or the method of equations), we can then find the principal stresses, principal angle 1, and the maximum shear stress.

SOLUTION

Strain gages: From Equations (E1) and (E4),

$$\varepsilon_{xx} = 900\ \mu \qquad \varepsilon_{yy} = 300\ \mu \qquad \gamma_{xy} = -577.4\mu \tag{E16}$$

Hooke's law: We note that for plane stress $\sigma_{zz} = 0$. Using Equations (3.12a) and (3.12b), we can write

$$\varepsilon_{xx} = \frac{\sigma_{xx} - \nu\sigma_{yy}}{30{,}000\text{ ksi}} = 900(10^{-6}) \quad \text{or} \quad \sigma_{xx} - 0.3\sigma_{yy} = 27\text{ ksi} \tag{E17}$$

$$\varepsilon_{yy} = \frac{\sigma_{yy} - \nu\sigma_{xx}}{30{,}000\text{ ksi}} = 300(10^{-6}) \quad or \quad \sigma_{yy} - 0.3\sigma_{xx} = 9\text{ ksi} \tag{E18}$$

Solving Equations (E17) and (E18), we obtain $\sigma_{xx} = 32.63$ ksi and $\sigma_{yy} = 18.79$ ksi From Equations (3.12d) and (E14) we obtain the shear stress as

$$\tau_{xy} = G\gamma_{xy} = (11,538)(-577.4)(10^{-6}) = -6.66\text{ ksi} \tag{E19}$$

Mohr's circle for stress

We draw the stress cube as shown in Figure 9.30*a* and record the coordinates of points *V* and *H* as

$$V(32.63, 6.66\circlearrowright) \qquad H(18.79, 6.66\circlearrowleft) \tag{E20}$$

We then draw Mohr's circle for stress as shown in Figure 9.30*b* and calculate the principal stresses. From the Pythagorean theorem we can find the radius *R*,

$$R = \sqrt{CB^2 + BV^2} = \sqrt{6.92^2 + 6.66^2} = 9.60 \tag{E21}$$

The principal stresses are the coordinates of points P_1 and P_2 in Figure 9.30*b*. As the point is on free surface, the state is in plane stress. Hence the third principal stress is zero,

$$\sigma_1 = 25.71 + 9.60 = 35.31\text{ ksi} \qquad \sigma_2 = 25.71 - 9.60 = 16.11\text{ ksi} \qquad \sigma_3 = 0 \tag{E22}$$

Using triangle *BCV* in Figure 9.30*b* we can find the principal angle θ_p, From Figure 9.30*b* we see that $\theta_1 = \theta_p$ and the direction is clockwise:

$$\cos 2\theta_p = \frac{CB}{CV} = \frac{6.92}{9.6} \qquad \text{or} \qquad 2\theta_p = 43.9° \qquad \theta_1 = \theta_p = 21.9°\text{ cw} \tag{E23}$$

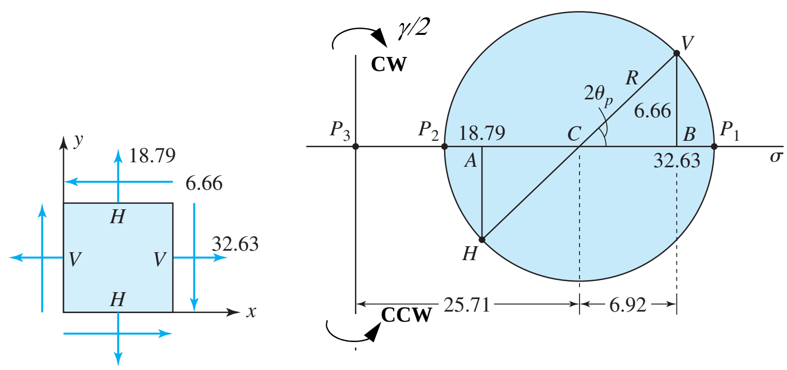


Figure 9.30 Mohr's circle in Example 9.10.

(a) (b)

ANS. $\sigma_1 = 35.3\text{ ksi(T)} \qquad \sigma_2 = 16.1\text{ ksi(T)} \qquad \sigma_3 = 0 \qquad \theta_1 = 21.9^o\text{ cw}$

The biggest circle will be between P_1 and P_3. The maximum shear stress is the radius of this circle and can be calculated as $\tau_{max} = (35.3\text{ ksi} - 0)/2 = 17.65\text{ ksi}$.

ANS. $\tau_{max} = 17.65\text{ ksi}$

COMMENT

1. As in Method 1, three concepts are combined, but the sequence in which the problem is solved is different. In Method 1 we used Mohr's circle (for strain) first and Hooke's law (in principal coordinates) second. In Method 2 we used Hooke's law (Cartesian coordinates) first and Mohr's circle (for stress) second. The number of calculations differs only with respect to ε_3, which is not calculated in Method 2.

EXAMPLE 9.11

The strain gage at point A recorded a value of $\varepsilon_A = -200\ \mu$. Determine the load P that caused the strain for the three cases shown in Figure 9.31. In each case the strain gage is 30° clockwise to the longitudinal axis (x axis). Use $E = 10{,}000$ ksi, $G = 4000$ ksi, and $\nu = 0.25$.

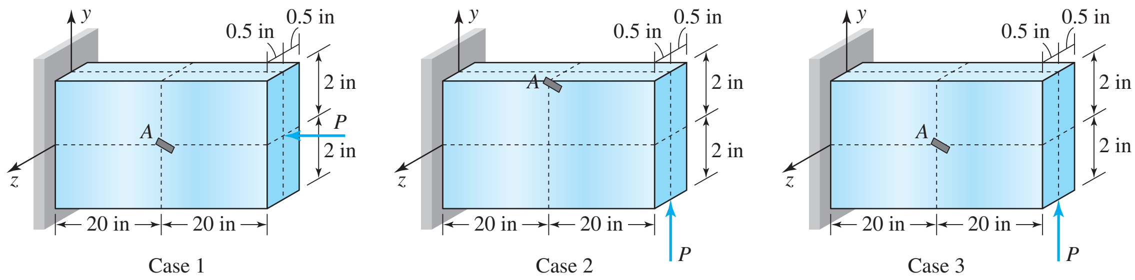


Figure 9.31 Three beams in Example 9.11.

PLAN

The axial stress σ_{xx} in case 1, the bending normal stress σ_{xx} in case 2, and the bending shear stress τ_{xy} in case 3 can be found in terms of P using Equations (4.8), (6.12), and (6.27), respectively. All other stress components are zero. Strains ε_{xx}, ε_{yy}, and γ_{xy} can be found in terms of P for each case, using the generalized Hooke's law. Substituting the strains and $\theta_A = -30°$ into Equation (9.4), the strain in the gage can be found in terms of P and equated to the given value of $-200\ \mu$ to obtain the value of P.

SOLUTION

Stress calculations: Recollect that A_s is the area between the free surface and point A, where shear stress is to be found. The cross-sectional area A, the area moment of inertia I_{zz}, and the first moment Q_z of the area A_s shown in Figure 9.32a can be calculated as

$$A = (1)(4) = 4\ \text{in}^2 \qquad I_{zz} = \frac{(1\ \text{in.})(4\ \text{in.})^3}{12} = 5.33\ \text{in.}^4 \qquad Q_z = (1\ \text{in.})(2\ \text{in.})(1\ \text{in.}) = 2\ \text{in.}^3 \tag{E1}$$

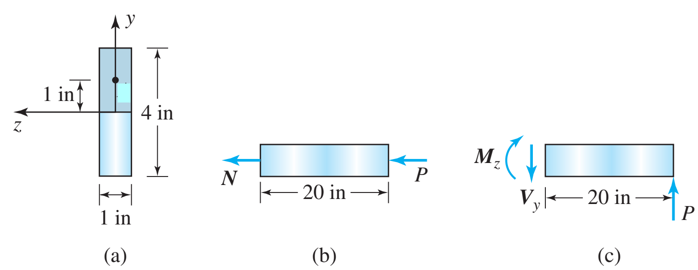


Figure 9.32 Calculation of geometric and internal quantities in Example 9.11

Figure 9.32b and c shows the free-body diagrams of the axial member and the beam after making the imaginary cut through point A. Using force and moment equilibrium equations, we find the internal forces and moment,

$$N = -P\ \text{kips} \qquad V_y = P\ \text{kips} \qquad M_z = 20P\ \text{in.}\cdot\text{kips} \tag{E2}$$

Substituting Equations (E1) and (E2) into Equation (4.8), we find the axial stress in case 1,

$$\sigma_{xx} = \frac{N}{A} = \frac{-P\ \text{kips}}{4\ \text{in.}^2} = -0.25P\ \text{ksi} \tag{E3}$$

Substituting Equations (E1), (E2), and $y = 2$ in into Equation (6.12), we find the bending normal stress in case 2,

$$\sigma_{xx} = -\frac{M_z y}{I_{zz}} = -\frac{(20P\ \text{in.}\cdot\text{kips})(2\ \text{in.})}{5.33\ \text{in.}^4} = -7.5P\ \text{ksi} \tag{E4}$$

Substituting Equations (E1), (E2), and $t = 1$ into Equation (6.27), we find the magnitude of τ_{xy} in case 3,

$$|\tau_{xy}| = \left|\frac{V_y Q_z}{I_{zz} t}\right| = \left|\frac{(P\ \text{kips})(2\ \text{in.}^3)}{(5.33\ \text{in.}^4)(1\ \text{in.})}\right| = 0.375P\ \text{ksi} \tag{E5}$$

Noting that τ_{xy} must have the same sign as V_y, we obtain the sign of τ_{xy} (see Section 6.6.6),

$$\tau_{xy} = 0.375P\ \text{ksi} \tag{E6}$$

Strain calculations: The only two nonzero stress components are given by Equations (E3), (E4), and (E6) for each case. Using the generalized Hooke's law [or Equations (4.13) and (6.29)], we obtain the strains for each case. Substituting the strains and $\theta_A = -30°$ into Equation (9.4) and equating the result to $-200\ \mu$ give the value of load P for each case:

- *Case 1:*

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{-0.25P\text{ ksi}}{10000\text{ ksi}} = -25P\,\mu \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} = 6.25P\,\mu \qquad \gamma_{xy} = 0 \tag{E7}$$

$$\varepsilon_A = (-25P\,\mu)\cos^2(-30°) + (6.25P\,\mu)\sin^2(-30°) = -17.19P\,\mu = -200\,\mu \qquad \text{or} \qquad P = 11.6\text{ kips} \tag{E8}$$

ANS. $P = 11.6$ kips

- *Case 2:*

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{-7.5P\text{ ksi}}{10000\text{ ksi}} = -750P\,\mu \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} = -\nu\varepsilon_{xx} = 187.5P\,\mu \qquad \gamma_{xy} = 0 \tag{E9}$$

$$\varepsilon_A = (-750P\,\mu)\cos^2(-30°) + (187.5P\,\mu)\sin^2(-30°) = -515.63P\,\mu = -200\,\mu \qquad \text{or} \qquad P = 0.39\text{ kips} \tag{E10}$$

ANS. $P = 0.39$ kips

- *Case 3:*

$$\varepsilon_{xx} = 0 \qquad \varepsilon_{yy} = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{0.375P\text{ ksi}}{4000\text{ ksi}} = 93.75P\,\mu \tag{E11}$$

$$\varepsilon_A = (93.75P\,\mu)\sin(-30°)\cos(-30°) = -40.59P\,\mu = -200\,\mu \qquad \text{or} \qquad P = 4.93\text{ kips} \tag{E12}$$

ANS. $P = 4.93$ kips

COMMENTS

1. This example demonstrates one of the basic principles used in the design of *load transducers*, also called *load cells*. Load transducers are used for measuring, applying, and controlling forces and moments on a structure. This example showed how one may measure a force by using strain gage readings and mechanics of materials formulas. The electrical signal from the strain gage can be processed and correlated with the intensity of the force and moment. It can be used to apply and control these quantities.
2. In this example the strain in the gage was caused by a single force. When there are multiple forces or moments acting on a structure, then to correlate strain gage readings to the applied forces and moments we need to supplement the formulas of mechanics and materials with the formulas for the Wheatstone bridge. See Section 9.6 for additional details on the Wheatstone bridge.
3. In Examples 9.5 and 9.10 and in this example we saw the use of the generalized Hooke's law. An alternative is to use formulas that are derived from the generalized Hooke's law. This is one important reason for memorizing the generalized Hooke's law.

PROBLEM SET 9.3

Strain gages

9.53 Show that upon substituting $\theta \pm 180°$ in place of θ, the strain transformation equation, Equation (9.4), is unchanged.

9.54 At a point on a free surface the strain components in the *x*, *y* coordinates are calculated as $\varepsilon_{xx} = 400\ \mu$in./in., $\varepsilon_{yy} = -200\ \mu$in./in., and $\gamma_{xy} = 500\ \mu$rad. Predict the strains that the strain gages shown in Figure P9.54 would record.

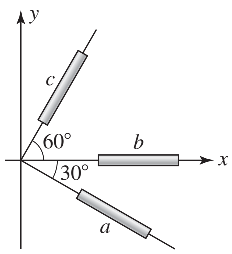


Figure P9.54

9.55 At a point on a free surface the strains recorded by the three strain gages shown in Figure P9.54 are $\varepsilon_a = 200\ \mu$in./in., $\varepsilon_b = 100\ \mu$in./in., and $\varepsilon_c = -400\ \mu$in./in. Determine strains ε_{xx}, ε_{yy}, and γ_{xy}.

9.56 At a point on a free surface of an aluminum machine component (E = 10,000 ksi and G = 4000 ksi) the stress components in the *x*, *y* coordinates were calculated by the finite-element method as σ_{xx} = 22 ksi (T), σ_{yy} = 15 ksi (C), and $\tau_{xy} = -10$ ksi. Predict the strains that the strain gages shown in Figure P9.56 would show.

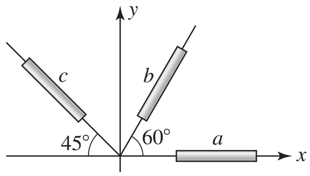


Figure P9.56

9.57 At a point on a free surface of aluminum (E = 10,000 ksi and G = 4000 ksi) the strains recorded by the three strain gages shown in Figure P9.56 are ε_a = –600 μin./in., ε_b = 500 μin./in., and ε_c = 400 μin./in. Determine stresses σ_{xx}, σ_{yy}, and τ_{xy}.

9.58 At a point on a free surface of a machine component (E = 80 GPa and G = 32 GPa) the stress components in the x, y coordinates were calculated by the finite-element method as σ_{xx} = 50 MPa (T), σ_{yy} = 20 MPa (C), and τ_{xy} = 96 MPa. Predict the strains that the strain gages shown in Figure P9.58 would show.

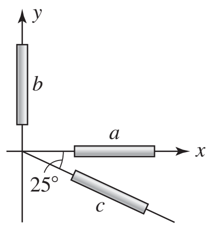


Figure P9.58

9.59 At a point on a free surface of a machine component (E = 80 GPa and G = 32 GPa) the strains recorded by the three strain gages shown in Figure P9.58 are ε_a = 1000 μm/m, ε_b = 1500 μm/m, and ε_c = –450 μm/m. Determine stresses σ_{xx}, σ_{yy}, and τ_{xy}.

9.60 On a free surface of steel (E = 210 GPa and ν = 0.28) the strains recorded by the three strain gages shown in Figure P9.60 are ε_a = –800 μm/m, ε_b = –300 μm/m, and ε_c = –700 μm/m. Determine the principal *strains*, principal angle 1, and the maximum shear *strain*

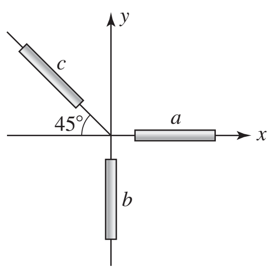


Figure P9.60

9.61 On a free surface of steel (E = 210 GPa and ν = 0.28) the strains recorded by the three strain gages shown in Figure P9.60 are ε_a = 200 μm/m, ε_b = 100 μm/m, and ε_c = 0. Determine the principal *stresses*, principal angle 1, and the maximum shear *stress*.

9.62 On a free surface of an aluminum machine component (E = 10,000 ksi and ν = 0.25) the strains recorded by the three strain gages shown in Figure P9.62 are ε_a = –100 μin./in., ε_b = 200 μin./in., and ε_c = 300 μin./in. Determine the principal *strains*, principal angle 1, and the maximum shear *strain*.

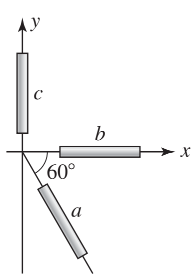


Figure P9.62

9.63 On a free surface of an aluminum machine component (E = 10,000 ksi and ν = 0.25) the strains recorded by the three strain gages shown in Figure P9.62 are ε_a = 500 μin./in., ε_b = 500 μin./in., and ε_c = 500 μin./in. Determine the principal *stresses*, principal angle 1, and the maximum shear *stress*.

Strain gages on structural elements

9.64 An aluminum (E = 70 GPa, G = 28 GPa) 50-mm × 50-mm square bar is axially loaded with a force F = 100 kN as shown in shown in Figure P9.64. Determine the strain that will be recorded by the strain gage.

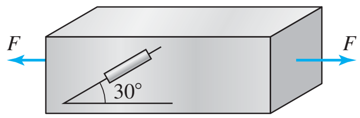


Figure P9.64

9.65 An aluminum (E = 70 GPa, G = 28 GPa) 50-mm × 50-mm square bar is axially loaded as shown in shown in Figure P9.64. Determine applied force F when the gage shows a reading of 200 μ.

9.66 A circular steel (E = 30,000 ksi, ν = 0.3) bar has a diameter of 2 in. and is axially loaded as shown in Figure P9.66. If the applied axial force F = 100 kips, determine the strain the gage will show.

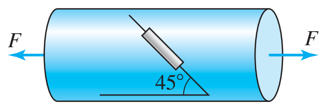


Figure P9.66

9.67 A circular steel (E = 30,000 ksi, ν = 0.3) bar has a diameter of 2 in. and is axially loaded as shown in Figure P9.66. Determine the applied axial force F when the strain gage shows a reading of 1000 μin./in.

9.68 A circular shaft of 2-in diameter has a torque applied to it as shown in Figure P9.68.The shaft material has a modulus of elasticity of 30,000 ksi and a Poisson's ratio of 0.3. Determine the strain that will be recorded by a strain gage.

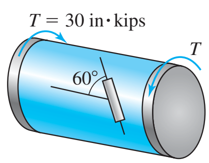


Figure P9.68

9.69 A circular shaft of 50-mm diameter has a torque applied to it as shown in Figure P9.69. The shaft material has a modulus of elasticity E = 70 GPa and a shear modulus G = 28 GPa. If the applied torque T = 500 N-m, determine the strain that the gage will show.

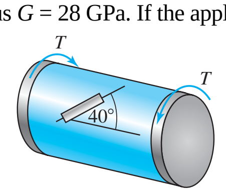


Figure P9.69

9.70 A circular shaft of 50-mm diameter has a torque applied to it as shown in Figure P9.69. The shaft material has a modulus of elasticity E = 70 GPa and a shear modulus G = 28 GPa. If the strain gage shows a reading of –600 μ, determine the applied torque T.

9.71 The steel cylindrical pressure vessel (E = 210 GPa and ν = 0.28) shown in Figure P9.71 has a mean diameter of 1000 mm. The wall of the cylinder is 10 mm thick and the gas pressure is 200 kPa. Determine the strain recorded by the two strain gages attached on the surface of the cylinder.

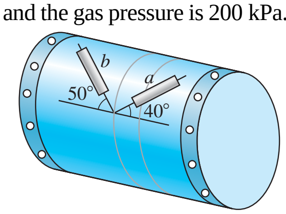


Figure P9.71

9.72 An aluminum beam (E = 70 GPa and ν = 0.25) is loaded by a force P = 10 kN and moment M = 5 kN·m at the free end, as shown in Figure P9.72. If the two strain gages shown are at an angle of 25° to the longitudinal axis, determine the strains in the gages.

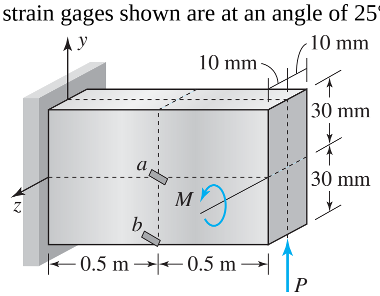
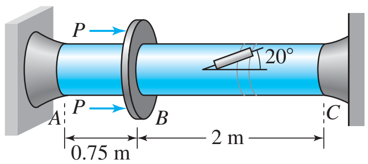


Figure P9.72

9.73 An aluminum beam (E = 70 GPa and ν = 0.25) is loaded by a force P and a moment M at the free end, as shown in Figure P9.72. Two strain gages at 30° to the longitudinal axis recorded the following strains: ε_a = –386 μm/m and ε_b = 4092 μm/m. Determine the applied force P and applied moment M.

9.74 A steel rod (E = 210 GPa and ν = 0.28) of 50-mm diameter is loaded by axial forces P = 100 kN, as shown in Figure P9.74. Determine the strain that will be recorded by the strain gage.

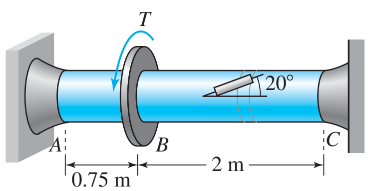
Figure P9.74



9.75 The strain gage mounted on the surface of the solid axial steel rod (E = 210 GPa and ν = 0.28) illustrated in Figure P9.74 showed a strain of –214 μm/m. If the diameter of the shaft is 50 mm, determine the applied axial force P.

9.76 A steel shaft (E = 210 GPa and ν = 0.28) of 50-mm diameter is loaded by a torque T = 10 kN·m, as shown in Figure P9.76. Determine the strain that will be recorded by the strain gage.

Figure P9.76

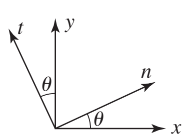


9.77 The strain gage mounted on the surface of the solid steel shaft (E = 210 GPa and ν = 0.28) shown in Figure P9.76 recorded a strain of 1088 μm/m. If the diameter of the shaft is 75 mm, determine the applied torque T.

Stretch yourself

In Problems 9.78 through 9.80, Equations (9.17.a) and (9.17.b) are transformation equations relating the x, y coordinates to the n, t coordinates of a point (Figure P9.77). Equations (9.17.c) and (9.17.d) are transformation equations relating displacements u and v *in the x and y directions to the displacements u_n and u_t in the n and t directions, respectively. Solve each problem using Equations (9.24a) through (9.24d).*

Figure P9.77



$$n = x\cos\theta + y\sin\theta \tag{9.17.a}$$

$$t = -x\sin\theta + y\cos\theta \tag{9.17.b}$$

$$u_n = u\cos\theta + \mathrm{v}\sin\theta \tag{9.17.c}$$

$$u_t = -u\sin\theta + \mathrm{v}\cos\theta \tag{9.17.d}$$

9.78 Starting with $\varepsilon_{nn} = \partial u_n/\partial n$ and using Equations (9.24*a*) through (9.24*d*) and the chain rule for differentiation, derive Equation (9.4).

9.79 Starting with $\varepsilon_{tt} = \partial u_t/\partial t$ and using Equations (9.24*a*) through (9.24*d*) and the chain rule for differentiation, derive Equation (9.5).

9.80 Starting with $\gamma_{nt} = \partial u_t/\partial n + \partial u_n/\partial t$ and using Equations (9.24*a*) through (9.24*d*) and the chain rule for differentiation, derive Equation (9.6).

9.81 Starting from Equation (9.15), show that for isotropic materials $E_x = E_y$ and $G_{xy} = E_x/2(1+\nu)$.

Computer problems

9.82 The displacements u and v in the x and y directions are given by the equations

$$u = [0.5(x^2 - y^2) + 0.5xy + 0.25x]10^{-3}\ \text{mm} \qquad \mathrm{v} = [0.25(x^2 - y^2) - xy]10^{-3}\ \text{mm}$$

Assuming plane strain, determine the principal strains, principal angle 1, and the maximum shear strain every 30° on a circle of radius 1 around the origin. Use a spreadsheet or write a computer program for the calculation.

9.83 On an aluminum beam (E = 70 GPa and ν = 0.25) two strain gages were attached to monitor loads P and w, which vary slowly over time (Figure P9.83). The strain gage readings are given in Table 9.83. Determine the values of P and w at the times the strains were measured.

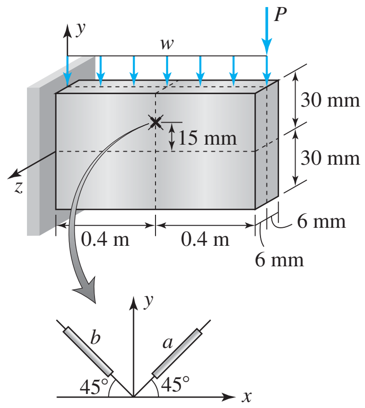


Figure P9.83

TABLE P9.83 **Strain values**

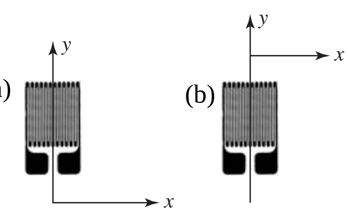
	ε_a (μ)	ε_b (μ)
1	1501	2368
2	1433	2276
3	1385	2193
4	1483	2336
5	1470	2331
6	1380	2191
7	1448	2282
8	1496	2366
9	1398	2223
10	1411	2228

QUICK TEST 9.2

Time: 15 minutes/Total: 20 points

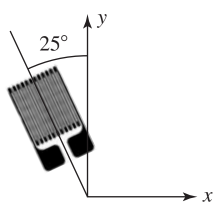
Grade yourself with the answers given in Appendix E. Each question is worth two points.

1. The strain gage recorded a strain of 800 μ. What is ε_{yy} for the two cases shown?



In Questions 2 through 4, report the smallest positive and the smallest negative angle θ that can be substituted in the strain transformation equation relating the strain gage reading to strains in Cartesian coordinates.

4.



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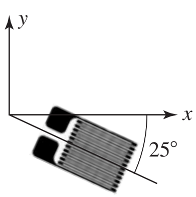

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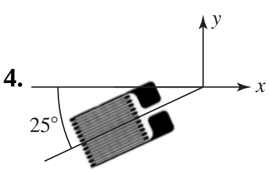
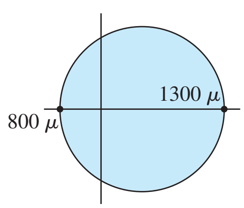


In Questions 5 through 7, Mohr's circles for strains for points in plane stress are as shown. The modulus of elasticity of the material is E = 10,000 ksi and Poisson's ratio is 0.25. What is the maximum shear strain in each question?

4.



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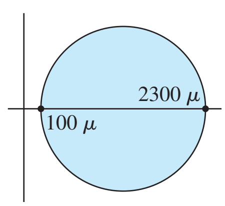

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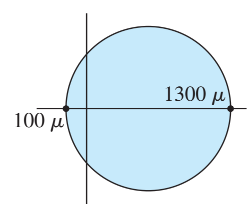


In Questions 8 through 10, answer true or false. If false, then give the correct explanation.

8. In plane strain there are two principal strains, but in plane stress there are three principal strains.

9. Since strain values change with the coordinate system, the principal strains at a point depend on the coordinate system used in finding the strains.

10. The principal coordinate axis for stresses and strains is always the same, irrespective of the stress–strain relationship.

MoM in Action: Load Cells

Load cells are everywhere in our lives, even if we do not call them by their name. A *load cell* is a device that measures, controls, or applies a force or a moment. Bathroom scales, tire pressure gauges, hydraulic presses, and pressure pads are all load cells, based on variety of mechanical principles.

Scales for weighing were the first kind of load cell. They have been in existence at least since Archimedes (Syracuse, Greece, 287– 212 BCE) stated the lever principle. These mechanical load cells can measure weights with high precision (Figure 9.33*a*) over a large range, provided the fulcrum point is a fine knife edge. A pointer, attached at the fulcrum point, allows the readings to be calibrated and amplified (Figure 9.33*a*). In this way measurements can be made from a few milligrams in chemistry laboratories to thousands of kilograms on truck scales. In spring scales, it is the extension of a spring that is calibrated. However, the familiar pointers from bathroom scales are now being replaced by digital readings.

Today most load cells are constructed using strain gages. Trucks that had to come to a stop for weighing by mechanical scales now simply drive over scales that have been strain gaged. The popularity of strain gages comes from two facts, one in the mechanics of materials and the other electrical: the formulas relating the force or moment on structural members to the strains (see Example 9.11) are very reliable; and the signal from strain gages can be processed for reading, storage, or control. A vast variety of load cells are manufactured ready for use; others are custom build for specific applications. Figure 9.33*b* shows a load cells built around axial-member.

(a) **(b)**

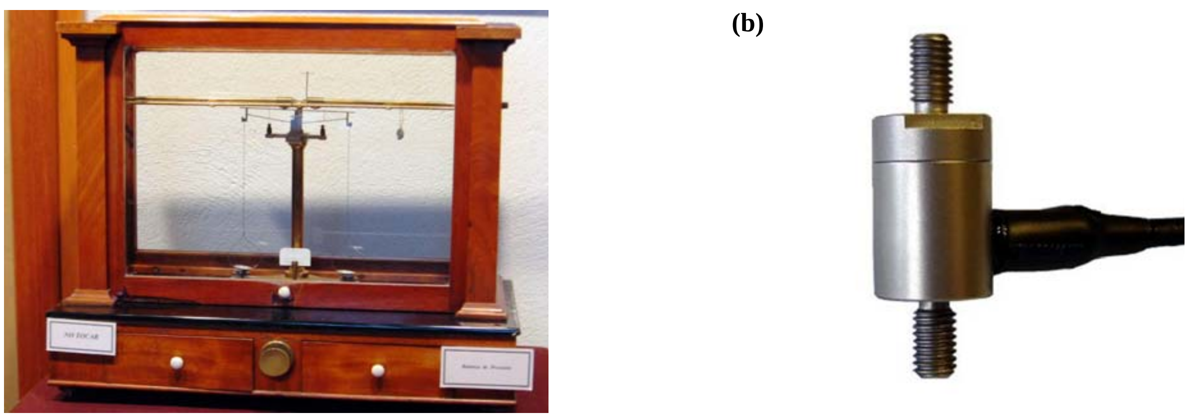


Figure 9.33 Load cells: (a) weighing scale (b) tension/compression (Courtesy Celsum Technologies Ltd.).

Load cells are used to maintain proper tension in manufacturing rolls of paper or metal sheets. They are also used for monitoring tension in the cables and compression in the towers of a suspension bridge. Load cells embedded in masonry can detect cracks in structures during construction and operation. Accurate drug dosages can be delivered by calibrating the weight of fluid to load cell readings. The field of robotics and assembly-line automation also uses a vast variety of load cells, from earthbound applications to the *Rovers* on the Moon and Mars.

For all their complexity and variety, from mundane applications to the cutting edge, the heart of a load cell is the predictable deformation of a structural member, according to the simple formulas we have studied in this book. Such is the breath and importance of mechanics of materials.

*9.6 CONCEPT CONNECTOR

The history of strain gages is interesting in its own right. As we see in Section 9.6.1, however, it also heralds the pitfalls for modern universities in maintaining the delicate balance between pure research for knowledge and its potential commercial benefits. Section 9.6.2 then looks ahead at how strain gage resistance is measured using a Wheatstone bridge.

9.6.1 History: Strain Gages

Two Americans invented the strain gage nearly simultaneously. In 1938 Arthur C. Ruge of the Massachusetts Institute of Technology (MIT) wanted to measure low-level strains in an elevated thin-walled water tank during an earthquake. He solved this problem by inventing the strain gage. When Ruge sought to register his invention with the MIT patent committee in 1939, the committee felt that the invention was unlikely to have significant commercial use and released the invention to him. Around the same time, Edward E. Simmons, then a graduate student at the California Institute of Technology, was studying the stress–strain characteristics of metals during impact. He invented the strain gage independently, as part of a dynamometer for measuring the power of impact. Caltech and Simmons waged a legal battle for the rights to the patent, but Simmons won because, as a student, he was not a salaried employee. Ruge and Simmons subsequently resolved their patent claims to each one's satisfaction.

Today strain gages are the most popular strain-measuring devices. Strain gages are also used in applications involving measurements or control of forces and moments. Pressure transducers, force transducers, torque transducers, load cells, and dynamometers are all examples of industrial applications of strain gages, whereas a bathroom scale is an example of a household product using strain gages. The popularity of strain gages comes from their cost-effectiveness in measuring strains as small as 1 μmm/mm to strains as large as 50,000 μmm/mm over a large range of temperatures.

The sensitivity of a strain gage is called the *gage factor,* which is the ratio of percentage change in resistance to percentage change in length (strain). Metal foil gages have gage factors of between 2 and 4. Ideally we would like a linear relationship between changes in resistance to strain—in other words, a constant gage factor over the range of measurements. To keep the value as close as possible to a constant, strain gages are constructed with different materials for different applications. The most common material is constantan or Advance, an alloy of copper (55%) and nickel (45%). The thermal conductivity of the two metals is such that the gage does not undergo significant thermal expansion over a large range of temperatures (–75°C to 175°C); the gage is thus said to be *self-temperature-compensated*. Annealed constantan is useful in large strain measurements (as high as 20%). For high-temperature applications, an alloy of iron (70%), chromium (20%), and aluminum (10%), called Armour D, is used. Strain gages using semiconductors (doped silicon wafers) have gage factors of between 50 to 200 and are used for small-strain measurements, but they require extreme care during installation because of the brittle nature of the silicon wafers.

9.6.2 Wheatstone Bridge Application to Strain Gages

Early strain gages were built by taking a very thin wire and going back and forth a number of times over a small area. This construction technique is based on the observation that the resistance R of a wire is related to its length L, its cross-sectional area A, and the material-specific resistance ρ by the expression $R = \rho L/A$. For a given value of strain, a longer wire results in a larger change in L, and hence a larger change in the resistance, which can be measured more easily. At the same time, the small cross-sectional area reduces the transverse effect of Poisson's ratio. Winding the long wire in a small region therefore leads to a better average strain value. Though the idea of using a long thin conductor in a small region still dictates the design of modern strain gages, the manufacturing process has changed. *Photoetching*, in which material is removed chemically to produce a desired pattern, has replaced winding a wire.

By measuring the change in resistance and knowing the gage factor, one can find the strain from a strain gage. The most common means of measuring changes in resistance is the Wheatstone bridge circuit, shown in Figure 9.34. The bridge was invented by Samuel Hunter Christie in 1833 and made popular by Charles Wheatstone in 1843.

The voltage V_0 in Figure 9.34 can be related to V as follows:

$$V_0 = V\frac{R_1 R_3 - R_2 R_4}{(R_1 + R_2)(R_3 + R_4)}$$

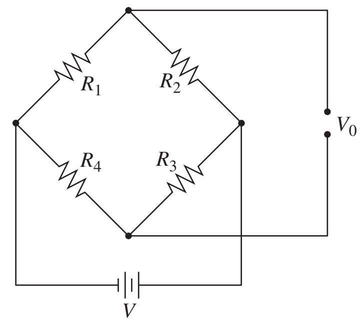


Figure 9.34 Wheatstone bridge circuit.

Clearly, if $R_1R_3 = R_2R_4$ then the voltage V_0 is be zero, and the bridge is said to be *balanced*. Suppose that one of the resistor is a strain gage—say, R_1. Before the material is loaded (and strained) the bridge is balanced. When the load is applied, the resistance R_1 changes. By adjusting the values of the other resistances by a known amount, we can again balance the bridge, and from $R_1R_3 = R_2R_4$ the resistance of R_1 can again be found. The strain can then be calculated from the change in R_1. A Wheatstone bridge is so important in strain measurements because it is sensitive to very small changes in resistance. And since we need to use only one of the resistances to balance the bridge, strains due to different causes can be separated by creative combinations of two or more gages.

9.7 CHAPTER CONNECTOR

In this chapter we studied the relationship of strains in different coordinate systems, and we found methods to determine the maximum normal strains and maximum shear strains. We noted that the principal axes form an orthogonal coordinate system. Hence we can determine the principal stresses from the principal strains by using the generalized Hooke's law. These principal stresses will be used in Chapter 10 to determine whether a material would fail.

We also learned about strain gages as a means of measuring strains at a point on a material. In Chapters 4 through 7 we studied one-dimensional structural elements and developed theories that let us compute the strains in an x, y, z coordinate system that an applied load produces. From these predicted strains, we are able to determine what a strain gage will record at any orientation. This same idea, of relating external loads to the reading of a strain gage, can be used in monitoring and controlling the applied forces and moments on a structure.

POINTS AND FORMULAS TO REMEMBER

- Strain transformation equations relate strains *at a point* in different coordinate systems:

$$\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad (9.4)$$

$$\gamma_{nt} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \qquad (9.6)$$

- Directions of the principal coordinates are the axes in which the shear strain is zero.
- Normal strains in principal directions are called *principal strains.*
- The greatest principal strain is called *principal strain 1.*
- The angles the principal axis makes with the global coordinate system are called *principal angles.*
- The angle of principal axis 1 from the x axis is only reported in describing the principal coordinate system in two-dimensional problems. Counterclockwise rotation from the x axis is defined as positive.
- Principal directions are orthogonal.
- Maximum and minimum normal strains at a point are the principal strains.
- The maximum shear strain in coordinate systems that can be obtained by rotating about one of the three axes (usually the z axis) is called *in-plane maximum shear strain*.
- The maximum shear strain at a point is the absolute maximum shear strain that can be obtained in a coordinate system by considering rotation about all three axes.
- Maximum shear strain exists in two coordinate systems that are 45° to the principal coordinate system.

$$\tan 2\theta_p = \frac{\gamma_{xy}}{\varepsilon_{xx} - \varepsilon_{yy}} \quad (9.7) \qquad \varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \quad (9.8) \qquad \left|\frac{\gamma_p}{2}\right| = \left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right| \quad (9.11)$$

- where θ_p is the angle to either principal plane 1 or 2, ε_1 and ε_2 are the principal stresses, γ_p is the in-plane maximum shear stress.
- $$\varepsilon_{nn} + \varepsilon_{tt} = \varepsilon_{xx} + \varepsilon_{yy} = \varepsilon_1 + \varepsilon_2 \qquad (9.9)$$
- $$\varepsilon_3 = \begin{cases} 0, & \text{plane strain} \\ -\dfrac{\nu}{1-\nu}(\varepsilon_{xx} + \varepsilon_{yy}) & \text{plane stress} \end{cases} \quad (9.10) \qquad \frac{\gamma_{max}}{2} = \left|\max\left(\frac{\varepsilon_1 - \varepsilon_2}{2}, \frac{\varepsilon_2 - \varepsilon_3}{2}, \frac{\varepsilon_3 - \varepsilon_1}{2}\right)\right| \quad (9.12)$$
- Each point on Mohr's circle represents a unique direction passing through the point at which the strains are specified. The coordinates of each point on the circle are the strains (ε_{nn}, $\gamma_{nt}/2$).
- The maximum shear strain at a point is the radius of the biggest of the three circles that can be drawn between the three principal strains.
- The principal directions for stresses and strains are the same for isotropic materials.
- Generalized Hooke's law in principal coordinates:
- $$\varepsilon_1 = \frac{\sigma_1 - \nu(\sigma_2 + \sigma_3)}{E} \quad (9.14.a) \qquad \varepsilon_2 = \frac{\sigma_2 - \nu(\sigma_3 + \sigma_1)}{E} \quad (9.14.b) \qquad \varepsilon_3 = \frac{\sigma_3 - \nu(\sigma_1 + \sigma_2)}{E} \quad (9.14.c)$$
- Strain gages measure only normal strains directly.
- Strain gages are bonded to a free surface, i.e., the strains are in a state of plane stress and not plane strain.
- Strain gages measure average strain at a point.
- The change in strain gage orientation by ±180° makes no difference to the strain values.

CHAPTER TEN
DESIGN AND FAILURE

Learning objectives

1. Learn the computation of stresses and strains on a structural member under combined axial, torsion, and bending loads.
2. Develop the design and analysis skills for structures constructed from one-dimensional members.

In countless engineering applications, the structural members are subjected a combination of loads. The propeller on a boat (Figure 10.1*a*) subjects the shaft to an *axial force* as it pushes the water backward, but also a *torsional load* as it turns through the water. Gravity subjects the Washington Monument (Figure 10.1*b*) to a *distributed axial load*, while the wind pressure of a storm subjects the monument to *bending loads*. In still other cases, we have to take into account that a structure is composed of more than one member. For example, wind pressure on a highway sign (Figure 10.1*c*) subjects the base of the sign to both *bending* and *torsional loads*. This chapter synthesizes and applies the concepts developed in the previous nine chapters to the design of structures subjected to combined loading.

(a)

(b)

(c)

Figure 10.1 Examples of combined loadings.

10.1 COMBINED LOADING

We have developed separately the theories for axial members (Section 4.2), for the torsion of circular shafts (Section 5.2), and for symmetric bending about the z axis (Section 6.2). All these are linear theories, which means that the superposition principle applies. In many problems a structural member is subject simultaneously to axial, torsional, and bending loads. The solution to the combined loading problems thus involves a superposition of stresses and strains at a point.

Equations (10.1), (10.2), (10.3a), and (10.3b), listed here for convenience as Table 10.1 summarizes the stress formulas derived in earlier chapters. Equations (10.4a) and (10.4b) extend of the formulas for symmetric bending about the z axis [Equations (10.3a), and (10.3b)] to symmetric bending about the y axis as we shall see in Section 10.1.3.

TABLE 10.1 **Stresses and strains in one-dimensional structural members**

	Stresses		Strains
Axial	$\sigma_{xx} = \dfrac{\boldsymbol{N}}{A}$	(10.1)	$\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ $\quad \varepsilon_{yy} = -\dfrac{\nu\sigma_{xx}}{E}$ $\quad \varepsilon_{zz} = -\dfrac{\nu\sigma_{xx}}{E}$
	$\sigma_{yy} = 0 \quad \sigma_{zz} = 0$		$\gamma_{xy} = 0 \quad \gamma_{yz} = 0 \quad \gamma_{xz} = 0$
	$\tau_{xy} = 0 \quad \tau_{yz} = 0 \quad \tau_{xz} = 0$		
Torsion	$\tau_{x\theta} = \dfrac{\boldsymbol{T}\rho}{J}$	(10.2)	$\gamma_{x\theta} = \dfrac{\tau_{x\theta}}{G}$
	$\sigma_{xx} = 0 \quad \sigma_{yy} = 0 \quad \sigma_{zz} = 0$		$\varepsilon_{xx} = 0 \quad \varepsilon_{yy} = 0 \quad \varepsilon_{zz} = 0$
	$\tau_{yz} = 0$		$\gamma_{yz} = 0$
Symmetric bending about z axis	$\sigma_{xx} = -\dfrac{\boldsymbol{M}_z y}{I_{zz}}$	(10.3a)	$\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ $\quad \varepsilon_{yy} = -\dfrac{\nu\sigma_{xx}}{E}$ $\quad \varepsilon_{zz} = -\dfrac{\nu\sigma_{xx}}{E}$
	$\tau_{xs} = -\dfrac{\boldsymbol{V}_y Q_z}{I_{zz} t}$	(10.3b)	$\gamma_{xs} = \dfrac{\tau_{xs}}{G}$
			$\gamma_{yz} = 0$
	$\sigma_{yy} = 0 \quad \sigma_{zz} = 0 \quad \tau_{yz} = 0$		
Symmetric bending about y axis	$\sigma_{xx} = -\dfrac{\boldsymbol{M}_y z}{I_{yy}}$	(10.4a)	$\varepsilon_{xx} = \dfrac{\sigma_{xx}}{E}$ $\quad \varepsilon_{yy} = -\dfrac{\nu\sigma_{xx}}{E}$ $\quad \varepsilon_{zz} = -\dfrac{\nu\sigma_{xx}}{E}$
	$\tau_{xs} = -\dfrac{\boldsymbol{V}_z Q_y}{I_{yy} t}$	(10.4b)	$\gamma_{xs} = \dfrac{\tau_{xs}}{G}$
			$\gamma_{yz} = 0$
	$\sigma_{yy} = 0 \quad \sigma_{zz} = 0 \quad \tau_{yz} = 0$		

To understand the principal of superposition for stresses, consider a thin hollow cylinder (Figure 10.2) subjected to combined axial, torsional, and bending loads. We first draw the stress cubes at four points *A*, *B*, *C*, and *D*. The stress direction on the stress cube can then be determined by inspection or using subscripts (as in Sections 5.2.5, 6.2.5, 6.6.1, and 6.6.3). The magnitude of the stress components follows from the formulas in Table 10.1.

We will use the following notation for the magnitude of the stress components:

- σ_{axial}—axial normal stress.
- $\sigma_{\text{bend-}y}$—normal stress due to bending about y axis.
- $\sigma_{\text{bend-}z}$—normal stress due to bending about z axis.
- τ_{tor}—torsional shear stress. (10.5)
- $\tau_{\text{bend-}y}$—shear stress due to bending about y axis.
- $\tau_{\text{bend-}z}$—shear stress due to bending about z axis.

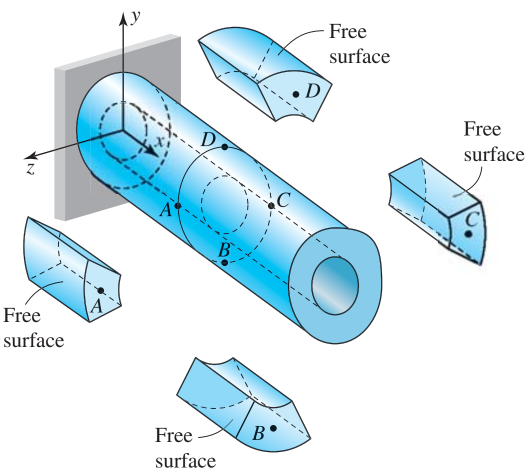


Figure 10.2 Thin hollow cylinder.

Because the surface of the shaft is a free surface, it is stress free. Hence, irrespective of the loading, no stresses act on this surface at the four points *A*, *B*, *C*, and *D* in Figure 10.2. The free surfaces at points *B* and *D* have outward normals in the *y* direction. Recall that the first subscript in each stress component is the direction of the outward normal to the surface on which the stress component acts. Thus τ_{yx}, which acts on this surface, has to be zero. Since $\tau_{xy} = \tau_{yx}$, it follows that τ_{xy} at points *B* and *D* will be zero *irrespective* of the loading. Similarly, the free surfaces at points *A* and *C* have outward normals in the *z* direction, and hence $\tau_{zx} = 0$. Thus, τ_{xz} is also zero at these points, irrespective of the loading.

10.1.1 Combined Axial and Torsional Loading

Figure 10.3 show the axial and torsional stresses on stress cubes at points *A*, *B*, *C*, and *D* due to individual loads. When both axial and torsional loads are present together, we do not simply add the two stress components. Rather we superpose or add the two *stress states.*

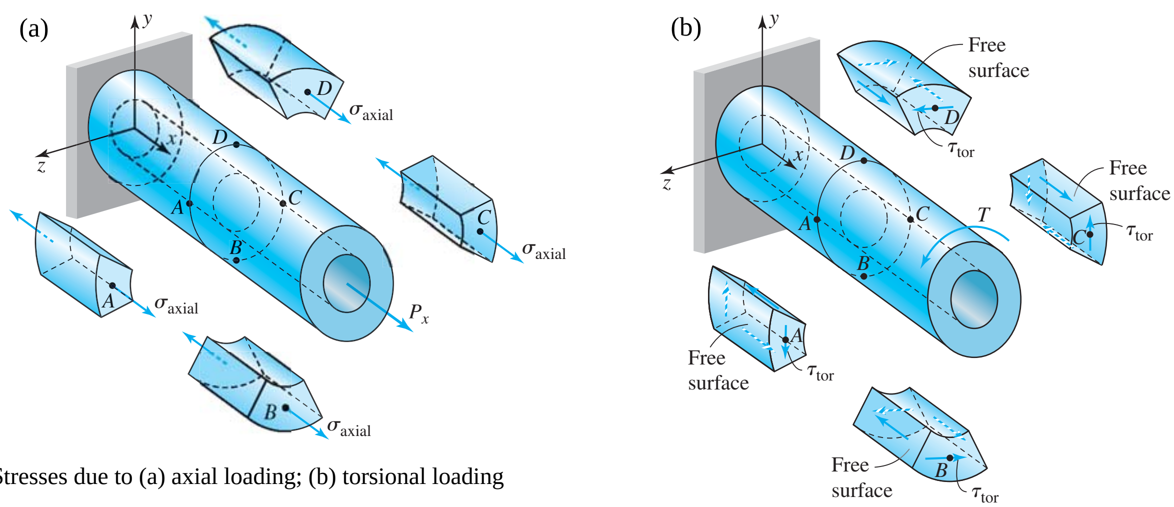


Figure 10.3 Stresses due to (a) axial loading; (b) torsional loading

What do we mean by superposing the stress states? To answer the question, consider two stress components σ_{xx} and τ_{xy} at point *C*. In axial loading, $\sigma_{xx} = \sigma_{axial}$ and $\tau_{xy} = 0$; in torsional loading $\sigma_{xx} = 0$ and $\tau_{xy} = \tau_{tor}$. When we add (or subtract), we add (or subtract) the same component in each loading. Hence, the total state of stress at point *C* is $\sigma_{xx} = \sigma_{axial} + 0 = \sigma_{axial}$ and $\tau_{xy} = 0 + \tau_{tor} = \tau_{tor}$. The state of stress at point *C* in combined loading (Figure 10.4) is thus very different from the states of stress in individual loadings (Figures 10.3*a* and *b*). Think how different is the Mohr's circle associated with the state of stress at point *C* in Figure 10.4 with those associated in Figures 10.3*a* and *b*. Example 10.1 further elaborates the differences in stress states and associated Mohr circle.

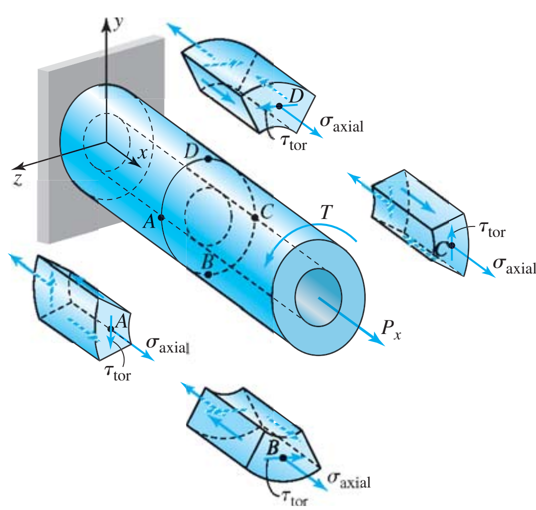


Figure 10.4 Stresses in combined axial and torsional loading.

10.1.2 Combined Axial, Torsional, and Bending Loads about *z* Axis

Figure 10.5*a* shows the thin hollow cylinder subjected to a load that bends the cylinder about the *z* axis. Points *B* and *D* are on the free surface. Hence the bending shear stress is zero at these points. Points *A* and *C* are on the neutral axis, and hence the bending normal stress is zero at these points. The nonzero stress components can be found from the formulas in Table 10.1, as shown on the stress cubes in Figure 10.5*a*. If we superpose the stress states for bending at the four points shown in Figure 10.5*a* and the stress states for the combined axial and torsional loads at the same points shown in Figure 10.4, we obtain the stress states shown in Figure 10.5*b*.

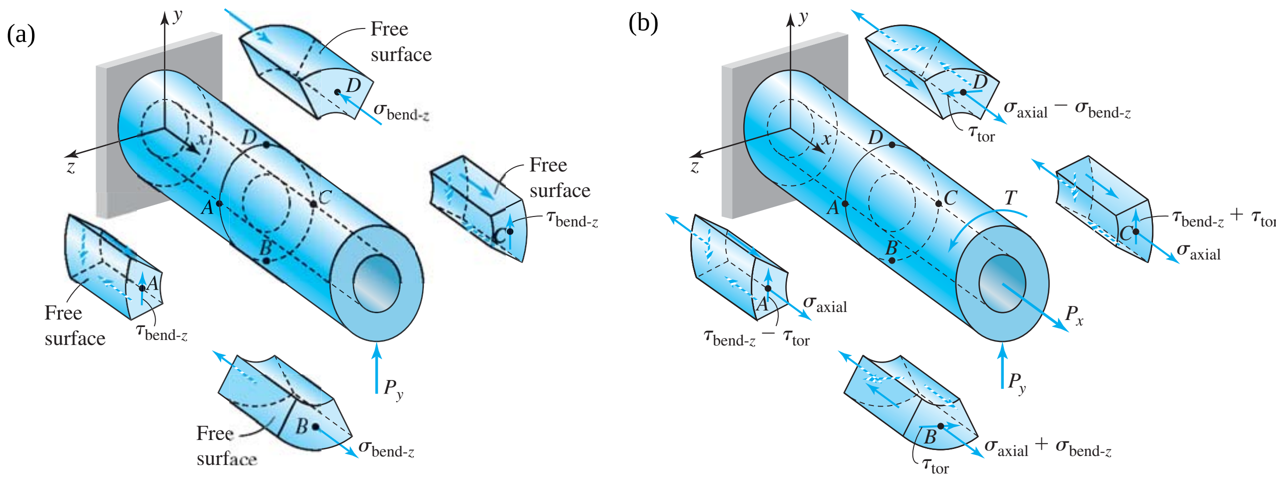


Figure 10.5 Stresses due to (a) bending about *z* axis; (b) Combined axial, torsional, and bending about *z* axis

In Figure 10.5*a*, the bending normal stress at point *D* is compressive, whereas the axial stress in Figure 10.4 is tensile. Thus, the resultant normal stress σ_{xx} is the difference between the two stress values, as shown in Figure 10.5*b*. At point *B* both the bending normal stress and the axial stress are tensile, and thus the resultant normal stress σ_{xx} is the sum of the two stress values. If the axial normal stress at point *D* is greater than the bending normal stress, then the total normal stress at point *D* will be in the direction as shown in Figure 10.5*b*. If the bending normal stress is greater than the axial stress, then the total normal stress will be compressive and would be shown in the opposite direction in Figure 10.5*b*.

At point *A* the torsional shear stress in Figure 10.4 is downward, whereas the bending shear stress in Figure 10.5*a* is upward. Thus, the resultant shear stress τ_{xy} is the difference between the two stress values, as shown in Figure 10.5*b*. At point *C* both the torsional shear stress and the bending shear stress are upward, and thus the resultant shear stress τ_{xy} is the sum of the two stress values. If the bending shear stress at point *A* is greater than the torsional shear stress, then the total shear stress at point *A* will be in the direction of positive τ_{xy}, as shown in Figure 10.5*b*. If the torsional shear stress is greater than the bending shear stress, then the total shear stress will be negative τ_{xy} and will be in the opposite direction in Figure 10.5*b*.

10.1.3 Extension to Symmetric Bending about *y* Axis

Before we combine the stresses due to bending about the *y* axis, consider the extension of the formulas derived for symmetric bending about the *z* axis. Assume that the *xz* plane is *also* a plane of symmetry, so that the loads lie in the plane of symmetry. Equations (10.4*a*) and (10.4*b*) for bending about the *y* axis can be obtained by interchanging the subscripts *y* and *z* in Equations (10.3*a*) and (10.3*b*). The sign conventions for the internal moment ***My*** and the shear force $\boldsymbol{V}_z$ in Equations (10.4*a*) and (10.4*b*) are then simple extensions of $\boldsymbol{M}_z$ and $\boldsymbol{V}_y$, as shown in Figure 10.6.

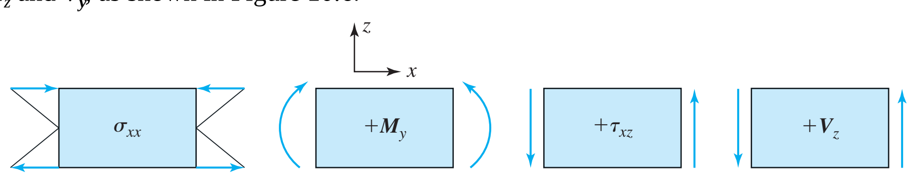


Figure 10.6 Sign convention for internal bending moments and shear force in bending about *y* axis.

Sign Convention: The positive internal moment $\boldsymbol{M}_y$ on a free-body diagram must be such that it puts a point in the positive z direction into compression.

Sign Convention: The positive internal shear force $\boldsymbol{V}_z$ on a free-body diagram is in the direction of positive shear stress τ_{xz} on the surface.

The direction of shear stress in Equation (10.4*b*) can be determined either by using the subscripts or by inspection, as we did for symmetric bending about the z axis. To use the subscripts, recall that the s coordinate is defined from the free surface (see Section 6.6.1) used in the calculation of Q_y. The shear flow (or shear stress) due to bending about the y axis only is drawn along the centerline of the cross section. Its direction must satisfy the following rules:

1. The resultant force in the z direction is in the same direction as $\boldsymbol{V}_z$.
2. The resultant force in the y direction is zero.
3. It is symmetric about the z axis. This requires that shear flow change direction as one crosses the y axis on the centerline. Sometimes this will imply that shear stress is zero at points where the centerline intersects the z axis.

10.1.4 Combined Axial, Torsional, and Bending Loads about *y* and *z* Axes

Figure 10.7*a* shows the thin hollow cylinder subjected to a load that bends the cylinder about the y axis. Points A and C are on the free surface, and hence bending shear stress is zero at these points. Points B and D are on the neutral axis, and hence the bending normal stress is zero at these points. The nonzero stress components can be found from the formulas in Table 10.1, as shown on the stress cubes in Figure 10.7*a*. If we superpose the stress states for bending at the four points shown in Figure 10.5*a* add the stress states for the combined axial and torsional loads at the same points shown in Figure 10.5*b*, we obtain the stress states shown in Figure 10.7*b*.

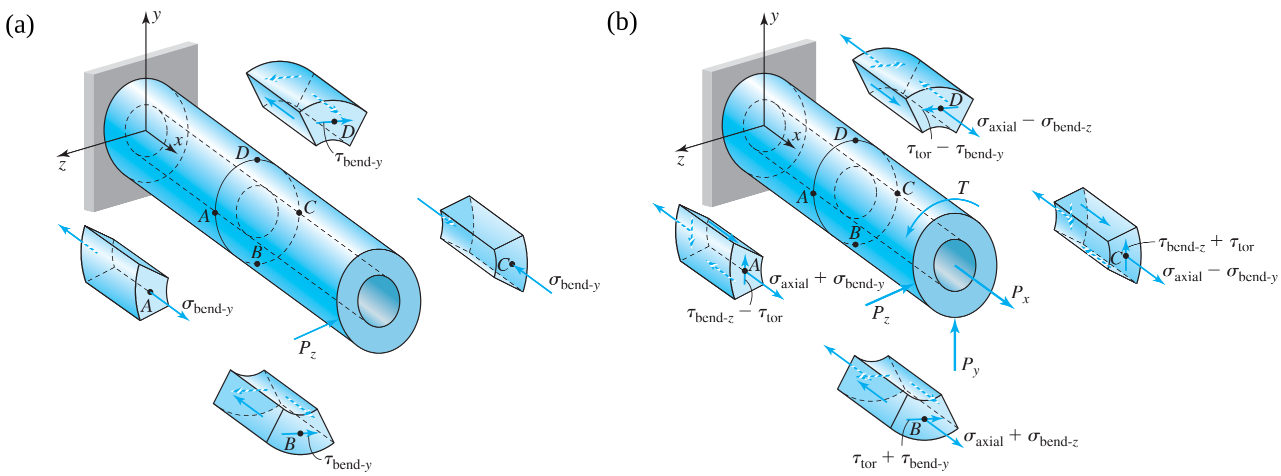


Figure 10.7 Stresses due to (a) bending about y axis; (b) combined axial, torsional, and bending about y and z axis.

Thus the complex stress states shown in Figure 10.7*b* can be obtained by first calculating the stresses due to individual loadings. We then simply superpose the stress states at each point.

10.1.5 Stress and Strain Transformation

To obtain strains in combined loading, we can superpose the strains given in Table 10.1. Alternatively, we can superpose the stresses, as discussed in the preceding sections and then use the generalized Hooke's law to convert these stresses to strains. The second approach is often preferable, because we may need to transform torsional shear stress $\tau_{x\theta}$ (see Section 5.2.5) and bending shear stress τ_{xs} (see Section 6.6.6) into the x, y, z coordinate system. (Remember that our stress and strain transformation equations were developed in the cartesian coordinates.) In Figure 10.7*b*, at points A and C the shear stress shown is positive τ_{xy}, at point B the shear stress shown is negative τ_{xz}, and at point D the shear stress shown is positive τ_{xz}. In general, it is important to show the stresses on a stress element before proceeding to stress or strain transformation.

In studying individual loading, we often had prefixes to stresses such as maximum axial normal stress, maximum torsional shear stress, maximum bending normal stress, maximum bending shear stress, or maximum in-plane shear stress. In this chapter, however, we are considering combined loading. Hence the *maximum normal stress* at a point will refer to the *principal stress* at the point, and the *maximum shear stress* will refer to the *absolute maximum shear stress*. This implies that *allowable normal stress* refers to the *principal stresses* and *allowable shear stress* refers to the *absolute maximum shear stress*. The allowable tensile normal stress refers to principal stress 1, assuming it is tensile. The allowable compressive normal stress refers to principal stress 2, assuming it is compressive.

10.1.6 Summary of Important Points in Combined Loading

We can now summarize the points to keep in mind when solving problems involving combined loading.

1. The problem of stress under combined loading can be simplified by first determining the states of stress due to individual loadings.
2. The superposition principle applies to stresses at a given point. That is, a stress component resulting from one loading can be added to or subtracted from a similar stress component from another loading. Stress components at *different* points cannot be added or subtracted. Neither can stress components that act on different planes or in different directions.
3. The stress formulas in Table 10.1 give the magnitude and the direction for each stress component, but only if the internal forces and moments are drawn on the free-body diagrams according to the prescribed sign conventions. If the directions of internal forces and moments are instead drawn so as to equilibrate external forces and moments, then the directions of the stress components must be determined by inspection.
4. In a given structure, the structural members may have different orientations. In using subscripts to determine the direction and signs of stress components, we therefore establish a local x, y, z coordinate system for each structural member such that the x direction is normal to the cross section. That is, the x direction is along the axis of the structural member.
5. Table 10.1 shows that stresses σ_{yy} and σ_{zz} are zero for the four cases listed, emphasizing that the theories are for one-dimensional structural members. Additional stress components are zero at free surfaces.
6. The state of stress in combined loading should be shown on a stress cube before applying stress or strain transformation.
7. The strains at a point can be obtained from the superposed stress values using the generalized Hooke's law. Since the normal stresses σ_{yy} and σ_{zz} are always zero in our structural members, the nonzero strains ε_{yy} and ε_{zz} are due to the Poisson effect; that is, $\varepsilon_{yy} = \varepsilon_{zz} = -\nu\varepsilon_{xx}$.

10.1.7 General Procedure for Combined Loading

A general procedure for calculating stresses in combined loading is as follows:

Step 1: Identify the equations in Table 10.1 relevant for the problem, and use the equations as a checklist for the quantities that must be calculated.

Step 2: Calculate the relevant geometric properties (A, I_{yy}, I_{zz}, J) of the cross section containing the points where stresses have to be found.

Step 3: At points where shear stress due to bending is to be found, draw a line perpendicular to the centerline through the point and calculate the first moments of the area (Q_y, Q_z) between the free surface and the drawn line. Record the s direction from the free surface toward the point where the stress is being calculated.

Step 4: Make an imaginary cut through the cross section and draw the free-body diagram. If subscripts are to be used in determining the directions of the stress components, draw the internal forces and moments according to our sign conventions. Use equilibrium equations to calculate the internal forces and moments.

Step 5: Using the equations identified in Step 1, calculate the individual stress components due to each loading. Draw the torsional shear stress $\tau_{x\theta}$ and the bending shear stress τ_{xs} on a stress cube using subscripts or by inspection. By examining the shear stresses in the x, y, z coordinate system, obtain τ_{xy} and τ_{xz} with proper signs.

Step 6: Superpose the stress components to obtain the total stress components at a point.

Step 7: Show the calculated stresses on a stress cube.

Step 8: Interpret the stresses shown on the stress cube in the x, y, z coordinate system before processing these stresses for the purpose of stress or strain transformation.

EXAMPLE 10.1

A hollow shaft that has an outside diameter of 100 mm, and an inside diameter of 50 mm is loaded as shown in Figure 10.8. For the three cases shown, determine the principal stresses and the maximum shear stress at point A. Point A is on the surface of the shaft.

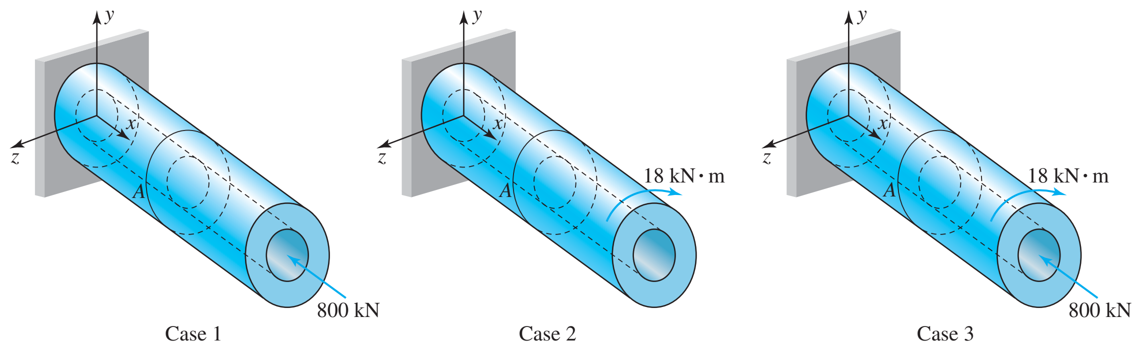


Figure 10.8 Hollow cylinder in Example 10.1.

PLAN

The axial normal stress in case 1 can be found from Equation 10.1. The torsional shear stress in case 2 can be found from Equation 10.2. The state of stress in case 3 is the superposition of the stress states in cases 1 and 2. The calculated stresses at point A can be drawn on a stress cube. Using Mohr's circle or the method of equations, we can find the principal stresses and the maximum shear stress in each case.

SOLUTION

Step 1: Equations (10.1) and (10.2) are used for calculating the axial stress and the torsional shear stress.

Step 2: The cross-sectional area A and the polar area moment J of a cross section can be found as

$$A = \frac{\pi}{4}[(100\text{ mm})^2 - (50\text{ mm})^2] = 5.89(10^3)\text{ mm}^2 \qquad J = \frac{\pi}{32}[(100\text{ mm})^4 - (50\text{ mm})^4] = 9.20(10^6)\text{ mm}^4 \tag{E1}$$

Step 3: This step is not needed as there is no bending.

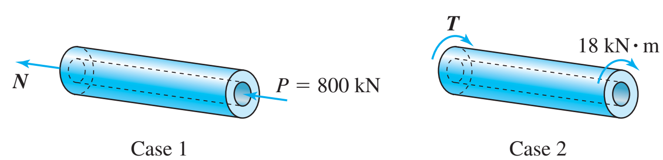


Figure 10.9 Free-body diagrams in Example 10.1.

Step 4: We draw the free-body diagrams in Figure 10.9 after making imaginary cuts. The internal axial force and the internal torque are drawn according to our sign convention. By equilibrium we obtain

$$\boldsymbol{N} = -800\text{ kN} \qquad \boldsymbol{T} = -18\text{ kN}\cdot\text{m} \tag{E2}$$

Step 5:

Case 1: The axial stress is uniform across the cross section and can be found from Equation 10.1,

$$\sigma_{xx} = \frac{\boldsymbol{N}}{A} = \frac{-800(10^3)\text{ N}}{5.89(10^{-3})\text{ m}^2} = -135.8(10^6)\text{ N/m}^2 = -135.8\text{ MPa} \tag{E3}$$

Case 2: The torsional shear stress varies linearly and is maximum on the surface ($\rho = 0.05$ m) of the shaft. It can be found from Equation 10.2,

$$\tau_{x\theta} = \frac{T\rho}{J} = \frac{[-18(10^3)\ \text{N}\cdot\text{m}](0.05\ \text{m})}{9.20(10^{-6})\ \text{m}^4} = -97.83(10^6)\ \text{N/m}^2 = -97.83\ \text{MPa} \tag{E4}$$

Steps 6, 7: We draw the stress cube and show the stresses calculated in Equations (E3) and (E4).
Case 1: The axial stress is compressive, as shown Figure 10.10*a*.
Case 2: From Equation (E4) we note that $\tau_{x\theta}$ is negative. The θ direction in positive counterclockwise with respect to the *x* axis, as shown in Figure 10.10*b*. At point *A* the outward normal to the surface is in the positive *x* direction and the positive θ direction at *A* is downward. Hence a negative $\tau_{x\theta}$ will be upward at point *A*, as shown in Figure 10.10*b*.

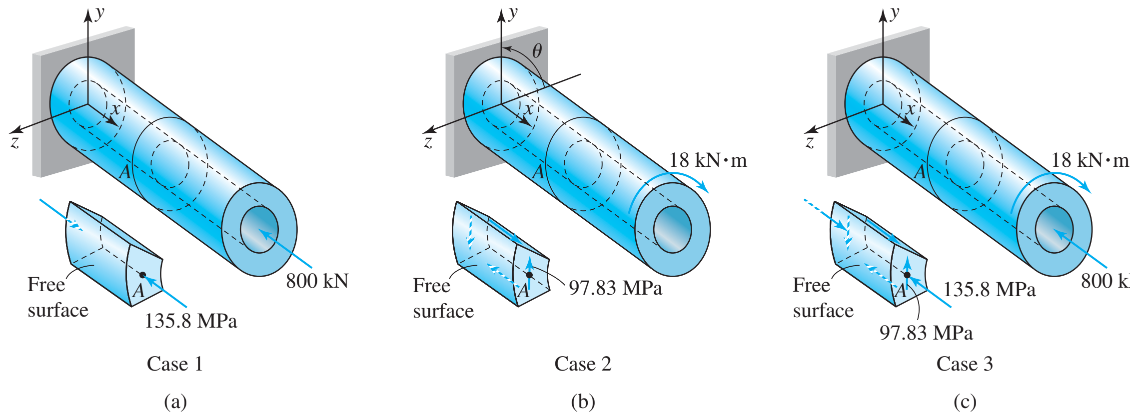


Figure 10.10 Stresses on stress cubes in Example 10.1.

Intuitive check: Figure 10.11 shows the hollow shaft with the applied torque on the right end and the reaction torque at the wall on the left end. The left part of the shaft would rotate counterclockwise with respect to the right part. Thus the surface of the cube at point *A* would be moving downward. The shear stress would oppose this impending motion by acting upward at point *A*, as shown in Figure 10.11, confirming the direction shown in Figure 10.10*b*.

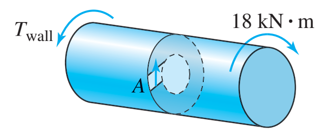


Figure 10.11 Direction of shear stress by inspection.

Case 3: The state of stress is a superposition of the states of stress shown on the stress cubes for cases 1 and 2 and is illustrated in Figure 10.10*c*.
Step 8: We can redraw the stress cubes in two dimensions and follow the procedure for constructing Mohr's circle for each case, as shown in Figure 10.12. The radius of the Mohr's circle can be found and the principal stresses and maximum shear stress calculated.

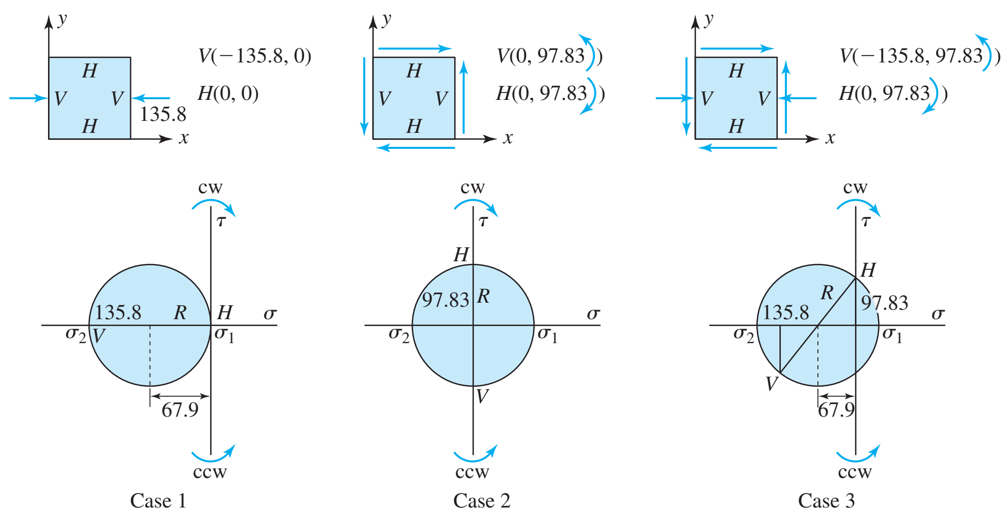


Figure 10.12 Mohr's circles in Example 10.1.

- *Case 1:* $R = 67.9\ \text{MPa}$.

ANS. $\sigma_1 = 0$ $\sigma_2 = 135.8\ \text{MPa (C)}$ $\sigma_3 = 0$ $\tau_{max} = 67.9\ \text{MPa}$

- *Case 2:* $R = 97.83\ \text{MPa}$.

ANS. $\sigma_1 = 97.8\ \text{MPa (T)}$ $\sigma_2 = 97.8\ \text{MPa (C)}$ $\sigma_3 = 0$ $\tau_{max} = 97.8\ \text{MPa}$

- *Case 3:* $R = \sqrt{(67.9\ \text{MPa})^2 + (97.83\ \text{MPa})^2} = 119.1$, thus $\sigma_{1,2} = -67.9\ \text{MPa} \pm 119.1\ \text{MPa}$

ANS. $\sigma_1 = 51.2\ \text{MPa (T)}$ $\sigma_2 = 187\ \text{MPa (C)}$ $\sigma_3 = 0$ $\tau_{max} = 119.1\ \text{MPa}$

COMMENTS

1. The results for the three cases show that the principal stresses and the maximum shear stress for case 3 cannot be obtained by superposition of the principal stresses and the maximum shear stress calculated for cases 1 and 2. Figure 10.12 emphasizes this graphically. Mohr's circle of case 3 cannot be obtained by superposing Mohr's circle for cases 1 and 2. The superposition principle is not applicable to principal stresses because the *principal planes* for the three cases are different. We cannot add (or subtract) stresses on different planes. If we had calculated the stresses for the three cases on the same plane, then we could apply the superposition principle.
2. Substituting $\sigma_{xx} = -135.8$ MPa, $\tau_{xy} = +97.8$ MPa, and $\sigma_{yy} = 0$ into Equation (8.7), we can find σ_1 and σ_2 for case 3

$$\sigma_{1,2} = \frac{(-135.8\ \text{MPa}) + 0}{2} \pm \sqrt{\left(\frac{-135.8\ \text{MPa}}{2}\right)^2 + (97.8\ \text{MPa})^2} = -67.9\ \text{MPa} \pm 119.1\ \text{MPa} \tag{E5}$$

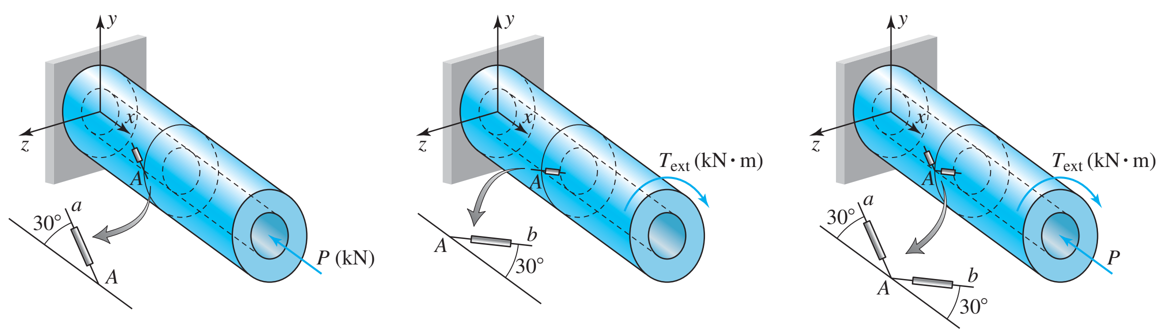
Noting that $\sigma_3 = 0$, we can find τ_{max} from Equation (8.13),

$$\tau_{max} = \left|\max\left(\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2}\right)\right| \tag{E6}$$

The results of Equations (E5) and (E6) are same as those obtained from the Mohr's circle.

EXAMPLE 10.2

A hollow shaft has an outside diameter of 100 mm and an inside diameter of 50 mm, is shown in Figure 10.13. Strain gages are mounted on the surface of the shaft at 30° to the axis. For each case determine the applied axial load P and the applied torque T_{ext} if the strain gage readings are $\varepsilon_a = -500\ \mu$ and $\varepsilon_b = 400\ \mu$. Use $E = 200$ GPa, $G = 80$ GPa, and $\nu = 0.25$.



Case 1 Case 2 Case 3

Figure 10.13 Hollow cylinder in Example 10.2.

PLAN

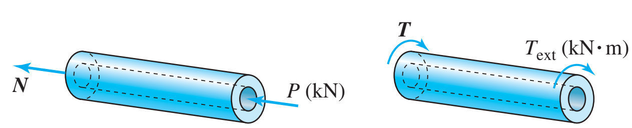
The stresses at point A in terms of P and T_{ext} can be found as in Example 10.1. Using the generalized Hooke's law, we can find the strains in terms of P and T_{ext}. From the strain transformation equation, Equation (9.4), the normal strain in direction of the strain gage can be found in terms of P and T_{ext}. The values of P and T_{ext} can be determined from the given strain gage readings.

SOLUTION

Step 1: Equations 10.1 and 10.2 will be used for calculating the axial stress and the torsional shear stress.
Step 2: From Example 10.1, the cross-sectional area A and the polar area moment J of a cross section are

$$A = 5.89(10^3)\ \text{mm}^2 \qquad J = 9.20(10^6)\ \text{mm}^4 \tag{E1}$$

Step 3: This step is not needed as there is no bending.



Case 1 Case 2

Figure 10.14 Free-body diagrams in Example 10.2.

Step 4: We make an imaginary cut and draw the free-body diagrams in Figure 10.14. By equilibrium we obtain

$$\boldsymbol{N} = -P\ \text{kN} \qquad \boldsymbol{T} = -T_{ext}\ \text{kN·m} \tag{E2}$$

Step 5:

Case 1: The axial stress is uniform across the cross section and can be found from Equation (10.1),

$$\sigma_{xx} = \frac{N}{A} = \frac{-P(10^3)\ \text{N}}{5.89(10^{-3})\ \text{m}^2} = -0.17P(10^6)\ \text{N/m}^2 \tag{E3}$$

Case 2: The torsional shear stress on the surface ($\rho = 0.05$ m) of the shaft can be found from Equation (10.2),

$$\tau_{x\theta} = \frac{T\rho}{J} = \frac{[-T_{ext}(10^3)\ \text{N}\cdot\text{m}](0.05\ \text{m})}{9.20(10^{-6})\ \text{m}^4} = -5.435T_{ext}(10^6)\ \text{N/m}^2 \tag{E4}$$

Steps 6, 7: Figure 10.15 shows the stresses on the stress elements calculated using Equations (E4) and (E5), as in Example 10.1.

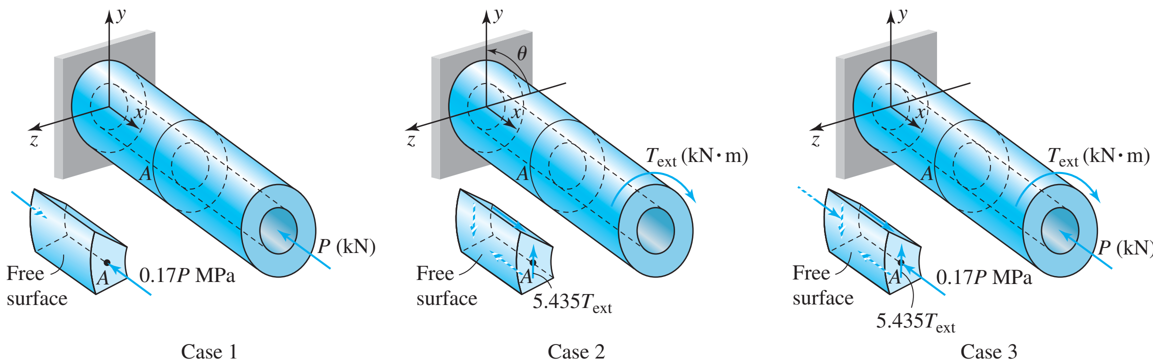


Figure 10.15 Stresses on stress cubes in Example 10.2.

Step 8:

Case 1: We note that the only nonzero stress is the axial stress given in Equation (E4). From the generalized Hooke's law we obtain the strains,

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} = \frac{-0.170P(10^6)\ \text{N/m}^2}{200(10^9)\ \text{N/m}^2} = -0.85P(10^{-6}) = -0.85P\ \mu \qquad \varepsilon_{yy} = -\nu\varepsilon_{xx} = -0.25(-0.85P\ \mu) = 0.213P\ \mu \tag{E5}$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0 \tag{E6}$$

Case 2: From Figure 10.15 we note that the shear stress $\tau_{xy} = +5.435T_{ext}$. The normal stresses are all zero. From the generalized Hooke's law we obtain the strains,

$$\varepsilon_{xx} = 0 \qquad \varepsilon_{yy} = 0 \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} = \frac{5.435T_{ext}(10^6)\ \text{N/m}^2}{80(10^9)\ \text{N/m}^2} = 67.94T_{ext}\ \mu \tag{10.6}$$

Case 3: The state of strain is the superposition of the state of strain for cases 1 and 2,

$$\varepsilon_{xx} = -0.85P\ \mu \qquad \varepsilon_{yy} = 0.213P\ \mu \qquad \gamma_{xy} = 67.94T_{ext}\ \mu \tag{E7}$$

Load calculations

Case 1: Substituting $\theta_a = 150°$ or $-30°$ and ε_{xx}, ε_{yy}, and γ_{xy}, into the strain transformation equation, Equation (9.4), we can find the normal strain in terms of P and equated it to the given value of $\varepsilon_a = -500\ \mu$. The value of P can be found as

$$\varepsilon_a = (-0.85P\ \mu)\cos^2(-30^\circ) + (0.213P\ \mu)\sin^2(-30^\circ) = -500\ \mu \qquad \text{or} \qquad (-0.638\ \mu + 0.053\ \mu)P = -500\ \mu \tag{E8}$$

ANS. $P = 855$ kN

Case 2: Substituting $\theta_b = 30°$ and ε_{xx}, ε_{yy}, and γ_{xy}, into the strain transformation equation, Equation (9.4), we can find the normal strain in terms of T_{ext} and equate it to the given value of $\varepsilon_b = 400\ \mu$. The value of T_{ext} can be found as

$$\varepsilon_b = (67.94T_{ext}\ \mu)\sin(30^\circ)\cos(30^\circ) = 400\ \mu \qquad \text{or} \qquad 29.42\ \mu T_{ext} = 400\ \mu \tag{E9}$$

ANS. $T_{ext} = 13.6\ \text{kN}\cdot\text{m}$

Case 3: Substituting $\theta_a = -30°$, $\theta_b = 30$, and Equations (E12), (E13), and (E14) into the strain transformation equation, Equation (9.4), and using the given strain values, we obtain

$$\varepsilon_a = (-0.85P\ \mu)\cos^2(-30°) + (0.213P\ \mu)\sin^2(-30°) + (67.94T_{ext}\ \mu)\sin(-30°)\cos(-30°) = -500\mu \quad \text{or}$$

$$-0.585P - 29.42T_{ext} = -500 \tag{E10}$$

$$\varepsilon_b = (-0.85P\ \mu)\cos^2(30°) + (0.213P\ \mu)\sin^2(30°) + (67.94T_{ext}\ \mu)\sin(30°)\cos(30°) = 400\ \mu$$

$$-0.585P + 29.42T_{ext} = 400 \tag{E11}$$

Equations (E10) and (E11) can be solved simultaneously to obtain the result.

ANS. $P = 85.4\ \text{kN}$ $T_{ext} = 15.3\ \text{kN}\cdot\text{m}$

COMMENTS

1. The values of P and T_{ext} for combined loading are different than the values obtained for individual loadings. The next comment explains why.
2. If we had been given P and T_{ext} and were required to predict the strains in the gages, we could have calculated strains along the strain gage direction for individual loads and superposed to get the total strain in the gages for combined loading. But as the results in this example demonstrate, the strains in the gages (or the total strain) for combined loading cannot be separated into strain due to axial load and strain due to torsion. Loads P and T_{ext} affect both strain gages simultaneously, and these effects cannot be decoupled into effects of individual loadings.
3. In this example and the previous one we solved the problem by separating axial and torsion problems and calculated internal axial force and internal torque using separate free-body diagrams. We could have used a single free-body diagram, as shown in Figure 10.16, to calculate the internal quantities. In subsequent examples we shall construct a single free-body diagram for the calculation of the internal quantities,

$$N = -P\ \text{kN} \qquad T = -T_{ext}\ \text{kN}\cdot\text{m}$$

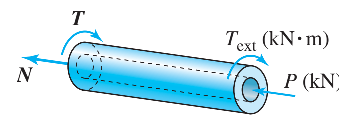


Figure 10.16 Single free-body diagram for combined loading.

This choice is not only less tedious but may be necessary. A single force may produce axial, torsion, and bending, which cannot be separated on a free-body diagram.

EXAMPLE 10.3

A box column is constructed from $\frac{1}{4}$-in.-thick sheet metal and subjected to the loads shown in Figure 10.17. (a) Determine the normal and shear stresses in the x, y, z coordinate system at points A and B and show the results on stress cubes. (b) A surface crack at point B is oriented as shown. Determine the normal and shear stresses on the plane containing the crack.

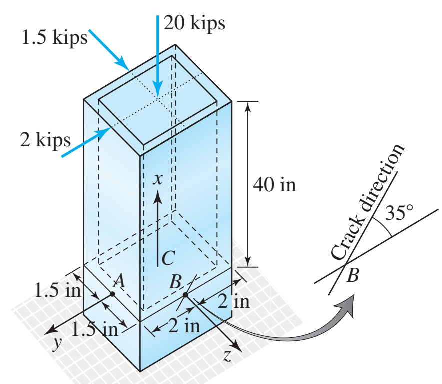


Figure 10.17 Beam and loading in Example 10.3.

PLAN

(a) We can follow the procedure in Section 10.1.7. The 20-kips force is an axial force, whereas the 2-kips and 1.5-kips forces produce bending about the z and y axes, respectively. Thus Equations 10.1, (10.3*a*), (10.3*b*), (10.4*a*), and (10.4*b*) will be used for calculating stresses. These formulas can be used as a checklist of the quantities that must be calculated in finding the individual stress components. By superposition the total stress at points A and B can be obtained. (b) Using the method of equations or Mohr's circle, the normal and shear stresses on the plane containing the crack can be found from the stresses determined at point B.

SOLUTION

Step 1: Equations 10.1, (10.3*a*), (10.3*b*), (10.4*a*), and (10.4*b*) will be used for calculating the stress components.
Step 2: The geometric properties of the cross section can be found as

$$A = (4\ \text{in.})(3\ \text{in.}) - (3.5\ \text{in.})(2.5\ \text{in.}) = 3.25\ \text{in.}^2 \tag{E1}$$

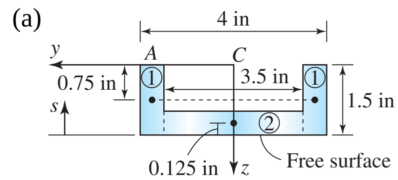
$$I_{yy} = \frac{1}{12}(4\ \text{in.})(3\ \text{in.})^3 - \frac{1}{12}(3.5\ \text{in.})(2.5\ \text{in.})^3 = 4.443\ \text{in.}^4 \qquad I_{zz} = \frac{1}{12}(3\ \text{in.})(4\ \text{in.})^3 - \frac{1}{12}(2.5\ \text{in.})(3.5\ \text{in.})^3 = 7.068\ \text{in.}^4 \tag{E2}$$

Step 3: At points A and B we draw a line perpendicular to the centerline of the cross section. We may then obtain the area A_s needed for the calculations of Q_y and Q_z at points A and B as shown in Figure 10.18:

$$t_A = t_B = 0.25 \text{ in.} + 0.25 \text{ in.} = 0.5 \text{ in.} \tag{E3}$$

$$(Q_y)_A = 2(1.5 \text{ in.})(0.25 \text{ in.})(0.75 \text{ in.}) + (3.5 \text{ in.})(0.25 \text{ in.})(1.5 \text{ in.} - 0.125 \text{ in.}) = 1.766 \text{ in.}^3 \qquad (Q_z)_A = 0 \tag{E4}$$

$$(Q_y)_B = 0 \qquad (Q_z)_B = (2 \text{ in.})(2 \text{ in.})(0.25 \text{ in.})(1 \text{ in.}) + (2.5 \text{ in.})(0.25 \text{ in.})(2 \text{ in.} - 0.125 \text{ in.}) = 2.172 \text{ in.}^3 \tag{E5}$$



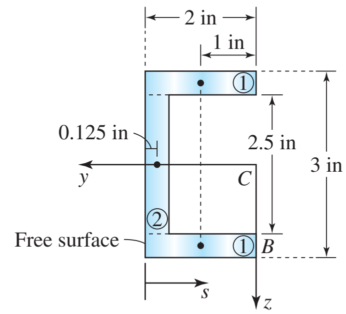


Figure 10.18 Calculation of Q_y and Q_z at (a) point A; (b) point B in Example 10.3.

Step 4: We can make an imaginary cut through the cross section containing points A and B and draw the free-body diagram shown in Figure 10.19. Internal forces and moments are drawn according to our sign convention. From the equilibrium equations, the internal forces and moments can be found,

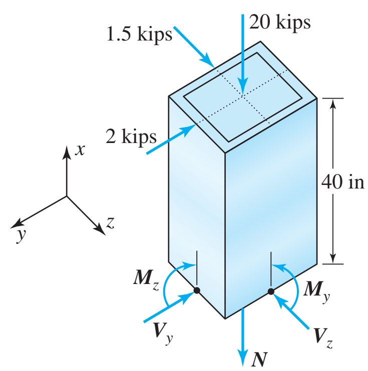


Figure 10.19 Free-body diagram in Example 10.3.

$$\boldsymbol{N} = -20 \text{ kips} \qquad \boldsymbol{V}_y = -2.0 \text{ kips} \qquad \boldsymbol{V}_z = 1.5 \text{ kips} \qquad \boldsymbol{M}_y = 60 \text{ in.}\cdot\text{kips} \qquad \boldsymbol{M}_z = -80 \text{ in.}\cdot\text{kips} \tag{E6}$$

Step 5: The stress components due to each loading are calculated next.

Axial stress calculations: The axial stresses at points A and B can be found from Equation (10.1) as

$$(\sigma_{xx})_{A,B} = \frac{\boldsymbol{N}}{A} = \frac{(-20 \text{ kips})}{3.25 \text{ in.}^2} = -6.154 \text{ ksi} \tag{E7}$$

Stresses due to bending about the y axis: We note that $z_A = 0$ and $z_B = 1.5$. From Equation (10.4a), we obtain

$$(\sigma_{xx})_A = 0 \qquad (\sigma_{xx})_B = -\frac{\boldsymbol{M}_y z_B}{I_{yy}} = -\frac{(60 \text{ in.}\cdot\text{kips})(1.5 \text{ in.})}{4.443 \text{ in.}^4} = -20.258 \text{ ksi} \tag{E8}$$

From Equation (10.4b) we obtain the shear stress at A and B,

$$(\tau_{xs})_A = -\frac{\boldsymbol{V}_z Q_y}{I_{yy} t} = -\frac{(1.5 \text{ kips})(1.766 \text{ in.}^3)}{(4.443 \text{ in.}^4)(0.5 \text{ in.})} = -1.192 \text{ ksi} \qquad (\tau_{xs})_B = 0 \qquad (\tau_{xy})_B = 0 \tag{E9}$$

From Figure 10.18*a* we note that the s direction is in the negative z direction at point A. Thus

$$(\tau_{xz})_A = -(\tau_{xs})_A = 1.19 \text{ ksi} \tag{E10}$$

Stresses due to bending about the z axis: We note that $y_A = 2$ and $y_B = 0$. From Equation (10.3a), we obtain

$$(\sigma_{xx})_A = -\frac{M_z y_A}{I_{zz}} = -\frac{(-80 \text{ in.}\cdot\text{kips})(2 \text{ in.})}{7.068 \text{ in.}^4} = 22.638 \text{ ksi} \qquad (\sigma_{xx})_B = 0 \tag{E11}$$

From Equation (10.3*b*), we obtain the shear stresses at points A and B,

$$(\tau_{xs})_A = 0 \qquad (\tau_{xz})_A = 0 \qquad (\tau_{xs})_B = -\frac{\boldsymbol{V}_y Q_z}{I_{zz} t} = -\frac{(-2 \text{ kips})(2.172 \text{ in.}^3)}{(7.068 \text{ in.}^4)(0.5 \text{ in.})} = 1.229 \text{ ksi} \tag{E12}$$

From Figure 10.18*b* we note that the s direction is in the negative y direction at point B. Thus

$$(\tau_{xy})_B = -(\tau_{xs})_B = -1.23 \text{ ksi} \tag{E13}$$

Step 6: Superposition

Normal stress calculations: The normal stress at point A can be obtained by superposing the values in Equations (E7), (E8), and (E11),

$$(\sigma_{xx})_A = -6.154 \text{ ksi} + 0 + 22.638 \text{ ksi} = 16.484 \text{ ksi} \tag{E14}$$

ANS. $(\sigma_{xx})_A = 16.5$ ksi (T)

Similarly, the normal stress at point B can be obtained by superposition of Equations (E7), (E8), and (E11),

$$(\sigma_{xx})_B = -6.154 \text{ ksi} - 20.258 \text{ ksi} + 0 = -26.412 \text{ ksi} \tag{E15}$$

ANS. $(\sigma_{xx})_B = 26.4$ ksi (C)

Intuitive check on normal stress calculations: The axial stress σ_{axial} due to a 20-kips force will be compressive. Figure 10.20 shows the exaggerated deformed shapes due to bending about the y and z axes. (These deformed shapes can actually be visualized without drawing the figures.) From Figure 10.20*a* it can be seen that the line passing through A will be in tension. That is, the normal stress due to bending about the z axis $\sigma_{bend\text{-}z}$ will be tensile. From 10.24*b* it can be seen that point A is on the neutral (bending) axis. Hence the normal stress due to bending about the y axis $\sigma_{bend\text{-}y} = 0$. Thus the total normal stress at point A is $(\sigma_{xx})_A = \sigma_{bend\text{-}z} - \sigma_{axial}$. Substituting the magnitude of $\sigma_{bend\text{-}z} = 22.638$ ksi and $\sigma_{axial} = 6.154$ ksi, we obtain the result in Equation (E14).

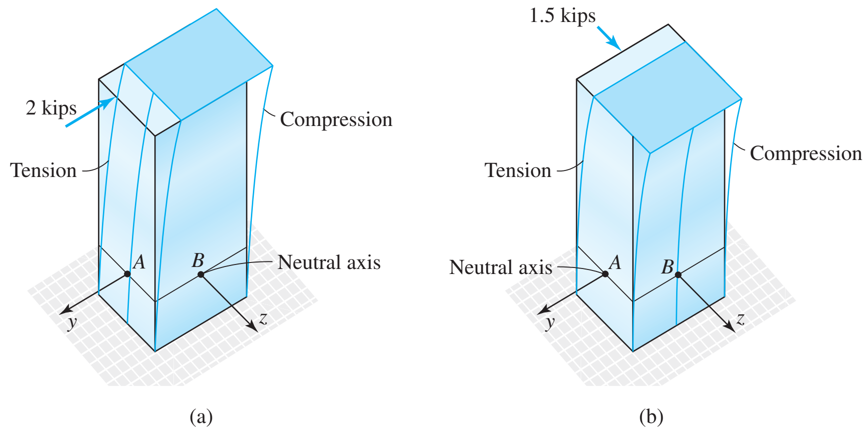


Figure 10.20 Determination of normal stress components by inspection for bending about (*a*) z axis; (*b*) y axis.

From Figure 10.20*b* it can be seen that the line passing though B will be in compression. That is, the normal stress due to bending about the y axis $\sigma_{bend\text{-}y}$ will be compressive. From Figure 10.20*a* it can be seen that point B is on the neutral (bending) axis; hence $\sigma_{bens\text{-}z} = 0$. Thus the total normal stress at point B can be written as $(\sigma_{xx})_B = -\sigma_{axial} - \sigma_{bend\text{-}y}$. Substituting the magnitude of $\sigma_{bend\text{-}y} = 20.258$ ksi and $\sigma_{axial} = 6.154$ ksi, we obtain the result in Equation (E15).

Shear stress calculations: The shear stresses at point A can be obtained by superposing the values in Equations (E10) and (E12). The shear stress at point B can be obtained by superposing the values in Equations (E9) and (E13).

ANS. $(\tau_{xz})_A = 1.2$ ksi $\quad (\tau_{xy})_B = -1.2$ ksi

Intuitive check on shear stress calculations: By inspection we deduce that the shear force on the bottom segment containing points A and B is in the negative y and positive z direction, as shown in Figure 10.21. We obtain the shear stress distribution (see Section 6.6.1) as shown. The direction of shear stress at point A and B are consistent with our results. Points A and B are on free surfaces with outward normals in y and z, respectively. Hence, $(\tau_{xy})_A = 0$ and $(\tau_{xz})_B = 0$. Thus the total shear stresses at A and B are $(\tau_{xz})_A = \tau_{bend\text{-}y}$ and $(\tau_{xy})_B = -\tau_{bend\text{-}z}$, consistent with our answers.

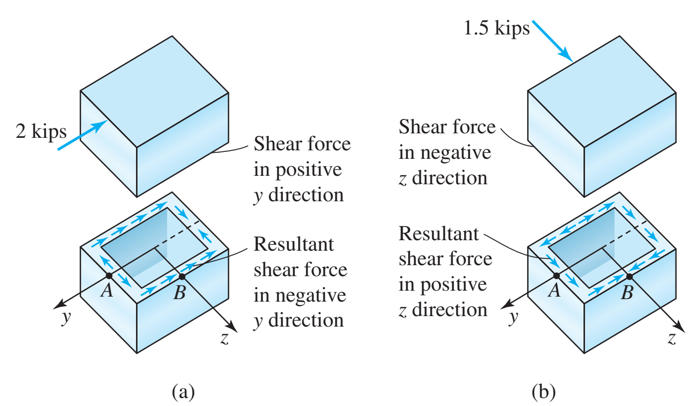


Figure 10.21 Direction of shear stress components by inspection for bending about (*a*) z axis; (*b*) y axis.

Step 7: The stresses at points A and B can now be drawn on a stress cube, as shown in Figure 10.22.

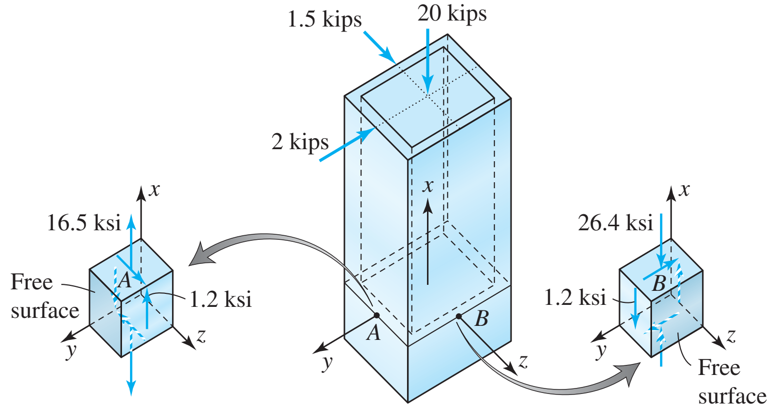


Figure 10.22 Stress cubes in Example 10.3.

Step 8: Figure 10.23 shows the plane containing the crack. From geometry we conclude that the angle that the outward normal makes with the x axis is 35°. Substituting $\theta = 35°$, $(\sigma_{xx})_B = -26.4$ ksi, $(\tau_{xy})_B = -1.2$ ksi, and $(\sigma_{yy})_B = 0$ into Equations (8.1) and (8.2), we obtain the normal and shear stresses on the plane containing the crack,

$$\sigma_{nn} = (-26.4\text{ ksi})\cos^2 35° + 2(-1.2\text{ ksi})\sin 35° \cos 35° = -18.84\text{ ksi} \quad \text{(E16)}$$

$$\tau_{nt} = -(-26.4\text{ ksi})\cos 35° \sin 35° + (-1.2\text{ ksi})(\cos^2 35° - \sin^2 35°) = 11.99\text{ ksi} \quad \text{(E17)}$$

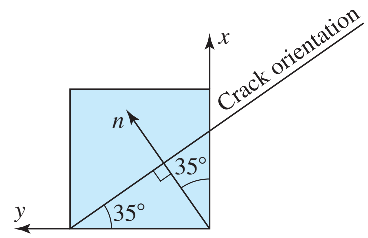


Figure 10.23 Angle of normal to plane containing crack.

ANS. $\sigma_{nn} = 18.84\text{ ksi (C)}$ $\tau_{nt} = 11.99\text{ ksi}$

COMMENTS

1. It may seem that the intuitive checks take as much effort as the calculation of the stresses by the procedural approach. But much of the description and diagrams here are for purpose of explanation only. Most of the intuitive check is by inspection. In the process you will develop an intuitive sense of the stresses under combined loading.
2. In place of three-dimensional free-body diagram of Figure 10.19, you may prefer drawing two perspectives of the free-body diagram shown in Figure 10.24. Figure 10.24*a* is constructed by looking down the y axis, whereas Figure 10.24*b* is the perspective looking down the z axis. Equation (E6) can be obtained from equilibrium.

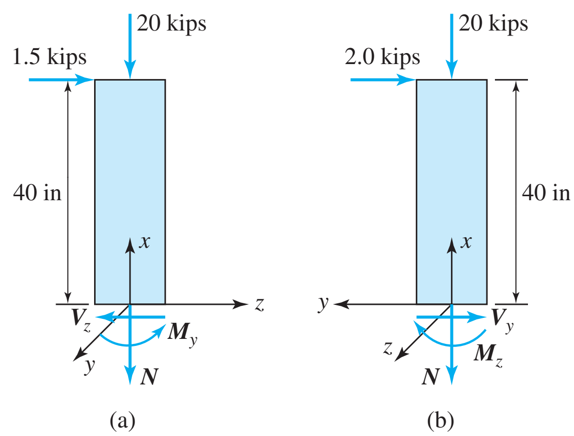


Figure 10.24 Two-dimensional free-body diagrams in Example 10.3.

3. In calculating bending stresses by inspection, be sure to use the correct area moment of inertia in the formula for rectangular cross sections: I_{yy} is not the same as I_{zz}. The subscripts emphasize that the moment of inertia to be used is the value about the bending axis.
4. The stresses on the plane containing the crack are used to assess whether a crack will grow and break the body.

EXAMPLE 10.4

A thin cylinder with an outer diameter of 100 mm and a thickness of 10 mm is loaded as shown in Figure 10.25. At point *A*, which is on the surface of the cylinder, determine the normal and shear stresses in the *x*, *y*, *z* coordinate system. Show your results on a stress cube.

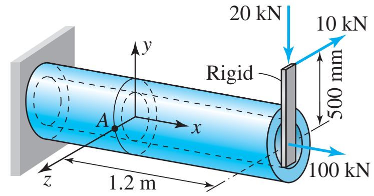


Figure 10.25 Geometry and loading in Example 10.4.

PLAN

We can follow the procedure outlined in Section 10.1.7. The 100 kN is an axial force. The 20-kN force will produce bending about the *z* axis. The 10-kN force will produce bending about the *y* axis and will also produce torque. Thus we need all the stress equations listed in Table 10.1. We can use these equations as a checklist of the quantities to calculate. and determine the stress at point *A* by superposition.

SOLUTION

Step 1: All the stress equations in Table 10.1 will be used.
Step 2: The geometric properties of the cross section can be found as

$$A = \pi[(50\text{ mm})^2 - (40\text{ mm})^2] = 2.827(10^3)\text{ mm}^2 \qquad J = \frac{\pi}{2}[(50\text{ mm})^4 - (40\text{ mm})^4] = 5.796(10^6)\text{ mm}^4 \tag{E1}$$

$$I_{yy} = I_{zz} = \frac{J}{2} = 2.898(10^6)\text{ mm}^4 \tag{E2}$$

Step 3: At points *A* we draw a line perpendicular to the centerline of the cross section to obtain the area A_s needed for the calculations of Q_y and Q_z at point *A*, as shown in Figure 10.26*a*.

$$t_A = 20\text{ mm} \qquad (Q_y)_A = 0 \tag{E3}$$

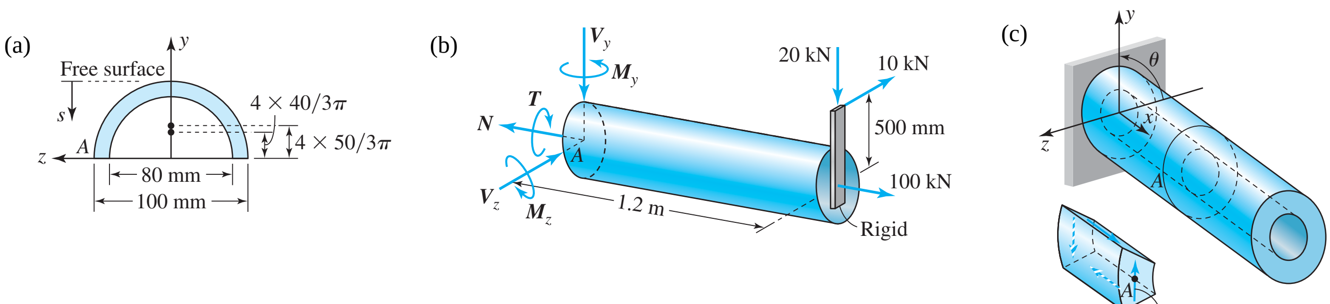


Figure 10.26 (a) Calculation of Q_z.(b) Free-body diagram. (c) Direction of torsional shear stress in Example 10.4.

To find $(Q_z)_A$, we use the formula $4r/3\pi$, given in Table C.2, for the location of the centroid for a half-disc of radius *r*. From Figure 10.26*a*, subtracting the first moment of the area of the inner disc of radius 40 mm from the first moment of the outer disc of radius 50 mm, we obtain

$$(Q_z)_A = \left[\frac{\pi(50\text{ mm})^2}{2}\right]\left[\frac{(4\text{ mm})(50\text{ mm})}{3\pi}\right] - \left[\frac{\pi(40\text{ mm})^2}{2}\right]\left[\frac{(4\text{ mm})(40\text{ mm})}{3\pi}\right] = 40.667(10^3)\text{ mm}^3 \tag{E4}$$

Step 4: We draw the free-body diagram shown in Figure 10.26*b* by making an imaginary cut at $x = 0$. The internal forces and moments are drawn according to our sign convention and can be obtained by equilibrium,

$$\boldsymbol{N} = 100\text{ kN} \qquad \boldsymbol{V}_y = -20\text{ kN} \qquad \boldsymbol{V}_z = -10\text{ kN} \qquad \boldsymbol{T} = -5\text{ kN}\cdot\text{m} \qquad \boldsymbol{M}_y = -12\text{ kN}\cdot\text{m} \qquad \boldsymbol{M}_z = -24\text{ kN}\cdot\text{m} \tag{E5}$$

Step 5: The stress components due to each loading are calculated next.
Axial stress calculations: From Equation (10.1) we obtain

$$(\sigma_{xx})_A = \frac{N}{A} = \frac{100(10^3)\text{ N}}{2.827(10^{-3})\text{ m}^2} = 35.373(10^6)\text{ N/m}^2 = 35.373\text{ MPa} \tag{E6}$$

Torsional shear stress calculations: Noting $\rho_A = 50\,(10^{-3})$ m we obtain from Equation (10.2),

$$(\tau_{x\theta})_A = \frac{T\rho_A}{J} = \frac{[-5(10^3)\text{ N}\cdot\text{m}][50(10^{-3})\text{ m}]}{5.796(10^{-6})\text{ m}^4} = -43.133(10^6)\text{ N/m}^2 \tag{E7}$$

The shear stress can be drawn on a stress cube using the subscripts, as shown in Figure 10.26*c*. The direction of shear stress in the *x*, *y* coordinate system is

$$(\tau_{xy})_A = 43.133\text{ MPa} \tag{E8}$$

Stresses due to bending about the y axis: Noting that $z_A = 50 \times 10^{-3}$ m, we obtain from Equation (10.4a),

$$(\sigma_{xx})_A = -\frac{M_y z_A}{I_{yy}} = -\frac{[-12(10^3)\ \text{N}\cdot\text{m}][50(10^{-3})\ \text{m}]}{2.898(10^{-6})\ \text{m}^4} = 207.04(10^6)\ \text{N/m}^2 = 207.04\ \text{MPa} \qquad \text{(E9)}$$

From Equation (10.4*b*) we obtain

$$(\tau_{xs})_A = 0 \qquad \text{or} \qquad (\tau_{xy})_A = 0 \qquad \text{(E10)}$$

Stresses due to bending about the z axis: Noting that $y_A = 0$, we obtain from Equation (10.3a)

$$(\sigma_{xx})_A = 0 \qquad \text{(E11)}$$

From Equation (10.3b) we obtain

$$(\tau_{xs})_A = -\frac{V_y(Q_z)_A}{I_{zz}t_A} = -\frac{[-20(10^3)\ \text{N}][40.667(10^{-6})\ \text{m}^3]}{[2.898(10^{-6})\ \text{m}^4][20(10^{-3})\ \text{m}]} = 14.033(10^6)\ \text{N/m}^2 \qquad \text{(E12)}$$

From Figure 10.26*a* we note that the *s* direction is in the negative *y* direction at point *A*. Thus,

$$(\tau_{xy})_A = -(\tau_{xs})_A = -14.033\ \text{MPa} \qquad \text{(E13)}$$

Step 6: Superposition

Normal stress calculations: The normal stress at point A can be obtained by superposing the values in Equations (E6), (E9), and (E12),

$$(\sigma_{xx})_A = 35.373\ \text{MPa} + 207.04\ \text{MPa} + 0 = 242.412\ \text{MPa} \qquad \text{(E14)}$$

ANS. $(\sigma_{xx})_A = 242.4\ \text{MPa (T)}$

Intuitive check on normal stress calculations: The axial stress σ_{axial} due to a 100-kN force will be tensile. Figure 10.27 shows the exaggerated deformed shapes due to bending about the *y* and *z* axes. From Figure 10.27*a*, it can be seen that line *AB* and hence the normal stress due to bending about the *y* axis $\sigma_{\text{bend-y}}$ will be tensile at point *A*. Hence the normal stress due to bending about the *z* axis $\sigma_{\text{bend-z}} = 0$ at point *A* is on the neutral (bending) axis in Figure 10.27*b*. Thus the total normal stress at point *A* can be written as $(\sigma_{xx})_A = \sigma_{\text{axial}} + \sigma_{\text{bend-y}}$, confirming our results.

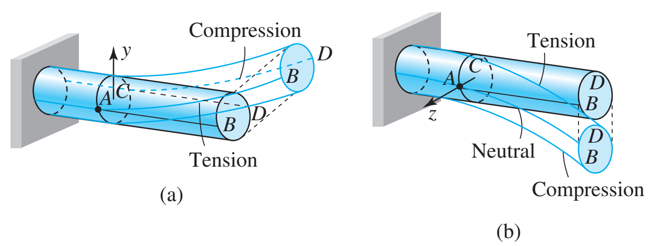


Figure 10.27 Determination of normal stress components by inspection from bending about (*a*) y axis; (*b*) z axis.

Shear stress calculations: The shear stress at point *A* can be obtained by superposing the values in Equations (E8), (E10), and (E13),

$$(\tau_{xy})_A = 43.133\ \text{MPa} + 0 - 14.033\ \text{MPa} = 29.10\ \text{MPa} \qquad \text{(E15)}$$

ANS. $(\tau_{xy})_A = 29.10\ \text{MPa}$

Intuitive check on shear stress calculations: Figure 10.28 shows the direction of shear stress due to torsion (see Section 5.2.5) and bending about *y* and *z* axis (see Section 6.6.1). We see that at point *A* the torsional shear stress τ_{tor} is upward, the shear stress due to bending about the *z* axis $\tau_{\text{bend-z}}$ is downward, and the shear stress due to bending about *y* axis $\tau_{\text{bend-y}}$ is zero. Thus the total shear stress at *A* can be written as $(\tau_{xy})_A = \tau_{\text{tor}} - \tau_{\text{bend-z}}$, confirming our results

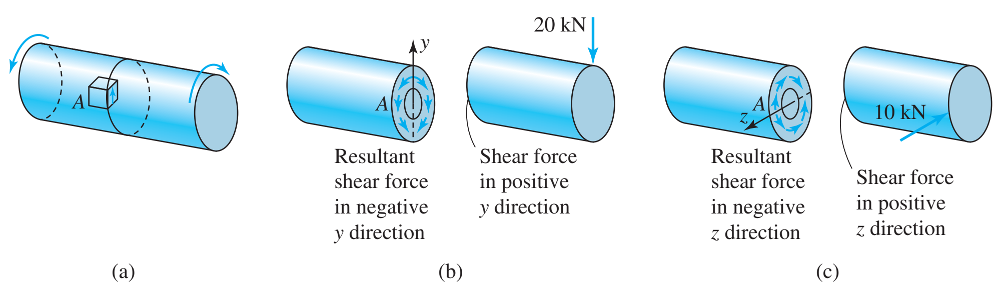


Figure 10.28 Direction of shear stress components by inspection from (*a*) torsion; (*b*) bending about *z* axis; (*c*) bending about *y* axis.

Step 7: Figure 10.29 shows the result of stresses on a stress cube.

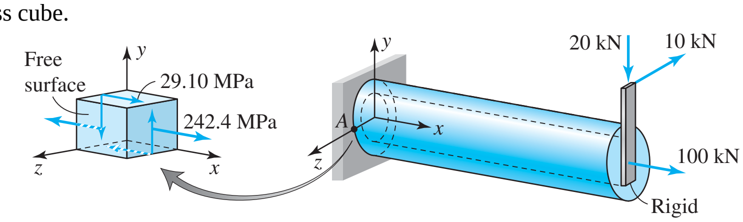


Figure 10.29 Results on stress cube in Example 10.4

COMMENTS

1. The stresses shown on the stress cube in Figure 10.29 can be processed further if necessary. We could find principal stresses as in Example 10.1, stresses on a plane as in Example 10.3, or strains along the direction of a gage as in Example 10.2.
2. The advantage of the procedure outlined in Section 10.1.7 is that it is methodical. It breaks a complex problem into a sequence of simple steps, as shown in this and previous examples. The shortcoming of this procedural approach is that it does not exploit any simplification that may be intrinsic to the problem.
3. Solving a problem by inspection has two distinct advantages: it helps build an intuitive understanding of stress behavior, and it can reduce the computational effort significantly.
4. The possibility of error is higher when solving the problem primarily by inspection because less is worked out on paper. For example, internal forces and moments are equal and opposite on the two surfaces created by an imaginary cut. It is easy to confuse one surface with another if we try to visualize all in our head, particularly in calculating of shear stress. Rough sketches can help.
5. You may find it more effective to solve part of the problem by a procedural approach and part by inspection. For example, you could solve for normal stresses but not shear stresses by inspection.

Consolidate your knowledge

1. Write a procedure you would use for solving combined loading problems.

EXAMPLE 10.5

The cylinder of 800-mm outer diameter shown in Figure 10.30 has a wall thickness of 15 mm. In addition to the axial and torsional loads, the cylinder is pressurized to 150 kPa. Determine the normal and shear stresses at point *A* on the center line of the cross section, and show them on a stress element in a cylindrical coordinate system.

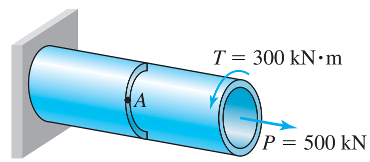


Figure 10.30 Cylinder and loading in Example 10.5*.

PLAN

The stress state at any point is from three sources: axial stress [Equation (10.1)]; torsional shear stress [Equation (10.2)]; and axial and hoop stresses due to pressure on the thin cylinder [Equations (4.28) and (4.29)].

SOLUTION

Step 1: Equations (10.1), (10.2), (4.28) and (4.29) will be used to find the stress components.
Step 2: The outer radius (R_o), the inner radius (R_i), the mean radius (R_m), the cross sectional area (A), and polar moment of inertia (J) can be found as

$$R_o = 0.4 \text{ m} \qquad R_i = 0.385 \text{ m} \qquad R_m = 0.3925 \text{ m} \tag{E1}$$

$$A = \pi(R_o^2 - R_i^2) = \pi[(0.4 \text{ m})^2 - (0.385 \text{ m})^2] = 36.99(10^{-3}) \text{ m}^2 \tag{E2}$$

$$J = \frac{\pi}{2}(R_o^4 - R_i^4) = \frac{\pi}{2}[(0.4 \text{ m})^4 - (0.385 \text{ m})^4] = 5.701(10^{-3}) \text{ m}^2 \tag{E3}$$

Step 3: This step is not needed as there is no bending.

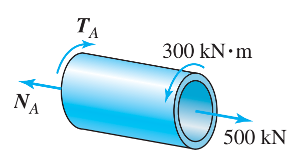


Figure 10.31 Free-body diagram in Example 10.5*.

Step 4: By equilibrium of free-body diagram in Figure 10.31 we obtain

$$N_A = 500 \text{ kN} \qquad T_A = 300 \text{ kN}\cdot\text{m} \tag{E4}$$

Step 5: The stress components due to each loading are calculated next.
Axial stress calculation: From Equation (10.1) we obtain

$$\sigma_{xx} = \frac{N_A}{A} = \frac{500(10^3) \text{ N}}{36.99(10^{-3}) \text{ m}^2} = 13.52(10^6) \text{ N/m}^2 = 13.52 \text{ MPa} \tag{E5}$$

Torsional shear stress: Noting $\rho_A = 0.3925$ m we obtain from Equation (10.2)

$$\tau_{x\theta} = \frac{T_A \rho_A}{J} = \frac{[300(10^3)\ \text{N}\cdot\text{m}](0.3925\ \text{m})}{5.701(10^{-3})\ \text{m}^4} = 20.65(10^6)\ \text{N/m}^2 = 20.65\ \text{MPa} \tag{E6}$$

Stresses due to pressure of thin-walled cylinders: Noting that the pressure is $p = 150$ kPa and t = 0.015 m we obtain from Equations (4.28) and (4.29)

$$\sigma_{xx} = \frac{pR_m}{2t} = \frac{[150(10^3)\ \text{N/m}^2](0.3925\ \text{m})}{2(0.015\ \text{m})} = 1.96 \times 10^6\ \text{N/m}^2 = 1.96\ \text{MPa} \tag{E7}$$

$$\sigma_{\theta\theta} = \frac{pR_m}{t} = 3.92\ \text{MPa} \tag{E8}$$

Step 6: Superposition

The normal stress at point *A* can be obtained by superposing the values in Equations (E5) and (E7),

$$\sigma_{xx} = 13.52\ \text{MPa} + 1.96\ \text{MPa} = 15.48\ \text{MPa} \tag{E9}$$

Figure 10.32 shows the stresses in Equations (E6), (E8), and (E9) on a stress element.

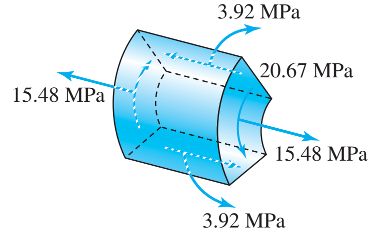


Figure 10.32 Stress element in Example 10.5*.

COMMENTS

1. From the stress state in Figure 10.32, we could find principal stresses, or strains in any direction.
2. We could have used Equation (5.26) in place of Equation (10.2), as the shaft is thin-walled, to obtain the same results

$$\tau = \frac{\boldsymbol{T}_A}{2tA_E} = \frac{[300(10^3)\ \text{N}\cdot\text{m}]}{2(0.015\ \text{m})[\pi(0.3925\ \text{m})^2]} = 20.67\ \text{MPa}$$

PROBLEM SET 10.1

Combined axial and torsion forces

10.1 Determine the normal and shear stresses in the seam of the shaft passing through point *A*, as shown in Figure P10.1 The seam is at an angle of 60° to the axis of a solid shaft of 2-in. diameter.

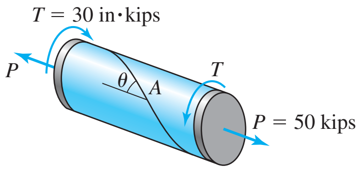


Figure P10.1

10.2 A 4-in.-diameter solid circular steel shaft is loaded as shown in Figure P10.2. Determine the shear stress and the normal stress on a plane passing through point *E*. Point *E* is on the surface of the shaft.

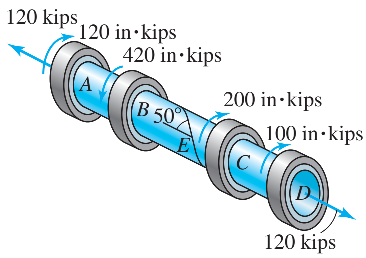


Figure P10.2

10.3 A solid shaft of 75-mm diameter is loaded as shown in Figure P10.3. The strain gage is 20° to the axis of the shaft and the shaft material has a modulus of elasticity E = 250 GPa and a Poisson ratio $\nu = 0.3$. If $T = 20$ kN·m and $P = 50$ kN, what strain will the strain gage show?

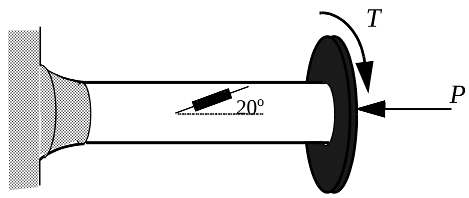


Figure P10.3

10.4 A solid shaft of 75-mm diameter is loaded as shown in Figure P10.3. The strain gage is 20° to the axis of the shaft and the shaft material has a modulus of elasticity E = 250 GPa and a Poisson ratio $\nu = 0.3$. If the strain gage shows a reading of −450 μm/m and $T = 10$ kN, determine the axial load P

10.5 A solid shaft of 75-mm diameter is loaded as shown in Figure P10.3. The strain gage is 20° to the axis of the shaft and the shaft material has a modulus of elasticity E = 250 GPa and a Poisson ratio $\nu = 0.3$. If the strain gage shows a reading of –300 μm/m and $P = 55$ kN, determine the applied torque T.

10.6 A solid shaft of 2-in. diameter is loaded as shown in Figure P10.6 The shaft material has a modulus of elasticity E = 30,000 ksi and a Poisson ratio $\nu = 0.3$. Determine the strains the gages would show if $P = 70$ kips and $T = 50$ in.·kips.

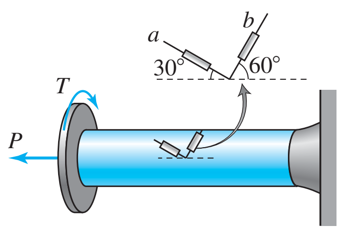


Figure P10.6

10.7 A solid shaft of 2-in. diameter is loaded as shown in Figure P10.6 The shaft material has a modulus of elasticity E = 30,000 ksi and a Poisson ratio $\nu = 0.3$. The strain gages mounted on the surface of the shaft recorded the strain values $\varepsilon_a = 2078\ \mu$ and $\varepsilon_b = -1410\ \mu$. Determine the axial force P and the torque T.

10.8 Two solid circular steel ($E_s = 200$ GPa, $G_s = 80$ GPa) shafts and a solid circular bronze shaft ($E_{br} = 100$ GPa, $G_{br} = 40$ GPa) are securely connected and loaded as shown Figure P10.8 Determine the maximum normal and shear stresses in the shaft.

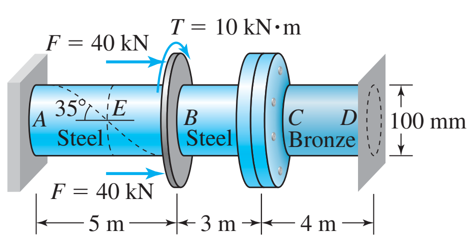


Figure P10.8

10.9 Determine the normal and shear stresses on a plane 35° to the axis of the shaft at point E in Figure P10.8. Point E is on the surface of the shaft.

Combined axial and bending forces

10.10 A 6-in. × 4-in. rectangular hollow member is constructed from a $\frac{1}{2}$-in.-thick sheet metal and loaded as shown in Figure P10.10. Determine the normal and shear stresses at points A and B and show them on the stress cubes for $P_1 = 72$ kips, $P_2 = 0$, and $P_3 = 6$ kips.

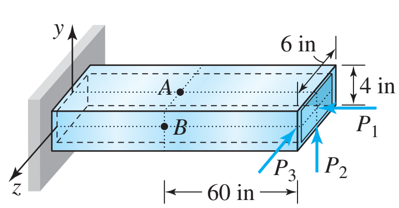


Figure P10.10

10.11 Determine the principal stresses and the maximum shear stress at points A and B in Figure P10.10 for $P_1 = 72$ kips, $P_2 = 3$ kips, and $P_3 = 0$.

10.12 Determine the strain shown by the strain gages in Figure P10.12 if $P_1 = 3$ kN, $P_2 = 40$ kN, the modulus of elasticity is 200 GPa, and Poisson's ratio is 0.3. The strain gages are parallel to the axis of the beam.

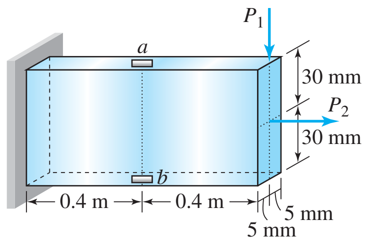


Figure P10.12

10.13 The strain gages shown in Figure P10.12 recorded the strain values $\varepsilon_a = 1000\ \mu$ and $\varepsilon_b = -750\ \mu$. Determine loads P_1 and P_2. The modulus of elasticity is 200 GPa and Poisson's ratio is 0.3.

10.14 Determine the strain shown by the strain gages in Figure P10.14 if $P_1 = 3$ kN, $P_2 = 40$ kN, the modulus of elasticity is 200 GPa, and Poisson's ratio is 0.3.

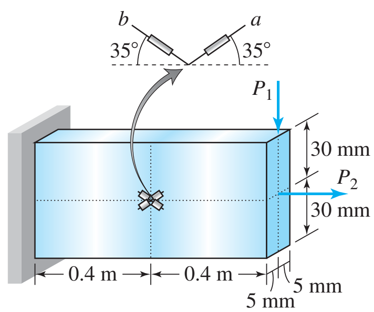


Figure P10.14

10.15 The strain gages shown in Figure P10.14 recorded the strain values $\varepsilon_a = 133\ \mu$ and $\varepsilon_b = 159\ \mu$. Determine loads P_1 and P_2. The modulus of elasticity is 200 GPa and Poisson's ratio is 0.3.

10.16 Determine the strain recorded by the gages at points A and B in Figure P10.16. Both gages are at 30° to the axis of the beam. The modulus of elasticity $E = 30{,}000$ ksi and $\nu = 0.3$.

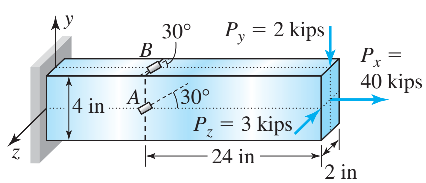


Figure P10.16

Combined axial, torsion, and bending forces

10.17 A thin cylinder with an outer diameter of 100 mm and a thickness of 10 mm is loaded as shown in Figure P10.17. Points A and B are on the surface of the shaft. Determine the normal and shear stresses at points A and B in the x, y, z coordinate system and show your results on stress cubes.

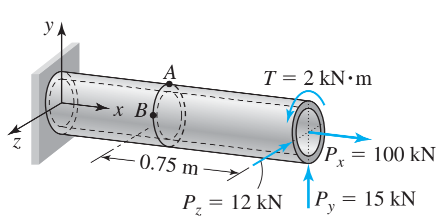


Figure P10.17

10.18 Determine the principal stresses and the maximum shear stress at point B on the shaft shown in Figure P10.17.

In problems 10.19 through 10.27, a pipe has outer diameter 120 mm and a thickness of 10 mm. All forces except the one given are zero. (a) Using the notation in Equation 10.5, determine by inspection the total normal and shear stresses at points A and B which are on the surface of the pipe. (b) Calculate the stresses and show on the stress cube.

Problem	***Loads***	***Use***
10.19	$P_x = 9$ kN	Figure P10.19 a =1.2 m, b =1.5m
10.20	$P_y = 12$ kN	Figure P10.19 a =1.2 m, b =1.5m
10.21	$P_z = 15$ kN	Figure P10.19 a =1.2 m, b =1.5m
10.22	$P_x = 9$ kN	Figure P10.22 a = 0.64 m, b = 0.5 m, c =0.3 m
10.23	$P_y = 12$ kN	Figure P10.22 a = 0.64 m, b = 0.5 m, c =0.3 m
10.24	$P_z = 15$ kN	Figure P10.22 a = 0.64 m, b = 0.5 m, c =0.3 m
10.25	$P_x = 9$ kN	Figure P10.25 a = 0.5 m, b = 0.8 m, c =0.7 m
10.26	$P_y = 12$ kN	Figure P10.25 a = 0.5 m, b = 0.8 m, c =0.7 m
10.27	$P_z = 15$ kN	Figure P10.25 a = 0.5 m, b = 0.8 m, c =0.7 m

Figure P10.19

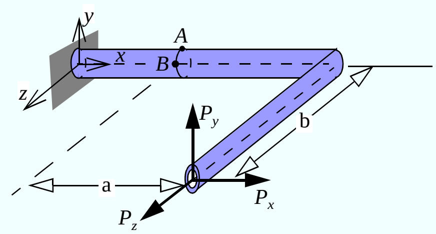


Figure P10.22

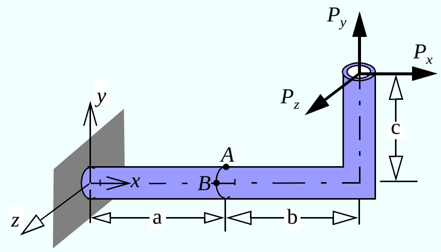
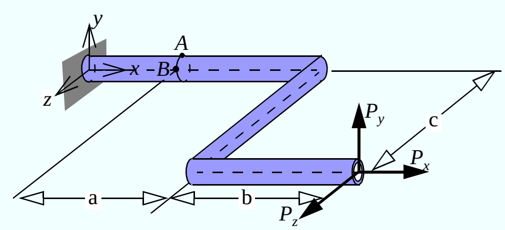


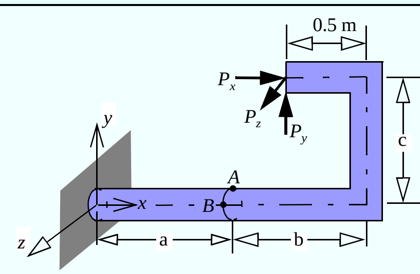
Figure P10.25



In problems 10.28 through 10.30, a pipe has outer diameter 120 mm and a thickness of 10 mm. All forces except the one given are zero. Determine the maximum normal and shear stress at points A and B.

Problem	***Loads***	***Use***
10.28	$P_x = 10$ kN	Figure P10.28 a = 0.7 m, b = 0.9 m, c =0.7 m
10.29	$P_y = 15$ kN	Figure P10.28 a = 0.7 m, b = 0.9 m, c =0.7 m
10.30	$P_z = 20$ kN	Figure P10.28 a = 0.7 m, b = 0.9 m, c =0.7 m

Figure P10.28



10.31 A pipe with an outside diameter of 2.0 in. and a wall thickness of $\frac{1}{4}$ in. is loaded as shown in Figure P10.31. Determine the normal and shear stresses at points *A* and *B* in the *x*, *y*, *z* coordinate system and show them on a stress cube. Points *A* and *B* are on the surface of the pipe. Use a =48 in. and b = 60 in.

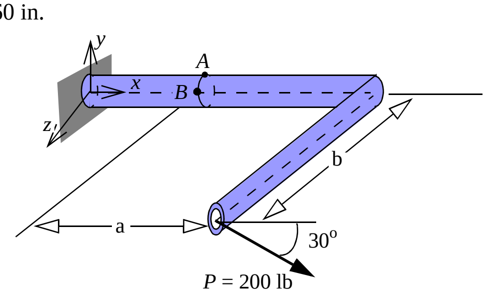


Figure P10.31

10.32 Determine the maximum normal stress and the maximum shear stress at point *B* on the pipe shown in Figure P10.31.

10.33 A pipe with an outside diameter of 40 mm and a wall thickness of 10 mm is loaded as shown in Figure P10.33. Determine the normal and shear stresses at points *A* and *B* in the *x, y, z* coordinate system and show them on a stress cube. Points *A* and *B* are on the surface of the pipe. Use a = 0.25 m, b = 0.4 m, and c = 0.1 m.

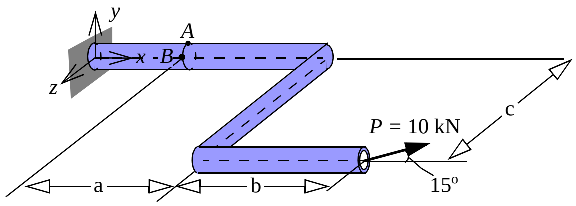


Figure P10.33

10.34 Determine the maximum normal stress and the maximum shear stress at point *B* on the pipe shown in Figure P10.33.

10.35 A bent pipe of 2-in. outside diameter and a wall thickness of $\frac{1}{4}$ in. is loaded as shown in Figure P10.35. If a = 16 in., b = 16 in.,and c = 10 in., determine the stress components at point *A*, which is on the surface of the shaft. Show your answer on a stress cube.

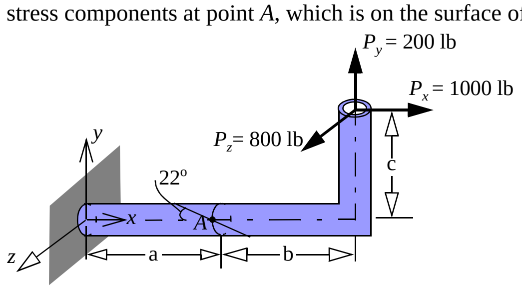


Figure P10.35

10.36 Determine the normal and shear stresses on a seam through point *A* that is 22° to the axis of the pipe shown in Figure P10.35.

10.37 The hollow steel shaft shown in Figure P10.37 has an outside diameter of 4 in. and an inside diameter of 3 in. Two pulleys of 24-in. diameter carry belts that have the given tensions. The shaft is supported at the walls using flexible bearings, permitting rotation in all directions. Determine the maximum normal and shear stresses in the shaft.

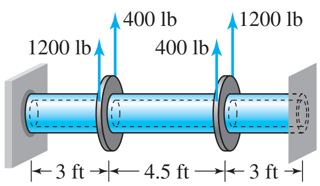


Figure P10.37

10.38 A thin hollow cylinder with an outer diameter of 120 mm and a wall thickness of 15 mm is loaded as shown in Figure P10.38. In *x, y, z* coordinate system, determine the normal and shear stresses at points *A* and *B* on the surface of the cylinder and show your results on stress cubes.

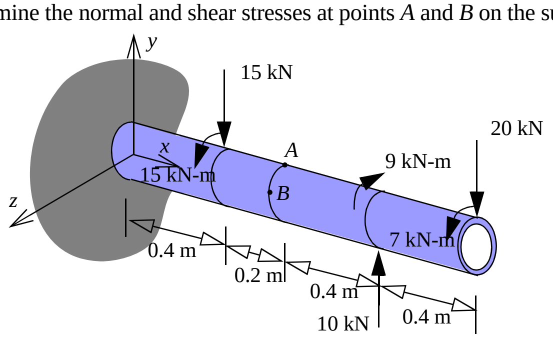


Figure P10.38

10.39 A 6 in. x 1 in. rectangular structural member is loaded as shown in Figure P10.39. Determine the maximum normal and shear stress in the member.

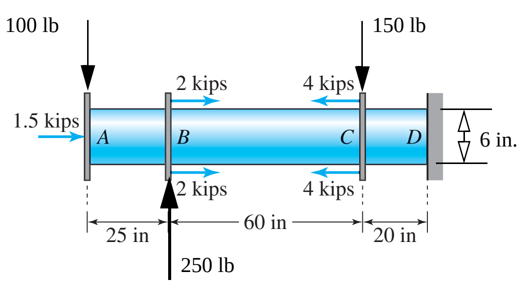


Figure P10.39

10.40 A thin cylinder is subjected to a uniform pressure of 300 psi and torques as shown in Figure P10.40. The cylinder has a outer radius of 10 in. and a wall thickness of 0.25 in. Determine the normal and shear stresses at point A and show them on a stress element in cartesian coordinates.

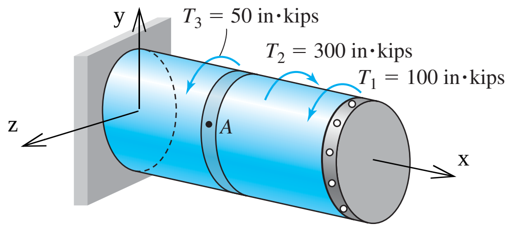


Figure P10.40

10.2 ANALYSIS AND DESIGN OF STRUCTURES

Most real structures are composed of many members joined together. Analyzing or designing these structures requires that we create a mathematical model to approximate the actual structure. Many decisions go into the creation of a model, including the proper modeling of joints and supports. Approximating joints by pins simplifies the analysis significantly, because pin joints do not transmit moments. In other words, we are neglecting the joints' intrinsic moment-carrying ability. Thus the model will predict higher internal forces, moments, and hence stresses than actually exist. This makes the pin joint approximation a conservative assumption.

Analyzing complex mathematical models requires numerical solutions. Here we shall consider simple structures made up of only a few members. Some members of the structure may be subjected to one type of combined loading, whereas other members may be subjected to another combination of loading.

There are two major steps in the solution of problems related to the analysis and design of structures:

1. Analysis of internal forces and moments that act on individual members.
2. Computation of stresses on members under combined loading.

For statically determinate structures, the internal forces and moments can be found using the principles of statics, as shown in Example 10.6. For statically indeterminate structures, we will also need the deformation equations developed earlier to complement the analysis skills learned in statics as we will see in Example 10.7.

10.2.1 Failure Envelope

In design, unlike analysis, the variable can assume a multitude of values. Each set of values corresponds to a different design. Which set of values gives the *best* design? The word *best*, of course, implies an objective. In engineering the objective is generally to minimize weight or cost. Algorithms that minimize an *objective function* (such as weight) subject to *constraints* (such as limits on stresses and displacements) are called *optimization methods*. Although optimization methods are beyond the scope of this book, we can develop an appreciation of the methods using the concept of failure envelope. A **failure envelope** separates the acceptable *design space* from the unacceptable values of the variables affecting design. More rigorous optimization algorithms would search the design space systematically, to minimize (or maximize) the objective function.

Consider a circular shaft that is subjected to axial loads and torsion, as shown in Figure 10.33*a*. Suppose the design limitation is that the maximum shear stress should not exceed 15 ksi. Further suppose that calculations show that the maximum shear stress at point A on the surface of the shaft is given by $\tau_{max} = 0.3183\sqrt{P^2 + 4T^2}$ ksi. Now τ_{max} should be less than or equal to 15 ksi, which gives us the following result: $P^2 + 4T^2 \le 2220$. We can now make a plot of T versus P, as shown in Figure 10.33*b*.

The shaded area consists of all possible values of T and P for which the maximum shear stress will be less than 15 ksi and hence represents our acceptable design space. The region beyond the shaded area represents values of T and P for which the shear stress is greater than 15 ksi and hence represents the failure space. On the curve $P^2 + 4T^2 = 2220$ all values of P and T

would result in a maximum shear stress of 15 ksi, and we are at incipient failure. This curve represents the failure envelope, which separates the design space from the failure space.

We can generalize this approach to a design problem containing *n* variables, which could be geometric variables, material constants, or loads, as in Figure 10.33. We may need to find the values of the variables to meet several design constraints. If one took each design variable and plotted it on an axis, then one would obtain a *n*-dimensional space containing all possible combinations of the *n* variables. Some of these combinations of the variables would result in failure. A failure envelope separates the space of acceptable values of these variables from the unacceptable values. On the failure envelope the values of the design variables correspond to impending failure. The sum total of all the design constraints defines the failure envelope. We shall also use the concept of failure envelope in Section 10.3 to describe failure theories in which the variables are principal stresses that are plotted on an axis.

Within the failure envelope we can compare different designs with respect to other criteria, such as cost, weight, and aesthetics. Example 10.8 elaborates the construction and use of the failure envelope.

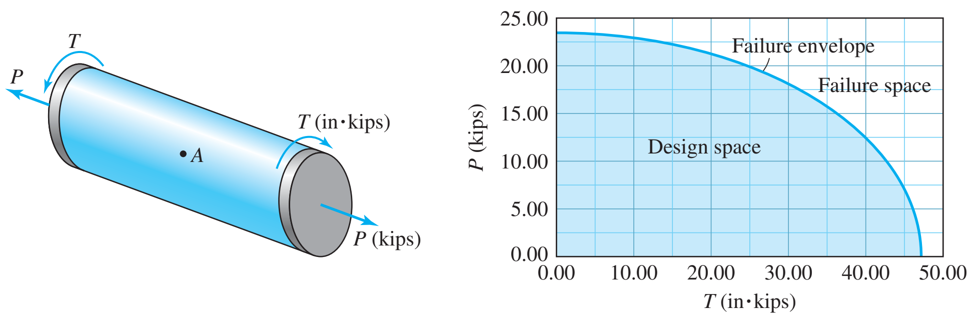


Figure 10.33 Failure envelope. (a) (b)

EXAMPLE 10.6

A hoist is to be designed for lifting a maximum weight $W = 300$ lb. Space considerations have established the length dimensions shown in Figure 10.34. The hoist will be constructed using lumber and assembled using steel bolts. The dimensions of the lumber cross sections are listed in Table 10.2. The bolted joints will be modeled as pins in single shear. Same-size bolts will be used in all joints. The allowable normal stress in the wood is 1.2 ksi and the allowable shear stress in the bolts is 6 ksi. Design the lightest hoist by choosing the lumber from Table 10.2 and the bolt size to the nearest $\frac{1}{8}$-in. diameter.

TABLE 10.2 **Dimensions of lumber**[a]

Cross-Section Dimensions	Cross-Section Dimensions
2 in. × 4 in.	4 in. × 8 in.
2 in. × 6 in.	6 in. × 6 in.
2 in. × 8 in.	6 in. × 8 in.
4 in. × 4 in.	8 in. × 8 in.
4 in. × 6 in.	

[a] The dimensions for finished lumber are slightly smaller and must be properly accounted in actual design.

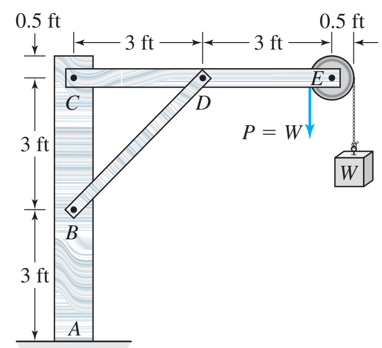


Figure 10.34 Hoist in Example 10.6.

PLAN

We analyze the problem in two steps. First we find the forces and moments on individual members, and then we find the stresses.

1. *BD* is a two-force axial member that will be in compression. Members *ABC* and *CDE* are multiforce members subjected to axial and bending loads. Free-body diagrams of members *ABC* and *CDE* will permit calculation of the forces at pin *C*, the axial force in *BD*—forces on pins *B* and *D* are thus known, as well as the reaction forces and the reaction moment at *A*.
2. We compute the maximum stresses from the forces calculated in step 1 and, using the limiting values on the maximum stresses, compute the dimensions of the pin and the wooden members. From the possible set of dimensions that satisfy the limiting criteria we choose those that will result in the lightest structure.

SOLUTION

Calculation of forces and moments on structural members: Figure 10.35 shows the free-body diagrams of members *CDE* and *ABC*. Using the moment equilibrium at point *C* in Figure 10.35*a*, we obtain

$$(N_{BD}\sin 45°)(3\text{ ft}) - (300\text{ lbs})(5.5\text{ ft}) - (300\text{ lbs})(6.5\text{ ft}) = 0 \quad \text{or} \quad N_{BD} = 1697\text{ lbs} \tag{E1}$$

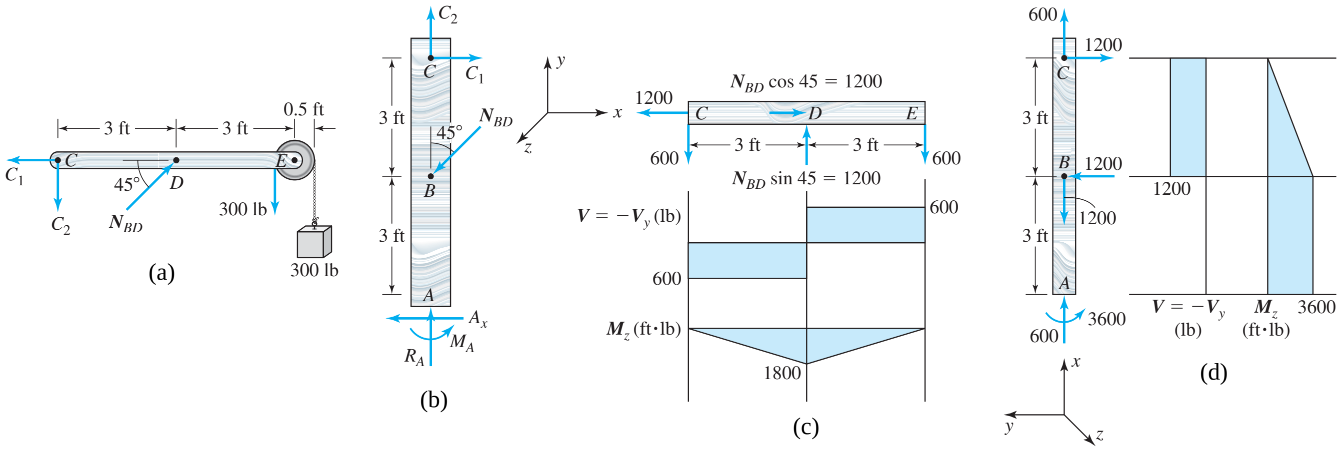


Figure 10.35 (a, b) Free-body diagram of member CDE and ABC. (c, d) Shear and moment diagrams of members CDE and ABC.

By equilibrium of forces in Figure 10.35*a*, we obtain

$$C_1 - N_{BD}\cos 45° = 0 \quad \text{or} \quad C_1 = 1200\text{ lbs} \tag{E2}$$

$$N_{BD}\sin 45° - C_2 - 600\text{ lbs} = 0 \quad \text{or} \quad C_2 = 600\text{ lbs} \tag{E3}$$

By equilibrium of forces and moment about point *A* in Figure 10.35*b*, we obtain

$$C_1 - N_{BD}\sin 45° - A_x = 0 \quad \text{or} \quad A_x = 0 \tag{E4}$$

$$R_A + C_2 - N_{BD}\cos 45° = 0 \quad \text{or} \quad R_A = 600\text{ lbs} \tag{E5}$$

$$M_A - C_1(6\text{ ft}) + (N_{BD}\cos 45°)(3\text{ ft}) = 0 \quad \text{or} \quad M_A = 3600\text{ ft}\cdot\text{lb} \tag{E6}$$

Bolt size calculations: The shear force acting on each bolt can be found from the forces calculated,

$$V_B = V_D = N_{BD} = 1697\text{ lbs} \qquad V_C = \sqrt{C_1^2 + C_2^2} = 1342\text{ lbs} \qquad V_E = 600\text{ lbs} \tag{E7}$$

The maximum shear stress will be in bolts *B* and *D*. This maximum shear stress should be less than 6 ksi. The cross-sectional area can be found and the diameter of the bolt calculated,

$$\tau_{max} = \frac{1697\text{ lbs}}{\pi d^2/4} \le 6000\text{ lbs/in.}^2 \quad \text{or} \quad d \ge 0.60\text{ in.} \tag{E8}$$

The nearest $\frac{1}{8}$-in. size that is greater than the numerical value in Equation (E8) is $d = 0.625$ in.

ANS. $\frac{5}{8}$-in.-size bolts should be used.

Lumber selection: The normal axial stress in member *BD* has to be less than 1200 psi. The cross-sectional area for member *BD* can be found as

$$\sigma_{BD} = \frac{N_{BD}}{A_{BD}} = \frac{1697\text{ lbs}}{A_{BD}} \le 1200\text{ lbs/in.}^2 \quad \text{or} \quad A_{BD} \ge 1.414\text{ in}^2 \tag{E9}$$

The 2-in. × 4-in. lumber has a cross-sectional area of 8 in.^{2}, which is the smallest cross section that meets the restriction of Equation (E9).

ANS. For member BD use lumber with the cross-section dimensions of 2 in. × 4 in**.**

Shear force and bending moment diagrams for members *ABC* and *CDE* can be drawn after resolving the force ***NBD*** into components parallel and perpendicular to the axis, as shown in Figure 10.35*c, d.* A local *x, y, z* coordinate system for each member is established to facilitate drawing the shear and moment diagrams.

From Figure 10.35*c* it can be seen that the maximum axial force is 1200 lbs tensile in segment *CD*. The bending moment is maximum on the cross section at *D* in member *CDE*. Due to bending, the top surface will be in tension and the bottom will be in compression. Thus the maximum normal stress in *CDE* will be at the top surface just before *D* and will be the sum of tensile stresses due to axial and bending loads. Using an axial force of 1200 lbs and a bending moment of 1800 ft·lbs = 21,600 in.·lbs, the maximum normal stress in *CDE* can be written as

$$\sigma_{CD} = \frac{1200\text{ lbs}}{A_{CDE}} + \frac{21{,}600\text{ in.}\cdot\text{lbs}}{S_{CDE}} \tag{E10}$$

where A_{CDE} and S_{CDE} are the cross-sectional area and the section modulus (with respect to the *z* axis) of member *CDE*.

From Figure 10.35*d* it can be seen that the axial force in *AB* is 600 lbs compressive and the axial force in *BC* is 600 lbs tensile. The bending moment is a maximum of 1800 ft · lbs = 43,200 in. · lbs throughout segment *AB*. Owing to bending, the right side of member *ABC* will be in compression and the left side will be in tension. Thus the maximum normal stress in member *ABC* will be on the right surface, just before *B* in segment *AB*, and will be the sum of compressive stresses due to axial and bending loads. With an axial force of 600 lbs and a bending moment of 3600 ft · lbs = 43,200 in. · lbs, the maximum compressive normal stress in *ABC* can be written as

$$\sigma_{AB} = \frac{600 \text{ lbs}}{A_{ABC}} + \frac{43{,}200 \text{ in.} \cdot \text{lbs}}{S_{ABC}} \tag{E11}$$

where A_{ABC} and S_{ABC} are the cross-sectional area and the section modulus (with respect to the *z* axis) of member *ABC*.

For the available lumber given in Table 10.2, the cross-sectional area *A* and the section modulus *S* can be determined assuming that the smaller dimension *a* is parallel to the *z* axis (bending axis or dimension out of the plane of the paper) and the larger dimension *b* is in the plane of the paper. With this stipulation $I_{zz} = ab^3/12$ and $y_{\max} = b/2$. Hence $S = I_{zz}/y_{\max} = ab^2/6$ (see the local coordinates in Figure 10.35*c*, *d*). Substituting the values of *A* and *S* into Equations (E10) and (E11), we find the stress values σ_{CD} and σ_{AB}. Table 10.2 can be created using a spreadsheet. Cross-section dimensions for which the normal stress is less than 1200 psi meet the strength limitation and are identified in bold in Table 10.2. But for the design of the lightest hoist we choose the cross section with the smallest area among the bold values.

ANS. For member *ABC*, use lumber with the cross-section dimensions of 4 in. × 8 in.
ANS. For member *CDE*, use lumber with the cross-section dimensions of 2 in. × 8 in.

TABLE 10.2 **Cross-section properties and stresses in Example 10.6**

a (in.)	*b* (in.)	$A = ab$ (in.^{2})	$S = ab^2/6$ (in.^{3})	σ_{CD} (psi)	σ_{AB} (psi)
2	4	8	5.3	4200.0	8175.0
2	6	12	12.0	1900.0	3650.0
2	8	16	21.3	1087.5	2062.5
4	4	16	10.7	2100.0	4087.5
4	6	24	24.0	950.0	1825.0
4	8	32	42.7	543.8	1031.3
6	6	36	36.0	633.3	1216.7
6	8	48	64.0	362.5	687.5
8	8	64	85.3	271.9	515.6

COMMENTS

1. Members in axial compression such as *BD* must be designed for strength as well as checked for buckling failure, as will be elaborated in the next chapter.
2. In actual design it may be preferable to use two pieces of 1-in. × 8-in. lumber for member *CDE* so that the pulley is in the middle of the two members. This will change pins at *C*, *D*, and *E* from single shear into double shear, thus also reducing the shear stresses in the pins.
3. Equation (E10) defines the curve of the failure envelope for member *CDE*. Substituting for the area and the section modulus in terms of *a* and *b*, and solving for *a* in terms of *b*, we obtain the equation of the curve defining the failure envelope as $a = 0.5/b + 216/b^2$. Figure 10.36 shows the failure envelope. As can be seen, the three possible solutions in Table 10.2 for member *CDE* fall in the design space and the remaining cross sections fall in the failure space. If we were to choose any value for *a* and *b*, then the failure envelope would identify all possible solutions.

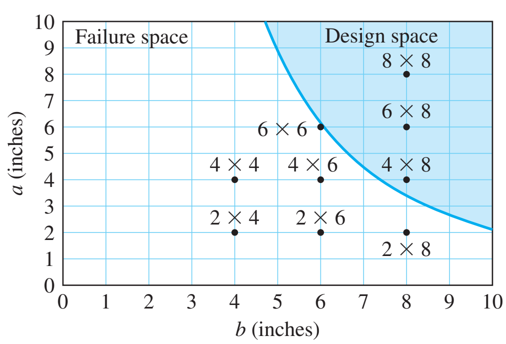


Figure 10.36 Failure envelope for member *CDE*.

4. The bending shear stress was not considered in selecting the lumber cross sections for members *ABC* and *CDE*. This is based on the consideration that the maximum bending normal stress is significantly (~10 times) greater than the maximum bending shear stress. Thus the principal stress at the top or bottom of the member, where the bending normal stress is maximum, will be greater than the principal shear stress at the neutral axis, where bending shear stress is maximum. We check these statements for the selected sizes of members *ABC* and *CDE* as follows.

For rectangular cross sections it can be shown (see comment 1 in Example 6.14) that the maximum shear stress in bending at a cross section is $\tau_{max} = 1.5\boldsymbol{V}/A$, where A is the cross-sectional area. From Figure 10.35 the maximum shear force is 600 lb and 1200 lb in members *CDE* and *ABC*, respectively. Substituting these shear force values and the values of 16 in.^{2} and 32 in.^{2} for the areas, we obtain the maximum shear stresses of $\tau_{CD} = 56.25$ psi and $\tau_{AB} = 56.25$ psi in members *CDE* and *ABC*, respectively. Comparing these maximum shear stress values to the maximum normal stress values of 1087.5 psi and 1031.3 psi for members *CDE* and *ABC*, which are given in Table 10.2, we conclude that the shear stresses can be ignored in the selection of lumber.

EXAMPLE 10.7

A rectangular wooden beam of 60 mm × 180 mm cross section is supported at the right end by an aluminum circular rod of 8-mm diameter, as shown in Figure 10.37. The allowable normal stress in the wood is 14 MPa and the allowable shear stress in aluminum is 60 MPa. The moduli of elasticity for wood and aluminum are $E_w = 12.6$ GPa and $E_{al} = 70$ GPa. Determine the maximum intensity w of the distributed load that the structure can support.

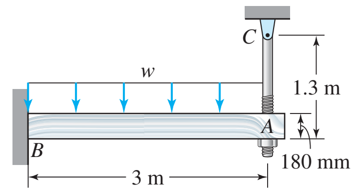


Figure 10.37 Beam in Example 10.7.

PLAN

We analyze the problem in two steps. First we find the forces and moments on individual members, and then we find the stresses.

1. To solve this statically indeterminate problem, the deflection of the beam at A can be equated to the axial deformation of the aluminum rod. This permits the calculation of the internal axial force in the aluminum rod in terms of w.
2. The axial stress in the aluminum rod in terms of w can be found from the internal axial force calculated in step 1. Using Mohr's circle, we can find the maximum shear stress in the axial rod in terms of w, and one limit on w can be obtained. The maximum bending moment at B can be found in terms of w and the maximum bending normal stress calculated in terms of w. Using the allowable value of 14 MPa, another limit on w can be found and a decision made on the maximum value of w.

SOLUTION

Calculation of forces on structural members: The area moment of inertia of the wood and the cross-sectional area of the aluminum rod can be calculated,

$$I_w = \frac{1}{12}(60\text{ mm})(180\text{ mm})^3 = 29.16(10^6)\text{ mm}^4 = 29.16(10^{-6})\text{ m}^4 \qquad A_{al} = \frac{\pi}{4}(8\text{ mm})^2 = 50.265\text{ mm}^2 = 50.265(10^{-6})\text{ m}^2 \qquad \text{(E1)}$$

Making an imaginary cut through the aluminum rod, we obtain the beam and loading shown in Figure 10.38*a*. The total loading on the beam can be considered as the sum of the two loadings shown in Figure 10.38*b* and *c*.

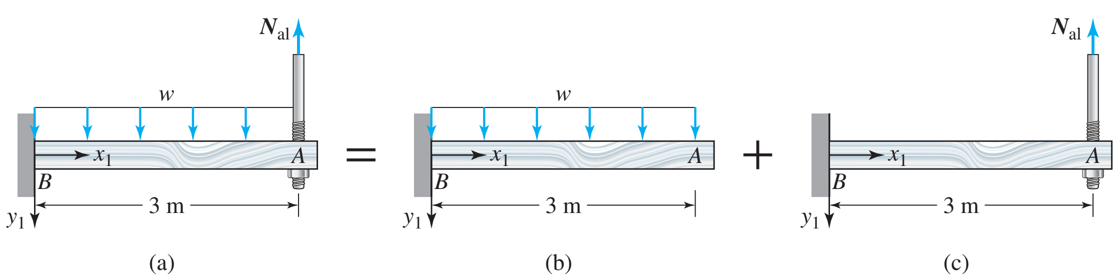


Figure 10.38 Superposition of deflection in Example 10.7.

Comparing the two beam loadings in Figure 10.38*b* and *c* to that shown for cases 1 and 3 in Table C.3, we obtain $P = -\boldsymbol{N}_{al}$ N and $p = w$ N/m, $a = 3$ m, $b = 0$, $E = 12.6\ (10^9)$ N/m^2, and $I = I_w = 29.16\ (10^{-6})$ m^4. Noting that v_{max} shown in Table C.3 for the cantilever beam occurs at point A, we can substitute the load values and superpose to obtain deflection v_A,

$$v_A = \frac{(w\text{ N/m})(3\text{ m})^4}{8[12.6(10^9)\text{ N/m}^2][29.16(10^{-6})\text{ m}^4]} + \frac{(-\boldsymbol{N}_{al}\text{ N})(3\text{ m})^3}{3[12.6(10^9)\text{ N/m}^2][29.16(10^{-6})\text{ m}^4]} \quad \text{or}$$

$$v_A = [27.56(10^{-6})\,w - 24.50(10^{-6})\,N_{al}]\text{ m} \qquad \text{(E2)}$$

The extension of the aluminum rod can be found using Equation (4.21),

$$\delta_{al} = \frac{N_{al}L_{al}}{E_{al}A_{al}} = \frac{(N_{al}\ \text{N})(1.3\ \text{m})}{[70(10^9)\ \text{N/m}^2][50.265(10^{-6})\ \text{m}^4]} = 0.369(10^{-6})N_{al}\ \text{m} \quad \text{(E3)}$$

The extension of the aluminum rod should equal the deflection of the beam at *A*. Equating Equations (E2) and (E3) gives the internal force N_{al} in terms of *w*,

$$27.56(10^{-6})w - 24.50(10^{-6})N_{al} = 0.369(10^{-6})N_{al} \qquad \text{or} \qquad N_{al} = 1.11w \quad \text{(E4)}$$

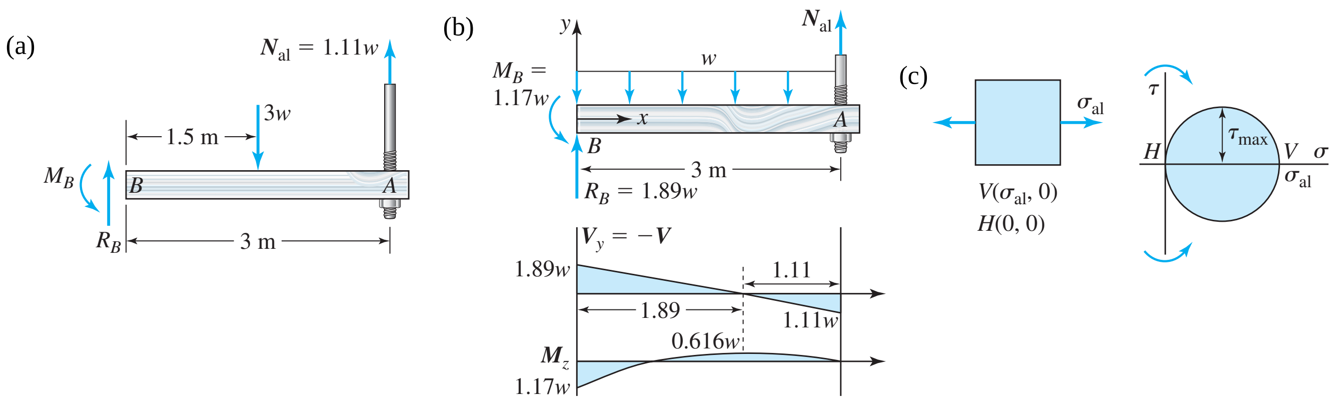


Figure 10.39 (a) Free-body diagram (b) Shear force and bending moment diagrams (c) Mohr's circle in Example 10.7.

Figure 10.39*a* shows the free-body diagram of the beam with the distributed force replaced by an equivalent force. By force and moment equilibrium we obtain

$$R_B - (3w) + N_{al} = 0 \qquad \text{or} \qquad R_B = 1.89w \quad \text{(E5)}$$

$$M_B - (3w)(1.5) + N_{al}(3) = 0 \qquad \text{or} \qquad M_B = 1.17w \quad \text{(E6)}$$

Figure 10.39*b* shows the shear and moment diagrams for the beam. From the moment diagram we see that the maximum moment is at the wall, and its value is

$$M_{max} = -1.17w \quad \text{(E7)}$$

Stress in aluminum: The axial stress in aluminum in terms of *w* is

$$\sigma_{al} = \frac{N_{al}}{A_{al}} = \frac{1.11w}{50.265(10^{-6})\ \text{m}^2} = 22.04(10^3)w\ \text{N/m}^2 \quad \text{(E8)}$$

Figure 10.39*c* shows the Mohr's circle for point on aluminum rod. The maximum shear stress maximum shear stress in should be less than 60 MPa,

$$\tau_{max} = \frac{\sigma_{al}}{2} = 11.02(10^3)w\ \text{N/m}^2 \le 60(10^6)\ \text{N/m}^2 \qquad \text{or} \qquad w \le 5.44(10^3)\ \text{N/m} \quad \text{(E9)}$$

Stress in wood: The bending normal stress will be maximum at the top and bottom surfaces at the wall; that is, $y_{max} = \pm 0.09$ m. The magnitude of the maximum bending normal stress from Equation (10.3*a*) should be less than 14 MPa, yielding another limit on *w*,

$$\sigma_w = \left|\frac{M_{max}y_{max}}{I_w}\right| = \frac{(1.17w\ \text{N}\cdot\text{m})(0.09\ \text{m})}{29.16(10^{-6})\ \text{m}^4} = 3.61(10^3)w\ \text{N/m}^2 < 14(10^6)\ \text{N/m}^2 \qquad \text{or} \qquad w \le 3.88(10^3)\ \text{N/m} \quad \text{(E10)}$$

The value in Equation (E10) also satisfies the inequality in Equation (E9) giving the maximum intensity of the distributed load.

ANS. $w_{max} = 3.88$ kN/m

COMMENT

1. At joint *A* we ensured continuity of displacement by enforcing the condition that the deformation of the axial member be the same as the deflection of the beam. We also enforced equilibrium of forces by using the same force N_{al} in the axial member and acting on the beam. These two conditions, continuity of displacement and equilibrium of forces, must be satisfied by all joints in more complex structures.

EXAMPLE 10.8

A circular member was repaired by welding a crack at point A that was 30° to the axis of the shaft, as shown in Figure 10.40. The allowable shear stress at point A is 24 ksi and the maximum normal stress the weld material can support is 9 ksi (T). Calculations show that the stresses at point A are $\sigma_{xx} = 9.55P_2$ ksi (T) and $\tau_{xy} = -6.79P_1$ ksi. (a) Draw the failure envelope for the applied loads P_1 and P_2. (b) Determine the values of loads P_1 and P_2. (c) If $P_1 = 2$ kips and $P_2 = 1.5$ kips, determine the factor of safety.

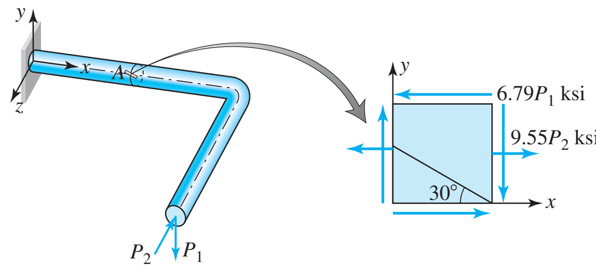


Figure 10.40 Problem geometry in Example 10.8.

PLAN

(a) By substituting the given stresses into Equation (8.12) we obtain the maximum shear stress in the material in terms of P_1 and P_2. Noting that the maximum shear stress is limited to 24 ksi, we obtain one equation relating P_1 and P_2. By substituting the given stresses into Equation (8.1) as well as the angle of the normal to the weld, we can obtain the normal stress on the weld, which gives us another equation relating P_1 and P_2. We can sketch both equations and obtain the failure envelope. (b) We can find the values of P_1 and P_2 that satisfy the two equations in part (a). (c) We can calculate the maximum in-plane shear stress in the material and the normal stress in the weld from the equations obtained in part (a) and compute two factors of safety. The lower value is the factor of safety for the system.

SOLUTION

(a) Substituting the values of the given stresses into Equation (8.12), we can obtain the maximum shear stress in the material. Noting that it should be less than 24 ksi, we obtain one equation on P_1 and P_2,

$$\tau_{max} = \sqrt{\left(\frac{9.55P_2\text{ ksi}}{2}\right)^2 + (-6.79P_1\text{ ksi})^2} = \sqrt{46.1P_1^2 + 22.8P_2^2}\text{ ksi} \le 24\text{ ksi} \qquad \text{or} \qquad 46.1P_1^2 + 22.8P_2^2 \le 576 \tag{E1}$$

The normal to the weld makes an angle of 60° to the x axis. Substituting the given stresses and $\theta = 60°$ into Equation (8.1), the normal stress on the weld must be less than 9 ksi (T), we obtain another equation on P_1 and P_2,

$$\sigma_{weld} = (9.55P_2\text{ ksi})\cos^2 60° + 2(-6.79P_1\text{ ksi})\cos 60° \sin 60° \qquad \text{or} \qquad \sigma_{weld} = (2.387P_2 - 5.881P_1)\text{ ksi} \le 9\text{ ksi} \tag{E2}$$

The maximum value of P_1 that will satisfy Equation (E1) corresponds to $P_2 = 0$. This maximum value of P_1 is 3.534 kips. We consider values of P_1 between zero and 3.534 in steps of 0.3 and solve for P_2 from Equation (E1). For the same values of P_1 we can also find values of P_2 from Equation (E2), as shown in Table 10.3, which was produced on a spreadsheet. We can plot the values in Table 10.3, as shown in Figure 10.41. The design space is the shaded region and the failure envelope is the boundary *ABCD*

TABLE 10.3 **Values of loads in Example 10.8**

P_1 (kips)	P_2 from Eq. (E1) (kips)	P_2 from Eq. (E2) (kips)
0.000	5.027	3.770
0.300	5.008	4.509
0.600	4.954	5.248
0.900	4.861	5.987
1.200	4.728	6.726
1.500	4.551	7.465
1.800	4.326	8.204
2.100	4.043	8.943
2.400	3.690	9.682
2.700	3.244	10.421
3.000	2.657	11.160
3.300	1.800	11.899

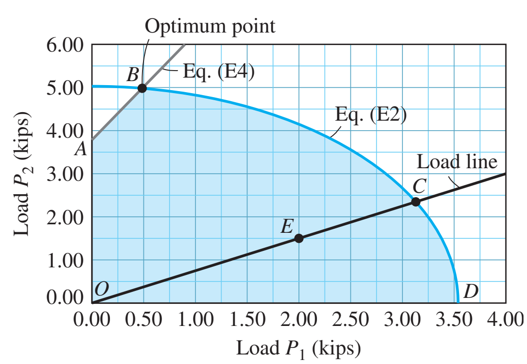


Figure 10.41 Failure envelope in Example 10.8.

(b) The values of P_1 and P_2 correspond to the maximum values of the loads that satisfy Equations (E1) and (E2). Using the equality sign in Equation (E2), we can solve for P_2,

$$P_2 = 3.770 + 2.4634P_1 \tag{E3}$$

We substitute Equation (E3) into Equation (E1) with the equality sign to obtain a quadratic equation in P_1,

$$46.11P_1^2 + 22.80(4.19 + 2.46P_1)^2 = 576 \quad \text{or} \quad 184.45P_1^2 + 423.2P_1 - 252 = 0 \tag{E4}$$

Solving Equation (E3) we obtain two roots for P_1, 0.4905 and –2.784. Only the positive root is admissible. Substituting $P_1 = 0.4905$ into Equation (E3), we obtain $P_2 = 4.978$.

ANS. $P_1 = 0.49 \text{ kips} \qquad P_2 = 4.98 \text{ kips}$

(c) Substituting $P_1 = 2$ kips and $P_2 = 1.5$ into Equations (E1) and (E2), we obtain

$$\tau_{max} = \sqrt{46.1 \times 2^2 + 22.8 \times 1.5^2} = 15.35 \text{ ksi} \qquad \sigma_{weld} = 2.387 \times 1.5 - 5.881 \times 2 = -8.18 \text{ ksi} \tag{E5}$$

For the given loads the normal stress in the weld is compressive. Hence it won't fail due to the specified failure in tension. The factor of safety is thus calculated from the maximum shear stress as

$$K = \frac{24 \text{ ksi}}{15.35 \text{ ksi}} = 1.56 \tag{E6}$$

ANS. $K = 1.56$

COMMENTS

1. In this example we generated the failure envelope using analytical equations. For more complex structures, the failure envelope can be created using numerical methods, such as the finite-element method described in Section 4.8.
2. In Figure 10.41 line *AB*, representing Equation (E1), would go downward if the direction of load P_1 were reversed [Substitute $-P_1$ in place of P_1 in Equation (E1)]. If line *AB* went downward, it would cut the design space considerably. Thus not only is the magnitude of the loads important in design, but the direction of the load can also be as critical. Failure envelopes can reveal such characteristics in a very visual manner.
3. The line joining the origin to point *E* is called *load line*, on which the loads vary proportionally. It's significance is that it can help give a graphical interpretation of the factor of safety. Along a load line, the distance of a point from the failure envelope is the margin of safety. In Figure 10.41 the factor of safety is the ratio of length *OC* to length *OE*. It can be verified that the coordinates of point *C* are $P_1 = 3.1263$ kips and $P_2 = 2.344$. Thus the length $OC = 3.9074$, whereas the length *OE* is 2.5. The factor of safety therefore is $K = 3.9074/2.5 = 1.56$, as before.

PROBLEM SET 10.2

Structural analysis and design

10.41 A brick chimney shown in Figure P10.41 has an outside diameter of 5 ft and a wall thickness of 6 in. The average specific weight of the brick and mortar is $\gamma = 120$ lb/ft^3. The height of the chimney is $H = 30$ ft. Determine the maximum wind pressure p that the chimney could withstand if there is to be no tensile stress.

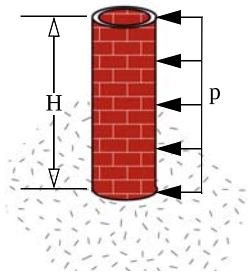


Figure P10.41

10.42 A brick chimney shown in Figure P10.41 has an outside diameter of 1.5 m and a wall thickness of 150 mm. The average specific density of the brick and mortar is $\gamma = 1800$ kg/m^3. The wind pressure acting on the chimney is $p = 800$ Pa. Determine the maximum height H of the chimney is if there is to be no tensile stress.

10.43 A hollow shaft that has an outside diameter of 100 mm and an inside diameter of 50 mm is loaded as shown in Figure P10.43. The normal stress and the shear stress in the shaft must be limited to 200 MPa and 115 MPa, respectively. (a) Determine the maximum value of the torque T that can be applied to the shaft. (b) Using the result of part (a), determine the strain that will be shown by the strain gage that is mounted on the surface at an angle of 35° to the axis of the shaft. Use $E = 200$ GPa, $G = 80$ GPa, and $\nu = 0.25$.

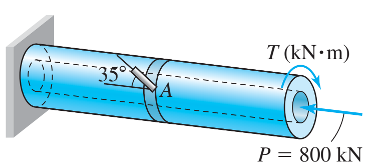


Figure P10.43

10.44 On the C clamp shown in Figure P10.44*a* determine the maximum clamping force *P* if the allowable normal stress is 160 MPa in tension and 120 MPa in compression.

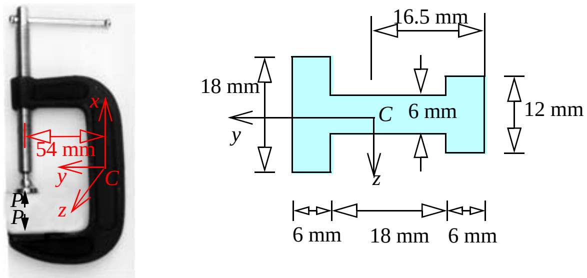


Figure P10.44

10.45 The T cross section of the beam was constructed by gluing two rectangular pieces together. A small crack was detected in the glue joint at section *AA*. Determine the maximum value of the applied load *P* if the normal stress in the glue at section *AA* is to be limited to 20 MPa in tension and 12 MPa in shear. The load *P* acts at the centroid of the cross section at *C*, as shown in Figure P10.45.

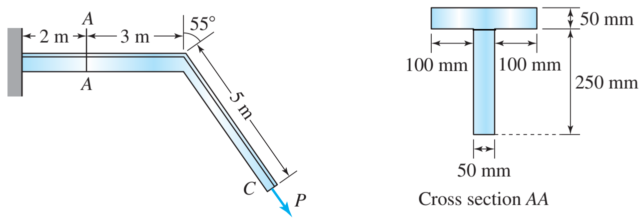


Figure P10.45

10.46 The bars in the pin connected structure shown in Figure P10.46 are circular bars of diameters that are available in increments of 5 mm. The allowable shear stress in the bars is 90 MPa. Determine the diameters of the bars for designing the lightest structure to support a force of $P = 40$ kN.

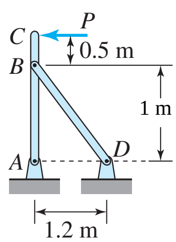


Figure P10.46

10.47 Member *AB* has a circular cross section with a diameter of 0.75 in. as shown in Figure P10.47 Member *BC* has a square cross section of 2 in. × 2 in. Determine the maximum normal stress in members *AB* and *BC*.

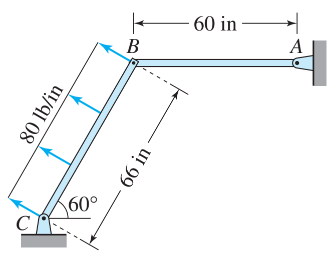


Figure P10.47

10.48 The members of the structure shown in Figure P10.48 have rectangular cross sections and are pin connected. Cross-section dimensions for members are 100 mm × 150 mm for *ABC*, 100 mm × 200 mm for *CDE*, and 100 mm × 50 mm for *BD*. The allowable normal stress in the members is 20 MPa. Determine the maximum intensity of the distributed load *w*.

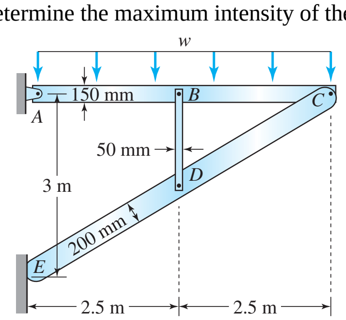


Figure P10.48

10.49 A hoist is to be designed for lifting a maximum weight $W = 300$ lb, as shown in Figure P10.49 The hoist will be installed at a certain height above ground and will be constructed using lumber and assembled using steel bolts. The lumber rectangular cross-section dimensions are listed in Table 10.2. The bolt joints will be modeled as pins in single shear. Same-size bolts will be used in all joints. The allowable normal stress in the wood is 1.2 ksi and the allowable shear stress in the bolts is 6 ksi. Design the lightest hoist by choosing the lumber from Table 10.2 and the bolt size to the nearest 1/8 in. diameter.

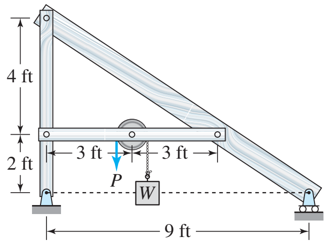


Figure P10.49

10.50 A rectangular wooden beam of 4-in. × 8-in. cross section is supported at the right end by an aluminum circular rod of $\frac{1}{2}$-in. diameter, as shown in Figure P10.50. The allowable normal stress in the wood is 1.5 ksi and the allowable shear stress in aluminum is 8 ksi. The moduli of elasticity for wood and aluminum are E_w = 1800 ksi and E_{al} = 10,000 ksi. Determine the maximum force P that the structure can support.

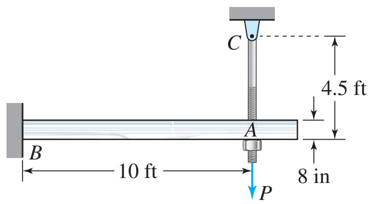


Figure P10.50

10.51 A steel pipe with an outside diameter of 1.5 in. and a wall thickness of $\frac{1}{4}$ in. is simply supported at D. A torque of T_{ext} = 30 in.·kips is applied as shown in Figure P10.51. If a = 12 in., b = 48 in., and c = 60 in., determine the normal and shear stresses at points A and B in the x, y, z coordinate system and show them on a stress cube. Points A and B are on the surface of the pipe. The modulus of elasticity is E = 30,000 ksi and Poisson's ratio is ν = 0.28.

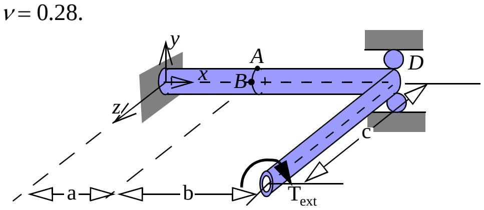


Figure P10.51

10.52 A composite beam is constructed by attaching steel strips at the top and bottom of a wooden beam, as shown in Figure P10.52 The beam is supported at the right end by an aluminum circular rod of 8-mm diameter. The allowable normal stresses in the wood and steel are 14 MPa and 140 MPa, respectively. The allowable shear stress in aluminum is 60 MPa. The moduli of elasticity for wood, steel, and aluminum are E_w = 12.6 GPa, E_s = 200 GPa, and E_{al} = 70 GPa, respectively. Determine the maximum intensity w of the distributed load that the structure can support.

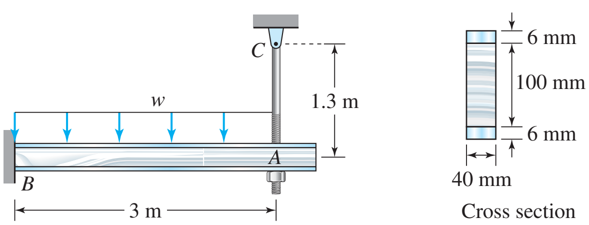


Figure P10.52

10.53 A park structure is modeled with pin joints at the points shown in Figure P10.53. Members BD and CE have cross-sectional dimensions of 6 in. × 6 in., whereas members AB, AC, and BC have cross-sectional dimensions of 2 in. × 8 in. Determine the maximum normal stress in each of the members due to the estimated snow load shown on the structure.

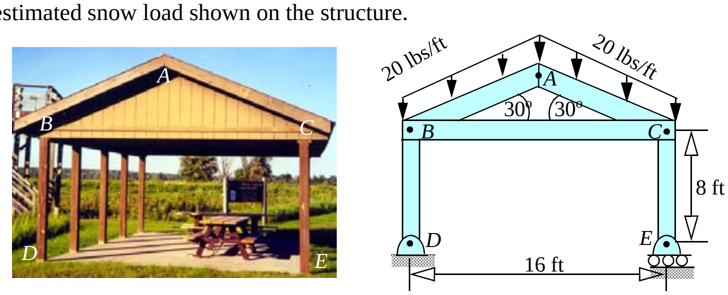


Figure P10.53

10.54 A highway sign uses a 16-in. hollow pipe as a vertical post and 12-in. hollow pipes for horizontal arms, as shown in Figure P10.54. The pipes are 1 in. thick. Assume that a uniform wind pressure of 20 lb/ft^2 acts on the sign boards and the pipes. Note that the pressure on the pipes acts on the projected area Ld, where L is the length of pipe and d is the pipe diameter. Neglecting the weight of the pipe, determine the normal and shear stresses at points A and B and show these stresses on stress cubes.

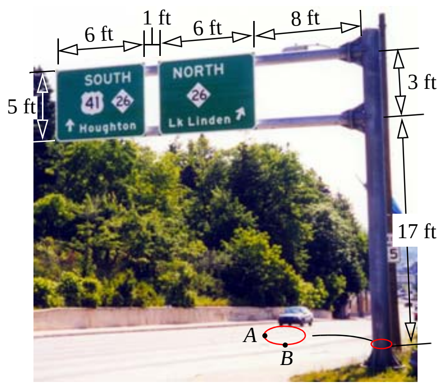


Figure P10.54

10.55 A bicycle rack is made from thin steel tubes of $\frac{1}{16}$-in. thickness and 1-in. outer diameter. The weight of the bicycles is supported by the belts from C to D and the members between C and B. Member AC carries negligible force and is neglected in the stress analysis, as shown on the model in Figure P10.55b. If the allowable normal stress in the steel tubes is 12 ksi and the allowable shear stress is 8 ksi, determine the maximum weight W to the nearest lb of each bicycle that can be put on the rack.

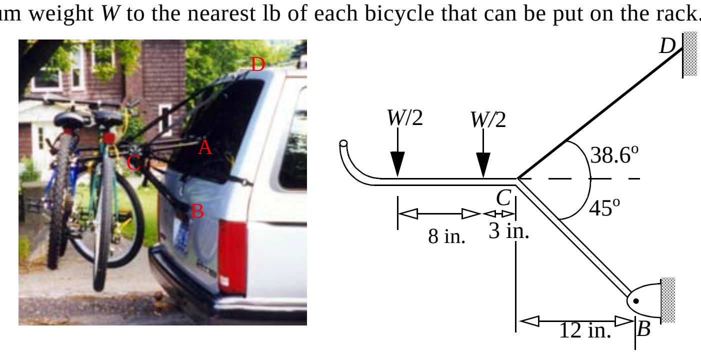


Figure P10.55

10.56 The hoist shown in Figure P10.56 was used to lift heavy loads in a mining operation. Member EF supported load only if the load being lifted was asymmetric with respect to the pulley; otherwise it carried no load and can be neglected in the stress analysis. If the allowable normal stress in steel is 18 ksi, determine the maximum load W that could be lifted using the hoist.[1]

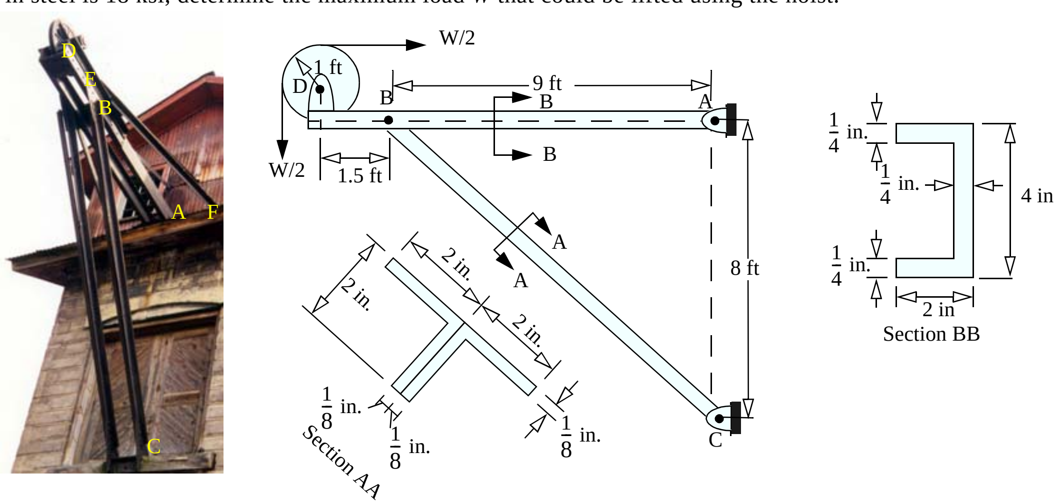


Figure P10.56

Failure envelopes

10.57 A solid shaft of 50-mm diameter is made from a brittle material that has an allowable tensile stress of 100 MPa, as shown in Figure P10.57. Draw a failure envelope representing the maximum permissible positive values of T and P.

[1]Though the load on section BB is not passing through the plane of symmetry, the theory of symmetric bending can still be used because of the structure symmetry.

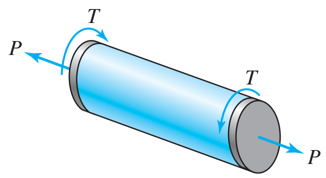


Figure P10.57

10.58 The shaft shown in Figure P10.57 is made from a ductile material and has an allowable shear stress of 75 MPa. Draw a failure envelope representing the maximum permissible positive values of *T* and *P*.

10.59 The shaft in Problem 10.57 is 1.5 m long and has a modulus of elasticity $E = 200$ GPa and a modulus of rigidity $G = 80$ GPa. Modify the failure envelope of Problem 10.57 to incorporate the limitation that the elongation cannot exceed 0.5 mm and the relative rotation of the right end with respect to the left end cannot exceed 3°.

10.60 A pipe with an outside diameter of 40 mm and a wall thickness of 10 mm is loaded as shown in Figure P10.60. At section *AA* the allowable shear stress is 60 MPa. Draw the failure envelope for the applied loads P_1 and P_2. Use a = 2.5 m, b = 0.4 m, c =0.1 m.

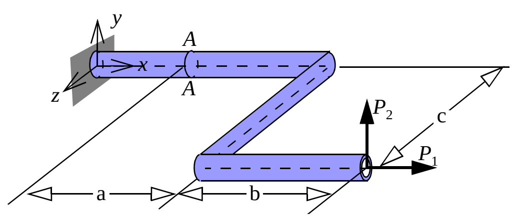


Figure P10.60

10.61 A bent pipe of 2-in. outside diameter and a wall thickness of $\frac{1}{4}$-in. is loaded as shown in Figure P10.61. The maximum shear stress the pipe material can support is 24 ksi. Draw the failure envelope for the applied loads P_1 and P_2. Use a = 16 in., b = 16 in., and c = 10 in.

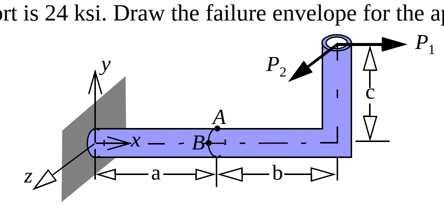


Figure P10.61

Computer problems

10.62 A hollow aluminum shaft of 5-ft length is to carry a torque of 200 in.·kips and an axial force of 100 kips. The inner radius of the shaft is 1 in. If the allowable shear stress in the shaft is 10 ksi, determine the outer radius of the lightest shaft.

10.63 The hollow cylinder shown in Figure P10.63 is fabricated from a sheet metal of 15-mm thickness. Determine the minimum outer radius to the nearest millimeter if the allowable normal stress is 150 MPa in tension or compression.

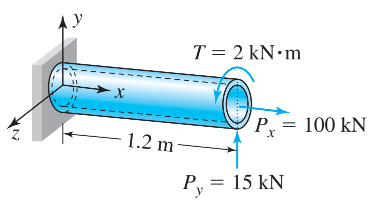


Figure P10.63

10.64 Table P10.64 shows the measured radii of the solid tapered member shown in Figure P10.64 at several points along the axis of the shaft. The member is subjected to a torque $T = 30$ kN · m and an axial force $P = 100$ kN. Plot the maximum normal and shear stresses as a function of *x*.

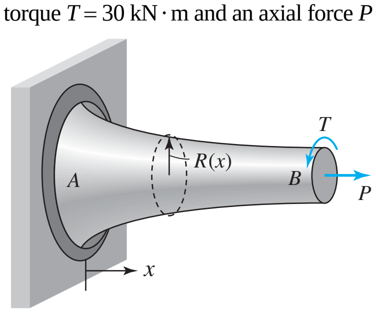


Figure P10.64

TABLE P10.64

x (m)	R(x) (mm)
0.0	100.6
0.1	92.7
0.2	82.6
0.3	79.6
0.4	75.9
0.5	68.8
0.6	68.0
0.7	65.9

TABLE P10.64

x (m)	R(x) (mm)
0.8	60.1
0.9	60.3
1.0	59.1
1.1	54.0
1.2	54.8
1.3	54.1
1.4	49.4
1.5	50.6

MoM in Action: Biomimetics

Nature produces the simplest and most efficient design by eliminating waste and inefficiency through the process of natural selection. This is the story about engineering design mimicking nature—that is, *Biomimetics*.

Tensegrity and adaptive, or smart, structures are just two of the latest in a long engineering history of mimicking nature. The flight of birds inspired the design of planes, while fish have inspired the sleek design of ship hulls. For a display of the technology of the Industrial Revolution, Joseph Paxton turned in 1851 to the structure of a lily pad. His wrought iron and glass building, the Crystal Palace, started an architectural trend. In Switzerland, George de Mestral invented Velcro in 1946 after observing the loops of seed-bearing burr clinging to his pants.

Tensegrity is the concatenation of tension and integrity. Tensegrity structures stabilize their shapes by continuous tension, like the camping tent in Figure 10.42*a*. Contrast these with stone arches, which achieve stability by continuous compression. Our own body is a tensegrity structure, with muscles supporting continual tension and bones in compression. Buckminster Fuller designed the first engineering tensegrity structure (Figure 10.42*b*) for the Expo 67 in Montreal, Canada. Such geodesic domes are structurally so efficient and stable that theoretically one could enclose New York City. Cells and the arrangement of carbon molecules called buckyballs in Fuller's honor are nature's tensegrity structures at the molecular level. They, in turn, are being emulated in carbon nano-tubes.

(a)

(b)

Figure 10.42 Tensegrity structures: (a) a tent; (b) Montreal bio-sphere (Courtesy Mr. Philipp Hienstorfer).

Cracked bones heal, which means that the body senses a crack and then sends the material needed to seal the crack. To emulate this in metallic and masonry structures, three elements are needed: a *sensor* to detect a crack; a *controller* to decide if the crack is a threat; and a *smart material* that could be activated by the controller to seal the crack. Sensors could be electrical (like strain gages), acoustic (ultrasound), piezoelectric (producing current when pressured), or fiber optics. The controller on a computer chip is a central processing unit with decision-making algorithms. Finally, smart materials are those whose properties can be significantly altered in a controlled fashion by external stimuli—such as changes in temperature, moisture, pH, stress, or electric and magnetic fields.

With these three elements, *adaptive* or *smart structures* can adapt to the environment. Adaptive crack sealing would have applications to aircraft, bridges, buildings, and medical implants. Buildings that adapt to earthquake motion, aircraft wings that change shape during flight, pumps that dispense insulin to people with diabetes—all are possibilities on the research frontier of smart structures.

The healing of bones is only one example of the adaptive nature of our bodies. Bones and muscles get stronger in response to stress, a response that is still to be understood and mimicked. The orthotropic nature of bones also have lessons for the design of composite materials. Biomimetics is the formal acknowledgement that nature is smart and we would be smart to mimic it.

10.3 FAILURE THEORIES

In principle, the maximum strength of a material is its atomic strength. In bulk materials, however, the distribution of flaws such as impurities, microholes, or microcracks creates a local stress concentration. As a result the bulk strength of a material is orders of magnitude lower than its atomic strength. Failure theories assume a homogeneous material, so that effects of flaws have been averaged[2] in some manner. With this assumption we can speak of average material strength values, which are adequate for most engineering design and analysis.

For a homogeneous, isotropic material, the characteristic failure stress is either the yield stress or the ultimate stress, usually obtained from the uniaxial tensile test (Section 3.1.1). However, in the uniaxial tension test there is only one nonzero stress component. How do we relate this the stress components in two- and three-dimensions? Attempt to answer this question are called *failure theories* although no one answer is applicable to all materials. A **failure theory** relates the stress components to the characteristic value of material failure.

TABLE 10.3 **Synopsis of failure theories**

	Ductile Material	Brittle Material
Characteristic failure stress	Yield stress	Ultimate stress
Theories	1. Maximum shear stress	1. Maximum normal stress
	2. Maximum octahedral shear stress	2. Coulomb–Mohr

We shall consider the four theories listed in Table 10.3. The maximum shear stress theory and the maximum octahedral shear stress theory are generally used for ductile materials, in which failure is characterized by yield stress. The maximum normal stress theory and Mohr's theory are generally used for brittle material, in which failure is characterized by ultimate stress.

10.3.1 Maximum Shear Stress Theory

Maximum shear stress theory predicts that the maximum shear stress alone accounts for failure:

A material will fail when the maximum shear stress exceeds the shear stress at the yield point obtained from a uniaxial tensile test.

The theory gives reasonable results for ductile materials. Figure 10.43 shows that the maximum shear stress at yield in a tension test is half that of the normal yield stress. We obtain the following the failure criterion:

$$\boxed{\tau_{max} \le \frac{\sigma_{yield}}{2}} \tag{10.7}$$

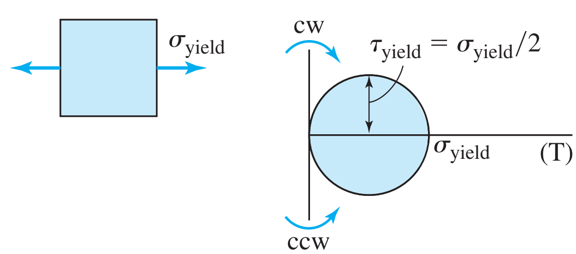


Figure 10.43 Shear stress at yield in tension test.

Equation (10.7) is also referred to as *Tresca's yield criterion*. The maximum shear stress at a point is given by Equation (8.13). If we substitute Equation (8.13) into Equation (10.7), we obtain

$$\boxed{\left|\max(\sigma_1 - \sigma_2,\ \sigma_2 - \sigma_3,\ \sigma_3 - \sigma_1)\right| \le \sigma_{yield}} \tag{10.8}$$

If we plot each principal stress on an axis, then Equation (10.8) gives us the failure envelope. For plane stress problems the failure envelope is shown in Figure 10.45 seen later.

[2]Micromechanics tries to account for the some of the flaws and nonhomogeneity in predicting the strength of a material, but extrapolating to macro levels requires some form of averaging.

10.3.2 Maximum Octahedral Shear Stress Theory

Figure P10.44 shows eight planes that make equal angles with the principal planes. These planes are called *octahedral planes* (from *octal*, meaning eight). The stress values on these planes are called *octahedral stresses*. The normal octahedral stress (σ_{oct}) and the octahedral shear stress (τ_{oct}) are given by Equations (8.16) and (8.17), written again here for convenience:

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{10.9}$$

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \tag{10.10}$$

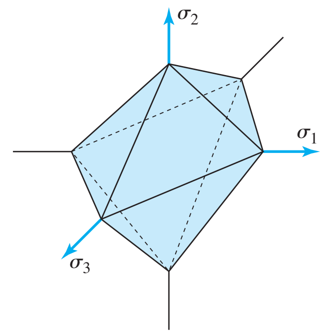


Figure 10.44 Octahedral Planes

The maximum octahedral shear stress theory for ductile materials states

A material will fail when the maximum octahedral shear stress exceeds the octahedral shear stress at the yield point obtained from a uniaxial tensile test.

Mathematically the failure criterion is

$$\tau_{oct} \le \overline{\tau_{yield}} \tag{10.11}$$

where $\overline{\tau_{yield}}$ is the octahedral shear stress at yield point in a uniaxial tensile test. Substituting $\sigma_1 = \sigma_{yield}$, $\sigma_2 = 0$, and $\sigma_3 = 0$ (the stresses at yield point in a uniaxial tension test) into the expression of octahedral shear stress, we obtain $\overline{\tau_{yield}} = \sqrt{2}\sigma_{yield}/3$. Substituting this and Equation (10.10) into Equation (10.11), we obtain

$$\frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \le \sigma_{yield} \tag{10.12}$$

The left-hand side of Equation (10.12) is referred to as *von Mises stress*. Because the von Mises stress σ_{von} is used extensively in the design of structures and machines, we formally define it as follows:

$$\boxed{\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}} \tag{10.13}$$

The failure criterion represented by Equation (10.12) is sometimes referred to as the *von Mises yield criterion* and is stated as follows:

$$\boxed{\sigma_{von} \le \sigma_{yield}} \tag{10.14}$$

At a point in a fluid the principal stresses are all compressive and equal to the hydrostatic pressure (p); that is, $\sigma_1 = \sigma_2 = \sigma_3 = -p$. Substituting this into Equation (10.9), we obtain $\sigma_{oct} = -p$; that is, octahedral normal stress corresponds to the hydrostatic state of stress. Thus this theory we are assumes that hydrostatic pressure has a negligible effect on the yielding of ductile material, a conclusion that is confirmed by experimental observation for very ductile materials like aluminum.

Equations (10.7) and (10.12) are failure envelopes[3] in a space in which the axes are principal stresses. For a plane stress ($\sigma_3 = 0$) problem we can represent these failure envelopes as in Figure 10.45. Notice that the maximum octahedral shear stress

[3]In drawing failure envelopes, the convention that $\sigma_1 > \sigma_2$ is ignored. If the convention were enforced, then there would be no envelope in the second quadrant, and only the envelope below a 45° line would be admissible in the third quadrant. A very strange looking envelope would result, rather than the symmetric envelope shown in Figure 10.45.

envelope encompasses the maximum shear stress envelope. Experiments show that, for most ductile materials, the maximum octahedral shear stress theory gives better results than the maximum shear stress theory. Still, the maximum shear stress theory is simpler to use.

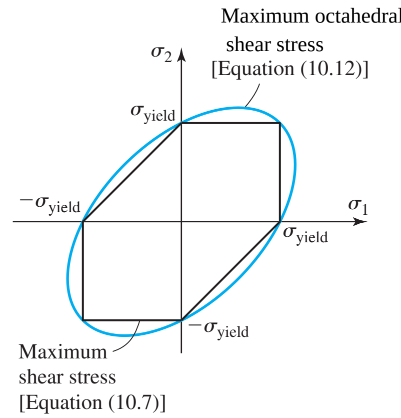


Figure 10.45 Failure envelopes for ductile materials in plane stress.

10.3.3 Maximum Normal Stress Theory

Maximum normal stress theory predicts that the maximum normal stress alone accounts for failure:

A material will fail when the maximum normal stress at a point exceeds the ultimate normal stress obtained from a uniaxial tension test.

The theory gives good results for brittle materials provided principal stress 1 is tensile, or if the tensile yield stress has the same magnitude as the yield stress in compression. Thus the failure criterion is given as

$$\boxed{|\max(\sigma_1, \sigma_2, \sigma_3)| \le \sigma_{ult}} \tag{10.15}$$

For most materials the ultimate stress in tension is usually far less than the ultimate stress in compression because microcracks tend to grow in tension and tend to close in compression. But the simplicity of the failure criterion makes the theory attractive, and it can be used if principal stress 1 is tensile and is the dominant principal stress.

10.3.4 Mohr's Failure Theory

The Mohr's failure theory predicts failure using material strength from three separate tests in which the ultimate stress in tension, compression, and shear are determined.

A material will fail if a stress state is on the envelope that is tangent to three Mohr's circles—corresponding to ultimate stress in tension, compression, and pure shear.

By experiments, we can determine separately the ultimate stress in tension σ_T, the ultimate stress in compression σ_C, and the ultimate shear stress in pure shear τ_U. Figure 10.46*a* shows the stress cubes and the corresponding Mohr's circle for three stress states. We then draw an envelope tangent to the three circles to represent the failure envelope. If Mohr's circle corresponding to a stress state just touches the envelope at any point, then the material is at incipient failure. If any part of Mohr's circle for a stress state is outside the envelope, then the material has failed at that point.

We can also plot the failure envelopes of Figure 10.46*a* using principal stresses as the coordinate axes. In plane stress this envelope is represented by the solid line in Figure 10.46*b*. For most brittle materials the pure shear test is often ignored. In such a case the tangent line to the circles of uniaxial compression and tension would be a straight line in Figure 10.46*a*. The resulting simplification for plane stress is shown as dotted lines in Figure 10.46*b* and is called *modified Mohr's theory*.

Figure 10.46*b* emphasizes the following:

1. If both principal stresses are tensile, then the maximum normal stress has to be less than the ultimate tensile strength.
2. If both principal stresses are negative, then the maximum normal stress must be less than the ultimate compressive strength.

3. If the principal stresses are of different signs, then for the modified Mohr's theory the failure is governed by

$$\boxed{\left|\frac{\sigma_2}{\sigma_C} - \frac{\sigma_1}{\sigma_T}\right| \le 1} \tag{10.16}$$

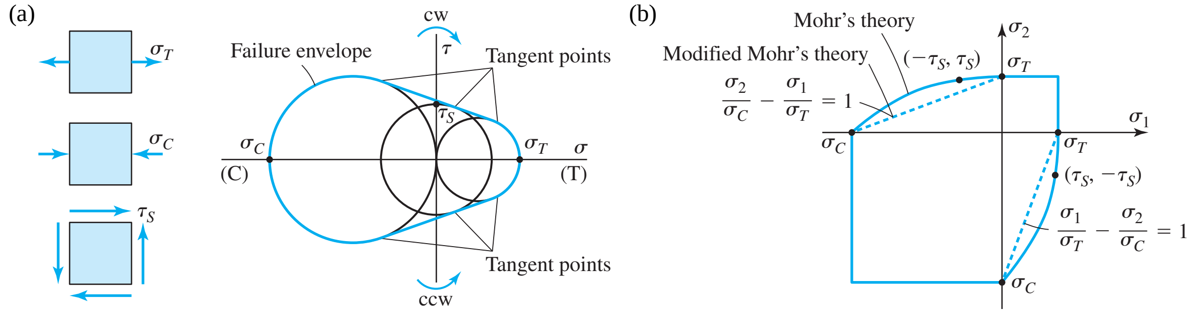


Figure 10.46 Failure envelopes for Mohr's failure theory in (a) normal and shear stress space; (b) principal stress space.

EXAMPLE 10.9

At a critical point on a machine part made of steel, the stress components were found to be σ_{xx} = 100 MPa (T), σ_{yy} = 50 MPa (C), and τ_{xy} = 30 MPa. Assuming that the point is in plane stress and the yield stress in tension is 220 MPa, determine the factor of safety using (a) the maximum shear stress theory; (b) the maximum octahedral shear stress theory.

PLAN

We can find the principal stresses and maximum shear stress by Mohr's circle or by the method of equations. (a)From Equation (10.7) we know that failure stress for the maximum shear stress theory is half the yield stress in tension. Using Equation (3.10) we can find the factor of safety. (b)We can find the von Mises stress from Equation (10.13), which gives us the denominator in Equation (3.10), and noting that the numerator of Equation (3.10) is the yield stress in tension, we obtain the factor of safety.

SOLUTION

For plane stress: $\sigma_3 = 0$

Mohr's circle method: We draw the stress cube, record the coordinates of planes *V* and *H*, draw Mohr's circle as shown in Figure 10.47*a*. The principal stresses and maximum shear stress are

$$R = \sqrt{(75\ \text{MPa})^2 + (30\ \text{MPa})^2} = 80.8\ \text{MPa} \tag{E1}$$

$$\sigma_1 = OC + OP_1 = 25\ \text{MPa} + 80.8\ \text{MPa} = 105.8\ \text{MPa} \qquad \sigma_2 = OC - OP_1 = 25\ \text{MPa} - 80.8\ \text{MPa} = -55.8\ \text{MPa} \tag{E2}$$

$$\tau_{\max} = R = 80.8\ \text{MPa} \tag{E3}$$

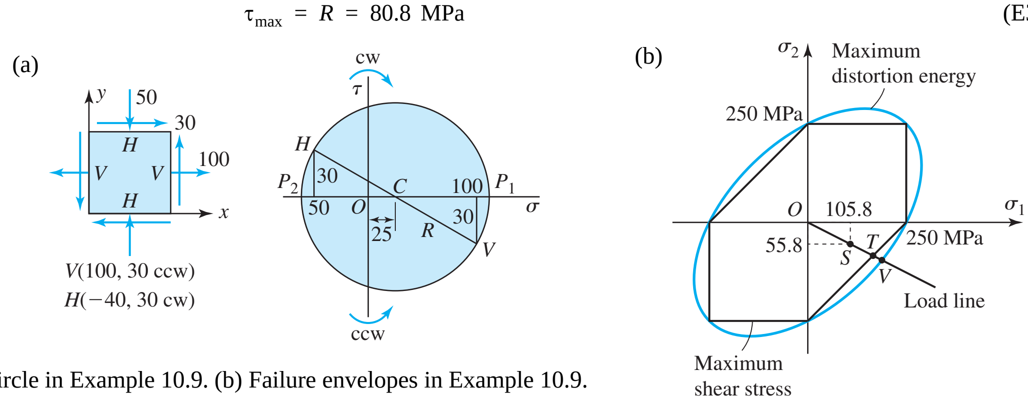


Figure 10.47 (a) Mohr's circle in Example 10.9. (b) Failure envelopes in Example 10.9.

Method of equations: From Equations (8.7) and (8.13) we can obtain the principal stresses and the maximum shear stress,

$$\sigma_{1,2} = \frac{100\ \text{MPa} - 50\ \text{MPa}}{2} \pm \sqrt{\left(\frac{100\ \text{MPa} + 50\ \text{MPa}}{2}\right)^2 + (30\ \text{MPa})^2} = 25\ \text{MPa} \pm 80.8\ \text{MPa} \tag{E4}$$

$$\sigma_1 = 105.8\ \text{MPa} \qquad \sigma_2 = -55.8\ \text{MPa} \qquad \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = 80.8\ \text{MPa} \tag{E5}$$

(a) The failure shear stress is half the yield stress in tension, that is, 110 MPa. Following Equation (3.10), we divide this value by the maximum shear stress [Equation (E3) or (E5)] to obtain the factor of safety $K_\tau = (110\ \text{MPa})/(80.8\ \text{MPa})$.

ANS. $K_\tau = 1.36$

(b) The von-Mises stress can be found from Equation (10.13),

$$\sigma_{von} = \frac{1}{\sqrt{2}}\sqrt{[105.8\ \text{MPa} - (-55.8\ \text{MPa})]^2 + (-55.8\ \text{MPa})^2 + (105.8\ \text{MPa})^2} = 142.2\ \text{MPa} \quad \text{(E6)}$$

The factor of safety is failure stress is 220 MPa divided by the von Mises stress, or $K_\sigma = (220\ \text{MPa})/(142.2\ \text{MPa})$

ANS. $K_\sigma = 1.55$

COMMENTS

1. The failure envelopes corresponding to the yield stress of 250 MPa are shown in Figure 10.47*b*. In comment 4 of Example 10.8 it was shown that graphically the factor of safety could be found by taking ratios of distances from the origin along the load line. If we plot the coordinates $\sigma_1 = 105.8$ MPa and $\sigma_2 = -55.8$ MPa, we obtain point *S*. If we join the origin *O* to point *S* and draw the line, we get the load line for the given stress values. It may be verified by measuring (or calculating coordinates of *T* and *V*) that the following is true: $K_\tau = OS/OT = 1.36$ and $K_\sigma = OS/OV = 1.55$.
2. Because the failure envelope for the maximum shear stress criterion is always inscribed inside the failure envelope of maximum octahedral shear stress criterion, the factor of safety based on the maximum octahedral shear stress will always be greater than the factor of safety based on maximum shear stress.

EXAMPLE 10.10

The stresses at a point on a free surface due to a load *P* were found to be $\sigma_{xx} = 3P$ ksi (C), $\sigma_{yy} = 5P$ ksi (T), and $\tau_{xy} = -2P$ ksi, where *P* is measured in kips. The brittle material has a tensile strength of 18 ksi and a compressive strength of 36 ksi. Determine the maximum value of load *P* that can be applied on the structure using the modified Mohr's theory.

PLAN

We can determine the principal stresses in terms of *P* by Mohr's circle or by the method of equations. As the given normal stresses are of opposite signs, we can expect that the principal stresses will have opposite signs. Using Equation (10.16) we can determine the maximum value of *P*.

SOLUTION

For plane stress: $\sigma_3 = 0$

Mohr's circle method: We draw the stress cube, record the coordinates of planes *V* and *H*, draw Mohr's circle as shown in Figure 10.48. The principal stresses and the maximum shear stress are

$$R = \sqrt{(4P\ \text{ksi})^2 + (2P\ \text{ksi})^2} = 4.47P\ \text{ksi} \quad \text{(E1)}$$

$$\sigma_1 = OC + OP_1 = P\ \text{ksi} + 4.47P\ \text{ksi} = 5.57P\ \text{ksi} \qquad \sigma_2 = OC - OP_1 = P\ \text{ksi} - 4.47P\ \text{ksi} = -3.37P\ \text{ksi} \quad \text{(E2)}$$

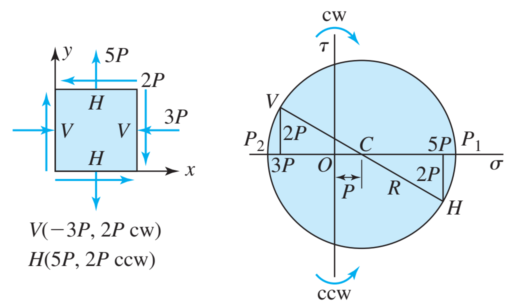


Figure 10.48 Calculation using Mohr's circle in Example 10.10.

Method of equations: From Equation (8.7) we can obtain the principal stresses as

$$\sigma_{1,2} = \frac{-3P\ \text{ksi} + 5P\ \text{ksi}}{2} \pm \sqrt{\left(\frac{-3P\ \text{ksi} - 5P\ \text{ksi}}{2}\right)^2 + (2P\ \text{ksi})^2} = P\ \text{ksi} \pm 4.47P\ \text{ksi} \quad \text{(E3)}$$

$$\sigma_1 = 5.57P\ \text{ksi} \qquad \sigma_2 = -3.37P\ \text{ksi} \quad \text{(E4)}$$

Substituting the principal stresses into Equation (10.16) and noting that $\sigma_T = 18$ ksi and $\sigma_C = 36$ ksi, we can obtain the maximum value of *P*,

$$\left|\frac{(-3.37P\ \text{ksi})}{(36\ \text{ksi})} - \frac{(5.57P\ \text{ksi})}{(18\ \text{ksi})}\right| \le 1 \qquad \text{or} \qquad 0.4031P \le 1 \qquad \text{or} \qquad P \le 2.481 \quad \text{(E5)}$$

ANS. $P_{\max} = 2.48 \text{ kips}$

COMMENT

1. We could not have used the maximum normal stress theory for this material, since the tensile and compressive strengths are significantly different.Here the compressive strength is the dominant strength, and not the tensile-strength.

PROBLEM SET 10.3

Failure theories

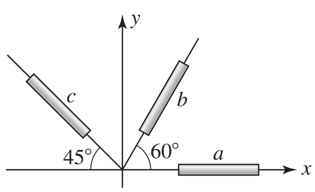
10.65 For a force P measured in kN, the stress components at a critical point that is in plane stress were found to be $\sigma_{xx} = 10P$ MPa (T), $\sigma_{yy} = 20P$ MPa (C), and $\tau_{xy} = 5P$ MPa. The material has a yield stress of 160 MPa as determined in a tension test. If yielding must be avoided, predict the maximum v force P using (a) maximum shear stress theory; (b) maximum octahedral shear stress theory.

10.66 For a force P, the stress components at a critical point that is in plane stress were found to be $\sigma_{xx} = 4P$ ksi (C), $\sigma_{yy} = 3P$ ksi (T), and $\tau_{xy} = -5P$ ksi. The material has a tensile rupture strength of 18 ksi and a compressive rupture strength of 32 ksi. Determine the maximum force P using the modified Mohr's theory.

10.67 A material has a tensile rupture strength of 18 ksi and a compressive rupture strength of 32 ksi. During usage a component made from this plastic showed the following stresses on a free surface at a critical point: $\sigma_{xx} = 9$ ksi (T), $\sigma_{yy} = 6$ ksi (T), and $\tau_{xy} = -4$ ksi. Determine the factor of safety using the modified Mohr's theory.

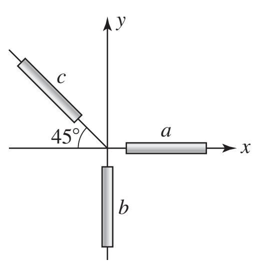
10.68 On a free surface of aluminum ($E =$ 10,000 ksi, $\nu = 0.25$, $\sigma_{\text{yield}} = 24$ ksi) the strains recorded by the three strain gages shown in Figure P10.68 are $\varepsilon_a = -600$ μin./in., $\varepsilon_b = 500$ μin/in, and $\varepsilon_c = 400$ μin./in. By how much can the loads be scaled without exceeding the yield stress of aluminum at the point? Use the maximum shear stress theory.

Figure P10.68



10.69 On a free surface of steel (E = 200 GPa, $\nu = 0.28$, $\sigma_{\text{yield}} = 210$ MPa) the strains recorded by the three strain gages shown in Figure P10.69 are $\varepsilon_a = -800$ μm/m, $\varepsilon_b = -300$ μm/m, and $\varepsilon_c = -700$ μm/m. By how much can the loads be scaled without exceeding the yield stress of steel at the point? Use the maximum octahedral shear stress theory.

Figure P10.69



10.70 A thin-walled cylindrical gas vessel has a mean radius of 3 ft and a wall thickness of $\frac{1}{2}$ in. The yield stress of the material is 30 ksi. Using the von Mises failure criterion, determine the maximum pressure of the gas inside the cylinder if yielding is to be avoided.

10.71 A thin cylindrical boiler can have a minimum mean radius of 18 in. and a maximum mean radius of 36 in. The boiler will be subjected to a pressure of 750 psi. A sheet metal with a yield stress of 60 ksi is to be used with a factor of safety of 1.5. Construct a failure envelope with the mean radius R and the sheet metal thickness t as axes. Use the maximum octahedral shear stress theory.

10.72 For plane stress show that the von Mises stress of Equation (10.13) can be written as

$$\sigma_{\text{von}} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\tau_{xy}^2} \tag{10.17}$$

Stretch Yourself

10.73 In Cartesian coordinates the von Mises stress in three dimensions is given by

$$\sigma_{\text{von}} = \sqrt{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx} + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2} \quad (10.18)$$

Show that for plane strain Equation (10.18) reduces to

$$\sigma_{\text{von}} = \sqrt{(\sigma_{xx}^2 + \sigma_{yy}^2)(1 + \nu^2 - \nu) - \sigma_{xx}\sigma_{yy}(1 + 2\nu - 2\nu^2) + 3\tau_{xy}^2} \quad (10.19)$$

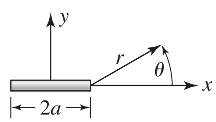
where ν is Poisson's ratio of the material.

10.74 Fracture mechanics shows that the stresses in model in the vicinity of the crack tip shown in Figure P10.74 are given by

$$\sigma_{xx} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad \sigma_{yy} = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \qquad \tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (10.20)$$

Notice that at $\theta = \pi$, that is, at the crack surface, all stresses are zero. In terms of K_I and r, obtain the von Mises stress at $\theta = 0$ and $\theta = \pi/2$, assuming plane stress.

Figure P10.74



10.4 CONCEPT CONNECTOR

In our problems thus far, we have relied on fixed values for the dimensions, material properties, and loads. Design that does not allow for variability in these parameters is called *deterministic*. Real structural members, however, are manufactured to dimensions only within a certain tolerance. Similarly, material properties may vary within a range, depending on material processing. Loads and support conditions, too, are at best an estimate, because wind pressure, snow weight, traffic loads, and other conditions are inherently variable.

Probabilistic design takes into account the variability in dimensions, material properties, and loads. Here we seek to achieve not just a given factor of safety but rather a specified *reliability*. This section offers a peek at how engineers approach probabilistic design. Section 10.4.1 discusses the concept of reliability, and Section 10.4.2 introduces a design methodology that incorporates it.

10.4.1 Reliability

Already in Section 3.3, we encountered the uncertainties regarding material properties, manufacturing processes, the control and estimate of loads, and so on. There we defined one measure of the margins of safety, the *factor of safety*. But choosing a factor of safety is always a compromise among several factors, including cost and human safety, based on experience. Such a compromise leaves an open question: *how reliable is our design?*

To understand the relationship between the factor of safety and reliability, suppose we wish to design an engine mount or other axial member with a factor of safety of 1.3. If the material strength of the axial member is 130 MPa, we have an allowable stress of 100 MPa.

The actual axial stress in the member, however, may be quite different, because of such factors as manufacturing tolerances, the variability of applied loads, temperature, and humidity. If we measured the axial stress in different members, we would get a range of values. We might ask instead, then, the frequency of occurrence of a given stress level. A plot of the number of members at that level would yield a distribution, perhaps like the left curve in Figure 10.49. Similarly, the material strength—that is, the failure stress for different batches of material—may vary due to impurities, material processing, and so on. The right curve in Figure 10.49 shows one possible distribution of material strengths.

In Figure 10.49, the mean axial stress in the members is 100 MPa, and the mean strength of all materials is 130 MPa. Naturally some axial members with stresses greater than 100 MPa will be made from materials that have failure stresses less than

130 MPa. Hence, those axial members are likely to fail. In the graph, these members occupy the region common to both distributions, labeled "possible failure." If we know the two distributions, we can always determine this possible failure region.

Statistical distributions are usually described by two parameters, their *mean value* and s*tandard deviation*. If the predicted reliability developed using these parameters is unacceptable, then the design can use a different factor of safety to obtain the desired reliability.

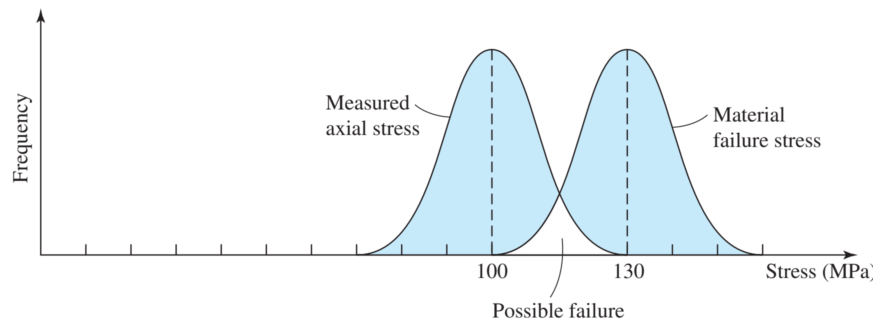


Figure 10.49 Load and resistance distribution curves.

10.4.2 Load and Resistance Factor Design (LRFD)

Load and resistance factor design (LRFD) allows civil engineers to design steel structures to a specified reliability. LRFD incorporates ideas from two other design methodologies, *allowable stress design* (ASD) and *plastic design* (PD).

ASD is based on elastic analysis, in which the factors of safety can vary with the primary function of the structural member. For example, a factor of safety of 1.5 is used for beams and 1.67 for tension members. These factors of safety are specified in building codes, usually based on statistical analysis.

Building codes also specify the kinds of load on a structure, and the methodologies that we are considering takes all these into account: the *dead load* (D) due to the weight of structural elements and other permanent features; *live load* (L) from people, equipment, and other movable objects during occupancy; *snow load* (S) and *rain or ice loads* (R) that appear on the roof of a structure; *roof load* L_r from cranes, air conditioners, and other movable objects during construction, maintenance, and occupancy; *wind load* W; and *earthquake load* E. In ASD, the sum total of the stresses from the various loads must be less than or equal to the allowable stress.

PD uses a single load factor for the design load on the structure. This factor varies with a combination of loads at hand. If only dead and live loads are considered, for example, then the load factor is 1.7 [written as 1.7 $(D + L)$]. If, however, the dead, live, and wind loads are considered, then the load factor is 1.3 [written as 1.3 $(D + L + W)$]. A *nonlinear analysis* can then determine the strength of the member at structure collapse. By nonlinear analysis, we mean that the stress values of many members fall in the plastic region—between the yield stress and ultimate stress. The member strengths must equal or exceed the required strengths calculated using factored loads.

The LRFD method overcomes shortcomings in both these methods. From the standpoint of consistent reliability in design, neither of the two methods is very accurate. In ASD the factor of safety is used to account for all variability in loads and material strength. In PD the variability in material strength is ignored. Furthermore, all loads do not have the same degree of variability.

TABLE 10.4 **Some resistance factors**

Tension members, failure due to yielding	0.90
Tension member, failure due to rupture	0.75
Axial compression	0.85
Beams	0.90
High-strength bolts, failure in tension	0.75

TABLE 10.5 **Load factors and load combinations**

$1.4D$
$1.2D + 1.6L + 0.5\ (L_r \text{ or } S \text{ or } R)$
$1.2D + 1.6\ (L_r \text{ or } S \text{ or } R) + (0.5L \text{ or } 0.8W)$
$1.2D + 1.3W + 0.5 + 0.5\ (L_r \text{ or } S \text{ or } R)$
$12D \pm 1.0E + 0.5L + 0.2S$
$0.9D \pm (1.3W \text{ or } 1.0E)$

In LRFD, the nominal failure strength of a member is multiplied by the appropriate resistance factor (from Table 10.4) to obtain the design strength. (The words *strength* and *resistance* for a material are often used interchangeably in LRFD.

Recall the historical evolution of the concept of strength from Section 1.6.) This accounts for variability in material strength and inaccuracies in dimensions and modeling.

The load factors in Table 10.7 account for the variability of the individual load components. It also takes into account the probability of combinations of loads acting together, such as live and snow loads. Using Table 10.5, factored loads are determined for a specific load combination. These factored loads are applied to the structure, and the member strength is calculated. This computed member strength must be less than or equal to the design strength computed using the resistance factor. Since variations of load and member strength are taken into account separately, LRFD gives a more consistent level of reliability.

10.5 CHAPTER CONNECTOR

This chapter synthesized and applied the concepts of all previous chapters. The use of subscripts and formulas to determine stress results in a systematic but slower approach to problem solving. Determining the stress directions by inspection can reduce the algebra significantly, but it requires care, depending on the problem being solved. For a given problem, it is important to find your own mix of these approaches. Whatever your preference, however, the importance of a systematic approach to the problem cannot be overstated. In the design and analysis of complex structures, without a systematic approach the chances of error rise dramatically.

So far we have based designs on material strength and structure stiffness. Instability, however, can cause a structure to fail at stresses far lower than the material strength. What is structure instability, and how can we incorporate it? The next chapter considers structure instability in the design of columns.

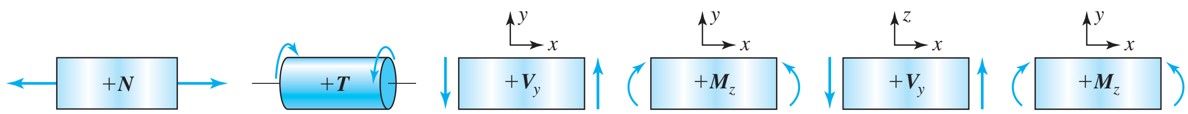
POINTS AND FORMULAS TO REMEMBER

- Superposition of stresses is addition or subtraction of stress components in the same direction acting on the same surface at a point.

$$\sigma_{xx} = \frac{N}{A} \quad (10.1) \qquad \tau_{x\theta} = \frac{T\rho}{J} \quad (10.2) \qquad \sigma_{xx} = -\frac{M_z y}{I_{zz}} \quad (10.3a) \qquad \tau_{xs} = -\frac{V_y Q_z}{I_{zz} t} \quad (10.3b)$$

$$\sigma_{xx} = -\frac{M_y z}{I_{yy}} \quad (10.4a) \qquad \tau_{xs} = -\frac{V_z Q_y}{I_{yy} t} \quad (10.4b)$$

- Sign convention for internal forces and moments:



- The internal forces and moments on a free-body diagram must be drawn according to the sign conventions if subscripts are to be used to determine the direction of stress components.
- The direction of the stress components must be determined by inspection if internal forces and moments are drawn on the free-body diagram to equilibrate the external forces and moments.
- A local x, y, z coordinate system can be established such that the x direction is along the axis of the long structural member.
- Stress components should be drawn on a stress cube and interpreted in the x, y, z coordinate system for use in the stress and strain transformation equations.
- Normal stresses perpendicular to the axis of the member are zero: $\sigma_{yy} = 0$, $\sigma_{zz} = 0$, and $\tau_{yz} = 0$.
- Normal strains perpendicular to the axis of the member can be obtained by multiplying the normal strains in the axis direction by Poisson's ratio.
- Superpose stresses, then use the generalized Hooke's law to obtain strains in combined loading:

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \qquad \varepsilon_{yy} = -\frac{\nu\sigma_{xx}}{E} \qquad \varepsilon_{zz} = -\frac{\nu\sigma_{xx}}{E} \qquad \gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{xz} = \frac{\tau_{xz}}{G} \qquad \gamma_{yz} = 0$$

- The allowable normal and shear stresses refer to the principal stresses and absolute maximum shear stress at a point, respectively.
- An individualized procedure that is a mix of subscripts, formulas, and inspection should be developed for analysis of stresses under combined loading.
- There are two major steps in the analysis and design of structures: (i) analysis of internal forces and moments that act on individual members; (ii) computation of stresses on members under combined loading.

CHAPTER ELEVEN
STABILITY OF COLUMNS

Learning objectives

1. Develop an appreciation of the phenomenon of buckling and the various types of structure instabilities.
2. Understand the use of buckling formulas in the analysis and design of structures.

Strange as it sounds, the column behind the steering wheel in Figure 11.1*a* is designed to fail: it is meant to *buckle* during a car crash, to prevent impaling the driver. In contrast, the columns of the building in Figure 11.1*b* are designed so that they do *not* buckle under the weight of a building.

Buckling is instability of columns under compression. Any axial members that support compressive axial loads, such as the weight of the building in Figure 11.1*b*, are called *columns*—and not all structural members behave the same. If a compressive axial force is applied to a long, thin wooden strip, then it bends significantly, as shown in Figure 11.1*c*. If the columns of a building were to bend the same way, the building itself would collapse. And when a column buckles, the collapse is usually sudden and catastrophic.

Under what conditions will a compressive axial force produce only axial contraction, and when does it produce bending? When is the bending caused by axial loads catastrophic? How do we design to prevent catastrophic failure from axial loads? As we shall see in this chapter, we can identify members that are likely to collapse by studying structure's equilibrium. Geometry, materials, boundary conditions, and imperfections all affect the stability of columns.

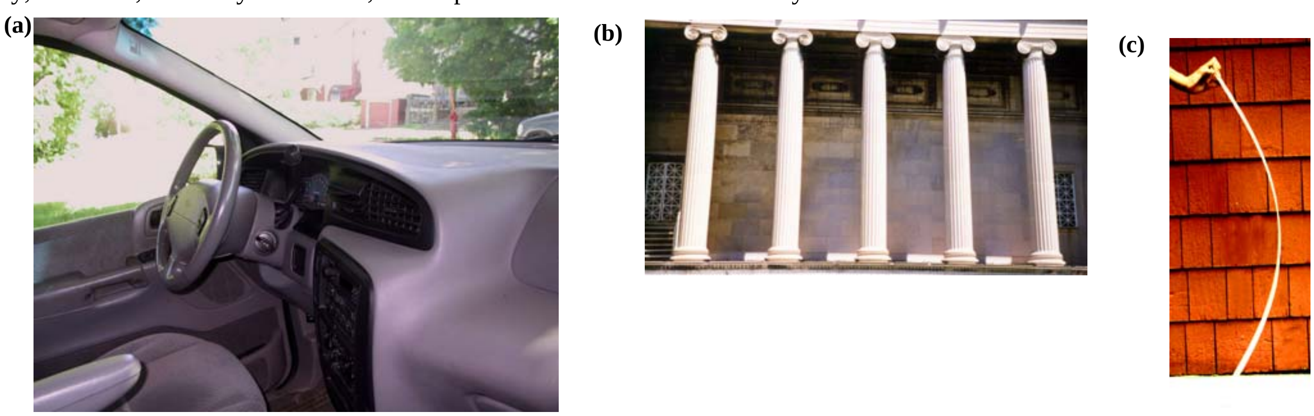


Figure 11.1 Examples of columns.

11.1 BUCKLING PHENOMENON

Buckling is an instability of equilibrium in structures that occurs from compressive loads or stresses. A structure or its components may fail due to buckling at loads that are far smaller than those that produce material strength failure. Very often buckling is a catastrophic failure. We discuss briefly some of the approaches and types of buckling in the following sections.

11.1.1 Energy Approach

We look at the energy approach using an analogy of a marble that is in equilibrium on different types of surfaces as shown in Figure 11.2. Left to itself, it will simply stay put. Suppose, however, that we disturb the marble to the shaded position in each case. When the surface is concave, as in Figure 11.2*a*, the marble will return to its equilibrium position — and it is said to be in *stable equilibrium*. When the surface is flat, as in Figure 11.2*b*, the marble will acquire a new equilibrium position. In this case the marble is said to be in a *neutral equilibrium*. Last when the surface is convex, as in Figure 11.2*c*, the marble will roll off. In this third case, a change in position also disturbs the equilibrium state and so the marble is said to be in *unstable equilibrium*.

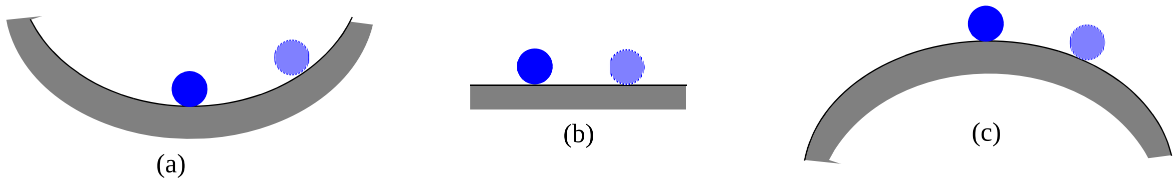


Figure 11.2 Equilibrium using marble (*a*) Stable. (*b*) Neutral. (*c*) Unstable.

The marble analogy in Figure 11.2 is useful in understanding one approach to the buckling problem, the *energy* method. Every deformed structure has a potential energy associated with it. This potential energy depends on the strain energy (the energy due to deformation) and on the work done by the external load. If the potential energy function is concave at the equilibrium position, then the structure is in stable equilibrium. If the potential energy function is convex, then the structure is in unstable equilibrium. The external load at which the potential energy function changes from concave to convex is called the *critical load* at which the buckling occurs. This energy method approach is beyond the scope of this book.

11.1.2 Eigenvalue Approach

To elaborate the eigenvalue approach in determining the load at which buckling occurs consider a rigid bar (Figure 11.3*a*) with a torsional spring at one end and a compressive axial load at the other end. Figure 11.3*b* shows the free-body diagram of the bar. Clearly, $\theta = 0$ is an equilibrium position. We call it a trivial solution to the problem. But at what value of P does there exist a nontrivial solution to the problem? This is the classical statement of an *eigenvalue problem*, and the *critical value* of P for which the nontrivial solution exists is called the *eigenvalue*. At this critical value of P the rod acquires a new equilibrium.

To determine the critical value of P, we consider the equilibrium of the moment at O in Figure 11.3*b*.

$$PL\sin\theta = K_\theta\theta \tag{11.1a}$$

For small angles we can approximate $\sin\theta \approx \theta$ and rewrite Equation (11.1a) as

$$(PL - K_\theta)\theta = 0 \tag{11.1b}$$

In Equation (11.1b) $\theta = 0$ is one solution, but if $PL = K_\theta$ then θ can have any non-zero value. Thus, the critical value of P is

$$P_{cr} = K_\theta / L \tag{11.1c}$$

You may be more familiar with eigenvalue problem in context of matrices. In problems 11.7 and 11.8 there are two unknown angles, and the problem can be cast in matrix form.

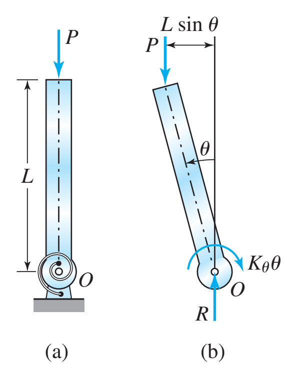


Figure 11.3 Eigenvalue problem.

11.1.3 Bifurcation Problem

To describe the bifurcation problem, we rewrite Equation (11.1a) as

$$PL/K_\theta = \theta/\sin\theta \tag{11.1d}$$

Figure 11.4 shows the plot of PL/K_θ versus θ. The equilibrium line separates the unstable region from the stable region. The bar remains in the vertical equilibrium position ($\theta = 0$) provided the load (P) increases are below point A and it will return to the vertical position if it is disturbed (rotated) slightly to the left or right. Any disturbance in equilibrium for load values above point A will send the bar to either to the left branch or to the right branch of the curve, where the it acquires a new equilibrium position. Point A is the *bifurcation point*, at which there are three possible solutions. The load P at the bifurcation point is called the *critical load*. Thus, we again see the same problem with a different perspective because of the methodology used in solving it.

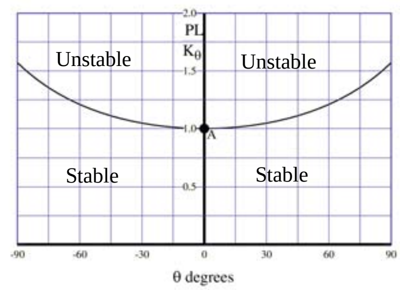


Figure 11.4 Bifurcation problem.

11.1.4 Snap Buckling

In *snap buckling* a structure jumps from one equilibrium configuration to a dramatically different equilibrium configuration. It is most often seen in shallow thin walled curved structures. To explain this phenomenon, consider a bar that can slide in a smooth slot. It has a spring attached to it at the right end and a force P applied to it at the left end, as shown in Figure 11.5. As we increase the force P, the inclination of the bar at the equilibrium position moves closer to the horizontal position. But there is an inclination at which the bar suddenly jumps across the horizontal line to a position below the horizontal line

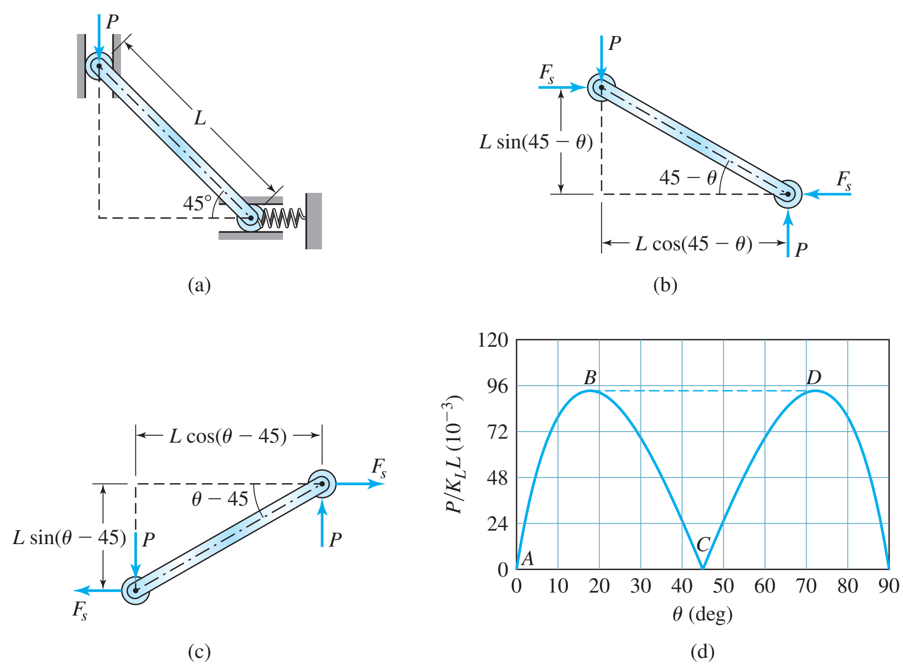


Figure 11.5 Snap buckling problem. (*a*) Undeformed position, $\theta = 0$. (*b*) $0 < \theta < 45°$. (*c*) $\theta > 45°$. (*d*) Load versus θ.

We consider the equilibrium of the bar before and after the horizontal line to understand the mathematics of snap buckling. Suppose the spring is in the instructed position, as shown in Figure 11.5*a*. We define the inclination of the bar by the angle θ measured from the undeformed position. Figure 11.5*b* and *c* shows the free-body diagrams of the bar before and after the horizontal position. The spring force must reverse direction as the bar crosses the horizontal position to ensure moment equilibrium. The deformation of the spring before the horizontal position is $L\cos(45^o - \theta) - L\cos 45^o$. Thus the spring force is $F_s = K_L[L\cos(45^o - \theta) - L\cos 45^o]$. By moment equilibrium we obtain

$$\frac{P}{K_L L} = [\cos(45° - \theta) - \cos 45°]\tan(45° - \theta), \qquad 0 < \theta < 45° \tag{11.2a}$$

In a similar manner, by considering the moment equilibrium in Figure 11.5*c*, we obtain

$$\frac{P}{K_L L} = [\cos(\theta - 45°) - \cos 45°]\tan(\theta - 45°), \qquad \theta > 45° \tag{11.2b}$$

Figure 11.5*d* shows a plot of $P/K_L L$ versus θ obtained from Equations (11.2a) and (11.2b). As we increase *P*, we move along the curve until we reach point *B*. At *B* rather than following paths *BC* and *CD*, the bar jumps (snaps) from point *B* to point *D*. It should be emphasized that each point on paths *BC* and *CD* represents an equilibrium position, but it is not a stable equilibrium position that can be maintained.

11.1.5 Local Buckling

The perspectives on the buckling problem in the previous sections were about structural stability. Besides the instability of a structure, however, we can have *local instabilities*. Figure 11.6*a* shows the crinkling of an aluminum can under compressive axial loads. This crinkling is the local buckling of the thin walls of the can. Figure 11.6*b* shows a thin cylindrical shaft under torsion. The stress cube at the top shows the torsional shear stresses. But if we consider a stress cube in principal coordinates, then we see that principal stress 2 is compressive. This compressive principal stress can also cause local buckling, though the orientation of the crinkles will be different than those from the crushing of the aluminum can.

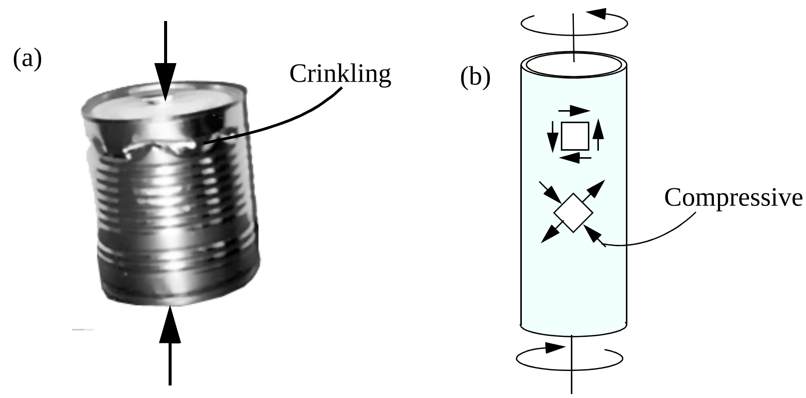


Figure 11.6 Local buckling. (*a*) Due to axial loads. (*b*) Due to torsional loads.

Consolidate your knowledge

1. Describe in your own words the various types of buckling.

PROBLEM SET 11.1

Stability of discrete systems

11.1 A linear spring that can be in tension or compression is attached to a rigid bar as shown in Figure P11.1. In terms of the spring constant k and the length of the rigid bar L, determine the critical load value P_{cr}

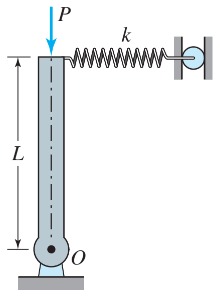


Figure P11.1

11.2 A linear spring that can be in tension or compression is attached to a rigid bar as shown in Figure P11.2. In terms of the spring constant k and the length of the rigid bar L, determine the critical load value P_{cr}.

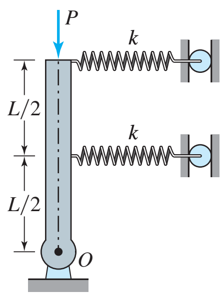


Figure P11.2

11.3 A linear spring that can be in tension or compression is attached to a rigid bar as shown in Figure P11.1. In terms of the spring constant k and the length of the rigid bar L, determine the critical load value P_{cr}.

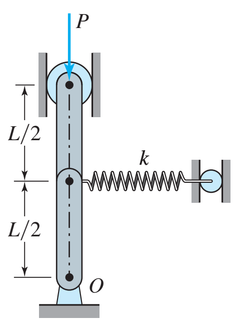


Figure P11.3

11.4 Linear deflection and torsional springs are attached to a rigid bar as shown Figure P11.4. The springs can act in tension or in compression and resist rotation in either direction. Determine the critical load value P_{cr}.

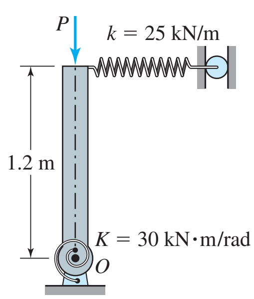


Figure P11.4

11.5 Linear deflection and torsional springs are attached to a rigid bar as shown Figure P11.5. The springs can act in tension or in compression and resist rotation in either direction. Determine the critical load value P_{cr}.

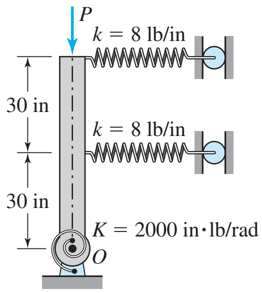


Figure P11.5

11.6 Linear deflection and torsional springs are attached to a rigid bar as shown Figure P11.6. The springs can act in tension or in compression and resist rotation in either direction. Determine the critical load value P_{cr}.

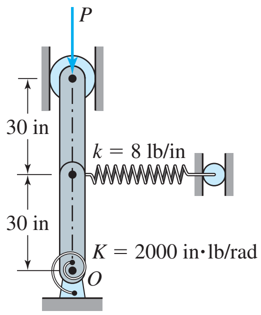


Figure P11.6

Stretch yourself

11.7 Two rigid bars are pin connected and supported as shown in Figure 11.7. The linear displacement spring constant is k = 25 kN/m and the linear rotational spring constant is K= 30 kN/rad. Using θ_1 and θ_2 as the angle of rotation of the bars AB and BC from the vertical, write the equilibrium equations in matrix form and determine the critical load P by finding the eigenvalues of the matrix. Assume small angles of rotation to simplify the calculations.

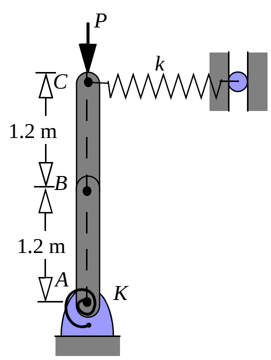


Figure P11.7

11.8 Two rigid bars are pin connected and supported as shown in Figure 11.8. The linear displacement spring constant is k = 8 lb/in. and the linear rotational spring constant is K= 2000 in.-lb/rad. Using θ_1 and θ_2 as the angle of rotation of the bars AB and BC from the vertical, write the equilibrium equations in matrix form and determine the critical load P by finding the eigenvalues of the matrix. Assume small angles of rotation to simplify the calculations.

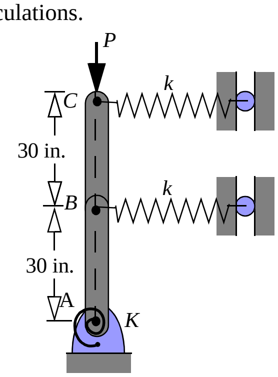


Figure P11.8

11.2 EULER BUCKLING

In this section we develop a theory for a straight column that is simply supported at either end. This theory was first developed by Leonard Euler (see Section 11.4) and is named after him.

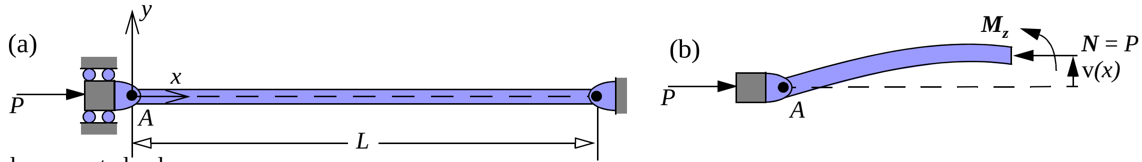


Figure 11.7 Simply supported column.

Figure 11.7*a* shows a simply supported column that is axially loaded with a force *P*. We shall initially assume that bending is about the *z* axis; as our equations in Chapter 7 on beam deflection were developed with just this assumption. We shall relax this assumption at the end to generate the formula for a critical buckling load.

Let the bending deflection at any location *x* be given by v(*x*), as shown in Figure 11.7*b*. An imaginary cut is made at some location *x*, and the internal bending moment is drawn according to our sign convention. The internal axial force ***N*** will be equal to *P*. By balancing the moment at point *A* we obtain $\boldsymbol{M}_z + P\text{v} = 0$. Substituting the moment–curvature relationship of Equation (7.1), we obtain the differential equation:

$$EI_{zz}\frac{d^2\text{v}}{dx^2} + P\text{v} = 0 \tag{11.3a}$$

If buckling can occur about any axis and not just the *z* axis, as we initially assumed, then the subscripts *zz* in the area moment of inertia should be dropped. The boundary value problem can be written using Equation (11.3a) as

- **Differential Equation**

$$\frac{d^2\text{v}}{dx^2} + \lambda^2\text{v} = 0 \tag{11.3b}$$

where

$$\lambda = \sqrt{\frac{P}{EI}} \tag{11.3c}$$

- **Boundary Conditions**

$$\text{v}(0) = 0 \tag{11.4a}$$

$$\text{v}(L) = 0 \tag{11.4b}$$

Clearly $\text{v} = 0$ would satisfy the boundary-value problem represented by Equations (11.3a), (11.4a), and (11.4b). This trivial solution represents purely axial deformation due to compressive axial forces. Our interest is to find the value of *P* that would cause bending; in other words, a nontrivial ($\text{v} \neq 0$) solution to the boundary-value problem. Alternatively, at what value of *P* does a nontrivial solution exist to the boundary-value problem? As observed in Section 11.1, this is the classical statement of an eigenvalue problem.

The solution to the differential equation, Equation (11.3b), is

$$\text{v}(x) = A\cos\lambda x + B\sin\lambda x \tag{11.5}$$

From the boundary condition (11.4*a*) we obtain

$$\text{v}(0) = A\cos(0) + B\sin(0) = 0 \qquad \text{or} \qquad A = 0 \tag{11.6a}$$

From boundary condition (11.4*b*),[1] we obtain

$$\text{v}(L) = A\cos\lambda L + B\sin\lambda L = 0 \qquad \text{or} \qquad B\sin\lambda L = 0 \tag{11.6b}$$

If $B = 0$, then we obtain a trivial solution. For a nontrivial solution the sine function must equal zero:

$$\sin\lambda L = 0 \tag{11.7}$$

Equation (11.7) is called the *characteristic equation*, or the *buckling equation*.

Equation (11.7) is satisfied if $\lambda L = n\pi$. Substituting for λ and solving for *P*, we obtain

$$P_n = \frac{n^2 \pi^2 EI}{L^2}, \qquad n = 1, 2, 3, \ldots \tag{11.8}$$

Equation (11.8) represents the values of load P (the eigenvalues) at which buckling would occur. What is the lowest value of P at which buckling will occur? Clearly, for the lowest value of P, n should equal 1 in Equation (11.8). Furthermore minimum value of I should be used. The critical buckling load is

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L^2}} \tag{11.9}$$

P_{cr}, the critical buckling load, is also called *Euler load*. *Buckling will occur about the axis that has minimum area moment of inertia*. The solution for v can be written as

$$v = B \sin\left(n\pi \frac{x}{L}\right) \tag{11.10}$$

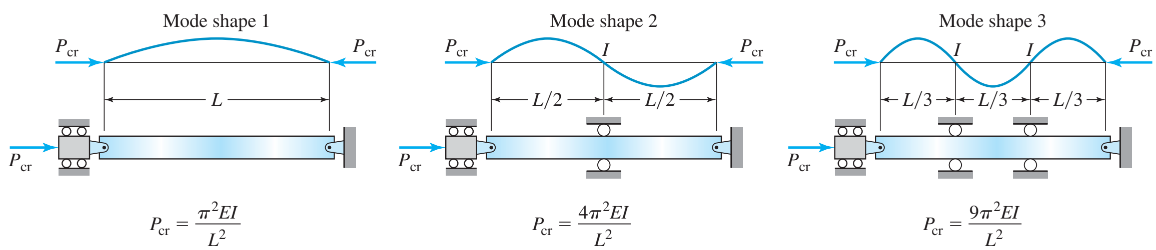


Figure 11.8 Importance of buckled modes.

Equation (11.10) represents the buckled mode (eigenvectors). Notice that the constant B in Equation (11.10) is undetermined. This is typical in eigenvalue problems. The importance of each buckled mode shape can be appreciated by examining Figure 11.8. If buckled mode 1 is prevented from occurring by installing a restraint (or support), then the column would buckle at the next higher mode at critical load values that are higher than those for the lower modes. Point I on the deflection curves describing the mode shapes has two attributes: it is an inflection point and the magnitude of deflection at this point is zero. Recall that the curvature d^2v/dx^2 at an inflection point is zero. Hence the internal moment $\boldsymbol{M_z}$ at this point is zero. If roller supports are put at any other points than the inflection points I, as predicted by Equation (11.10), then the boundary-value problem (see Problem 11.32) will have different eigenvalues (critical loads) and eigenvectors (mode shapes).

In many situations it may not be possible to put roller supports in order to change a mode to a higher critical buckling load. But buckling modes and buckling loads can also be changed by using elastic supports. Figure 11.9 shows a water tank on columns. The two rings are the elastic supports. Elastic supports can be modeled as springs and formulas for buckling loads developed as shown in Example 11.3.

Figure 11.9 Elastic supports on columns of a water tank.

[1] A matrix form may be more familiar for an eigenvalue problem. The boundary condition equations can be written in matrix form as

$$\begin{bmatrix} 1 & 0 \\ \cos\left(\sqrt{\frac{P}{EI_{zz}}}L\right) & \sin\left(\sqrt{\frac{P}{EI_{zz}}}L\right) \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For a nontrivial solution—that is, when A and B are not both zero—the condition is that the determinant of the matrix must be zero. This yields $\sin\left(\sqrt{(P/EI_{zz})}L\right) = 0$, in agreement with our solution.

Consolidate your knowledge

1. With the book closed derive the Euler buckling formula and comment on higher buckling modes.

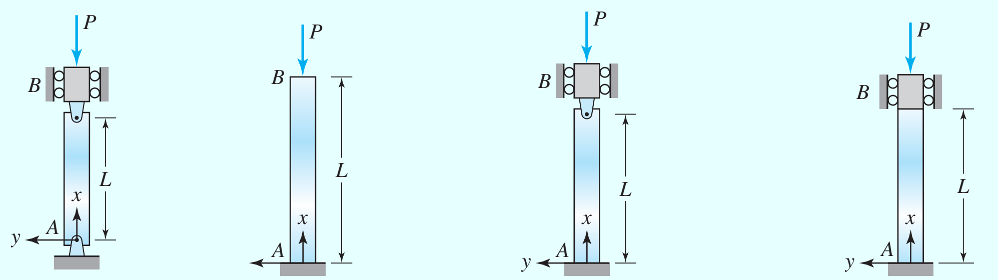
11.2.1 Effects of End Conditions

Equation (11.9) is applicable only to simply supported columns. However, the process used to obtain the formula can be used for other types of supports. Table 11.1 shows the critical elements in the derivation process and the results for three other supports. The formula for critical loads for all cases shown in Table 11.1 can be written as

$$\boxed{P_{cr} = \frac{\pi^2 EI}{L_{eff}^2}} \tag{11.11}$$

where L_{eff} is the effective length of the column. The effective length for each case is given in the last row of Table 11.1. This definition of effective length will permit us to extend results that will be derived in Section 11.3 for simply supported imperfect columns to imperfect columns with the supports shown in cases 2 through 4 in Table 11.1.

TABLE 11.1 **Buckling of columns with different supports**



	Case 1	Case 2	Case 3	Case 4[a]
	Pinned at both ends	One end fixed, other end free	One end fixed, other end pinned	Fixed at both ends
Differential equation	$EI\frac{d^2v}{dx^2} + Pv = 0$	$EI\frac{d^2v}{dx^2} + Pv = Pv(L)$	$EI\frac{d^2v}{dx^2} + Pv = R_B(L-x)$	$EI\frac{d^2v}{dx^2} + Pv = R_B(L-x) + M_B$
Boundary conditions	$v(0) = 0$ $v(L) = 0$		$v(0) = 0$ $\frac{dv}{dx}(0) = 0$ $v(L) = 0$	$v(0) = 0$ $\frac{dv}{dx}(0) = 0$ $v(L) = 0$ $\frac{dv}{dx}(L) = 0$
Characteristic equation $\lambda = \sqrt{\frac{P}{EI}}$	$\sin\lambda L = 0$	$\cos\lambda L = 0$	$\tan\lambda L = \lambda L$ [b]	$2(1-\cos\lambda L) - \lambda L\sin\lambda L = 0$
Critical load P_{cr}	$\frac{\pi^2 EI}{L^2}$	$\frac{\pi^2 EI}{4L^2} = \frac{\pi^2 EI}{(2L)^2}$	$\frac{20.13EI}{L^2} = \frac{\pi^2 EI}{(0.7L)^2}$	$\frac{4\pi^2 EI}{L^2} = \frac{\pi^2 EI}{(0.5L)^2}$
Effective length L_{eff}	L	$2L$	$0.7L$	$0.5L$

a. R_B and M_B are the force and moment reactions.

b. The roots of the equations have to be found iteratively. The two smallest roots of the equation are λL=4.4934 and λL=7.7253.

In Equation (11.9), I can be replaced by Ar^2, where A is the cross-sectional area and r is the minimum radius of gyration [see Equation (A.11)]. We obtain

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 E}{(L_{eff}/r)^2} \qquad (11.12)$$

where L_{eff}/r is the *slenderness ratio* and σ_{cr} is the compressive axial stress just before the column would buckle.

Equation (11.12) is valid only in the elastic region—that is, if $\sigma_{cr} < \sigma_{yield}$. If $\sigma_{cr} > \sigma_{yield}$, then elastic failure will be due to stress exceeding the material strength. Thus $\sigma_{cr} = \sigma_{yield}$ defines the failure envelope for a column. Figure 11.10 shows the failure envelopes for steel, aluminum, and wood using the material properties given in Table D.1. As nondimensional variables are used in the plots in Figure 11.10, these plots can also be used for metric units. Note that the slenderness ratio is defined using effective lengths; hence these plots are applicable to columns with different supports.

The failure envelopes in Figure 11.10 show that as the slenderness ratio increases, the failure due to buckling will occur at stress values significantly lower than the yield stress. This underscores the importance of buckling in the design of members under compression.

The failure envelopes, as shown in Figure 11.10, depend only on the material property and are applicable to columns of different lengths, shapes, and types of support. These failure envelopes are used for classifying columns as short or long.[2] Short column design is based on using yield stress as the failure stress. Long column design is based on using critical buckling stress as the failure stress. The slenderness ratio at point A for each material is used for separating short columns from long columns for that material. Point A is the intersection point of the straight line representing elastic material failure and the hyperbola curve representing buckling failure.

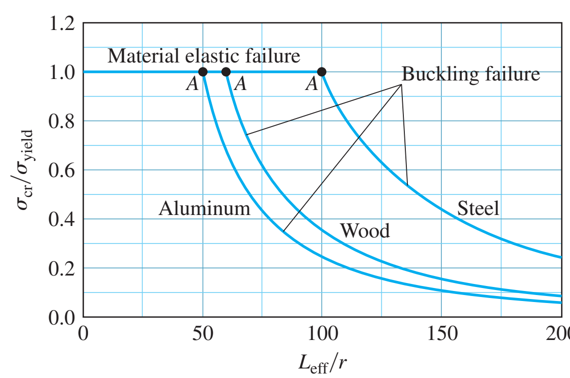


Figure 11.10 Failure envelopes for Euler columns.

EXAMPLE 11.1

A hollow circular steel column (E = 30,000 ksi) is simply supported over a length of 20 ft. The inner and outer diameters of the cross section are 3 in. and 4 in., respectively. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load. (d) If roller supports are added at the midpoint, what would be the new critical buckling load?

PLAN

(a) The area moment of inertia I for a hollow cylinder is same about all axes and can be found using the formula in Table C.2. From the value of I the radius of gyration can be found. The ratio of the given length to the radius of gyration gives the slenderness ratio. (b)In Equation (11.9) the given values of E and L, as well as the calculated value of I in part (a), can be substituted to obtain the critical buckling load P_{cr}. (c) Dividing P_{cr} by the cross-sectional area, the critical axial stress σ_{cr} can be found. (d)The column will buckle at the next higher buckling load, which can be found by substituting $n = 2$ and E, I, and L into Equation (11.8).

SOLUTION

(a) The outer diameter $d_o = 4$ in. and the inner diameter $d_i = 3$ in. From Table C.2 the area moment of inertia for the hollow cylinder, the cross-sectional area A, and the radius of gyration r can be calculated using Equation (A.11),

$$I = \frac{\pi(d_o^4 - d_i^4)}{64} = \frac{\pi[(4\text{ in.})^4 - (3\text{ in.})^4]}{64} = 8.590\text{ in.}^4 \qquad A = \frac{\pi(d_o^2 - d_i^2)}{4} = \frac{\pi[(4\text{ in.})^2 - (3\text{ in.})^2]}{4} = 5.498\text{ in.}^2 \qquad (E1)$$

[2]Intermediate column is a third classification used if the critical stress is between yield stress and ultimate stress. See Equation (11.26) and Problems 11.64 and 11.65 for additional details.

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{8.590\ \text{in.}^4}{5.498\ \text{in.}^2}} = 1.250\ \text{in.} \tag{E2}$$

The length $L = 20$ ft = 240 in. Thus the slenderness ratio is $L/r = (240\ \text{in.})/(1.25\ \text{in.})$.

ANS. $L/r = 192$

(b) Substituting $E = 30{,}000$ ksi, $L = 240$ in., and $I = 8.59$ in.^{4} into Equation (11.9), we obtain the critical buckling load,

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(30{,}000\ \text{ksi})(8.590\ \text{in.})}{(240\ \text{in.})^2} \tag{E3}$$

ANS. $P_{cr} = 44.15$ kips

(c) The axial stress at the critical buckling load can be found as

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{44.15\ \text{kips}}{5.498\ \text{in.}^2} \tag{E4}$$

ANS. $\sigma_{cr} = 8.03$ ksi (C)

(d) With the support in the middle, the buckling would occur in mode 2. Substituting $n = 2$ and E, I, and L into Equation (11.8) we obtain the critical buckling load,

$$P_{cr} = \frac{n^2\pi^2 EI}{L^2} = \frac{2^2\pi^2(30{,}000\ \text{ksi})(8.590\ \text{in.})}{(240\ \text{in.})^2} \tag{E5}$$

ANS. $P_{cr} = 176.6$ kips

COMMENTS

1. This example highlights the basic definitions of variables and equations used in buckling problems.
2. The middle support forces the column into the mode 2 buckling mode in part (d). Another perspective is to look at the column as two simply supported columns, each with an effective length of half the column or $L_{eff} = 120$ in. Substituting this into Equation (11.11), we obtain the same value as in part (d).

EXAMPLE 11.2

The hoist shown in Figure 11.11 is constructed using two wooden bars with modulus of elasticity $E = 1800$ ksi and ultimate stress of $\sigma_{ult} = 5$ ksi. For a factor of safety of $K = 2.5$, determine the maximum permissible weight W that can be lifted using the hoist for the two cases: (a) $L = 30$ in.; (b) $L = 60$ in.

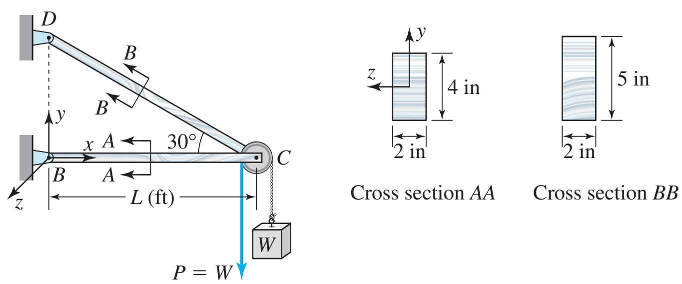


Figure 11.11 Hoist in Example 11.2.

PLAN

The axial stresses in the members can be found and compared with the calculated allowable values to determine a set of limits on W. By inspection we see that member BC will be in compression. Internal force in BC in terms of W can be found from free body diagram of the pulley and compared to critical buckling of BC to get another limit on W. The maximum value of W that satisfies the strength and buckling criteria can now be determined.

SOLUTION

The allowable stress in wood is

$$\sigma_{allow} = \sigma_{ult}/K = (5\ \text{ksi})/2.5 = 2\ \text{ksi} \tag{E1}$$

The free-body diagram of the pulley is shown in Figure 11.12 with the force in BC drawn as compressive and the force in CD as tensile. By equilibrium the internal axial forces

$$N_{CD}\ \sin 30° = 2W \qquad \text{or} \qquad N_{CD} = 4W \tag{E2}$$

$$N_{BC} = N_{CD} \cos 30° \quad \text{or} \quad N_{BC} = 3.464W \tag{E3}$$

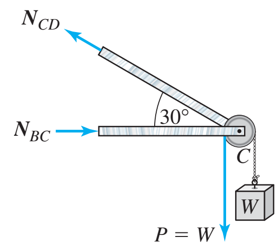


Figure 11.12 Free-body diagram in Example 11.2.

The cross-sectional areas for the two members are $A_{BC} = 8$ in.^{2} and $A_{CD} = 10$ in.^{2}. The axial stresses in terms of W can be found, and these should be less than the allowable stress of 2 ksi, from which we get two limits on W,

$$\sigma_{CD} = \frac{N_{CD}}{A_{CD}} = \frac{4W}{10\text{ in.}^2} \le 2\text{ ksi} \quad \text{or} \quad W \le 5.0\text{ kips} \tag{E4}$$

$$\sigma_{BC} = \frac{N_{BC}}{A_{BC}} = \frac{3.463W}{8\text{ in.}^2} \le 2\text{ ksi} \quad \text{or} \quad W \le 4.62\text{ kips} \tag{E5}$$

(a) We determine the minimum area moment of inertias for cross-section AA,

$$I_{yy} = \frac{1}{12}(4\text{ in.})(2\text{ in.})^3 = 2.667\text{ in.}^4 \qquad I_{zz} = \frac{1}{12}(2\text{ in.})(4\text{ in.})^3 = 10.67\text{ in.}^4 \tag{E6}$$

Substituting $E = 1800$ ksi, $L = 30$ in., and $I = 2.667$ in.^{4} into Equation (11.9), we obtain

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(1800\text{ ksi})(2.667\text{ in.}^4)}{(30\text{ in.})^2} = 52.63\text{ kips} \tag{E7}$$

N_{BC} should be less than the critical load P_{cr} divided by factor of safety K,

$$N_{BC} \le (P_{cr}/K) \quad \text{or} \quad 3.464W \le [(52.63\text{ kips})/2.5] \quad \text{or} \quad W \le 6.08\text{ kips} \tag{E8}$$

The maximum value of W must satisfy Equations (E4), (E5), and (E8).

ANS. $W_{max} = 4.6$ kips

(b) Substituting $E = 1800$ ksi, $L = 60$ in., and $I = 2.667$ in.^{4} into Equation (11.9), we obtain

$$P_{cr} = \frac{\pi^2 EI}{L^2} = \frac{\pi^2(1800\text{ ksi})(2.667\text{ in.}^4)}{(60\text{ in.})^2} = 13.159\text{ kips} \tag{E9}$$

N_{BC} should be less than the critical load P_{cr} divided by factor of safety K,

$$N_{BC} \le (P_{cr}/K) \quad \text{or} \quad 3.464W \le [(13.159\text{ kips})/2.5] \quad or \quad W \le 1.52\text{ kips} \tag{E10}$$

The maximum value of W must satisfy Equations (E4), (E5), and (E10).

ANS. $W_{max} = 1.5$ kips

COMMENTS

1. This example highlights the importance of identifying compression members such as BC, so that buckling failure is properly accounted for in design.
2. The example also emphasizes that the minimum area moment of inertia that must be used is Euler buckling. Had we used I_{zz} instead of I_{yy}, we would have found $P_{cr} = 52.7$ kips and incorrectly concluded that the failure would be due to strength failure and not buckling in case (b)
3. In case (a) material strength governed the design, whereas in case (b) buckling governed the design. If we had several bars of different lengths and different cross-sectional dimensions (such as in Problems 11.18 and 11.19), then it would save a significant amount of work to calculate the slenderness ratio that would separate long columns from short columns. Substituting $\sigma_{cr} = \sigma_{allow} = 2$ ksi into Equation (11.12), we find that $L/r = 94.2$ is the ratio that separates long columns from short columns. It can be checked that the slenderness ratio in case (a) is 51.9, hence material strength governed W_{max}. In case (b) the slenderness ratio is 103.9, hence buckling governed W_{max}.

EXAMPLE 11.3

Linear springs are attached at the free end of a column, as shown in Figure 11.13. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. Show that the critical load P_{cr} for (b) $k = 0$ and (c) $k = \infty$ is as given in Table 11.1 for cases 2 and 3, respectively.

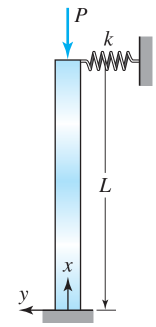


Figure 11.13 Column with elastic support in Example 11.3.

PLAN

The spring exerts a spring force $k\mathrm{v}_L$ at the upper end that must be incorporated into the moment equation, and hence into the differential equation. The boundary conditions are that the deflection and slope at $x = 0$ are zero. (a) The characteristic equation will be generated while solving the boundary-value problem. (b), (c) The roots of the characteristic equation for the two cases will give P_{cr}.

SOLUTION

By equilibrium of moment about point O in Figure 11.14, we obtain an expression for moment $\boldsymbol{M}_z$,

$$\boldsymbol{M}_z - P(\mathrm{v}_L - \mathrm{v}) + k\mathrm{v}_L(L-x) = 0 \qquad \text{or} \qquad \boldsymbol{M}_z + P\mathrm{v} = P\mathrm{v}_L - k\mathrm{v}_L(L-x) \tag{E1}$$

Substituting into Equation (7.1), we obtain the differential equation

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = P\mathrm{v}_L - k\mathrm{v}_L(L-x) \tag{E2}$$

(a) Using Equation (11.3c), Equation (E2) can be written as:

- **Differential equation:**

$$\frac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = \lambda^2\mathrm{v}_L - \frac{k\mathrm{v}_L}{EI}(L-x) \tag{E3}$$

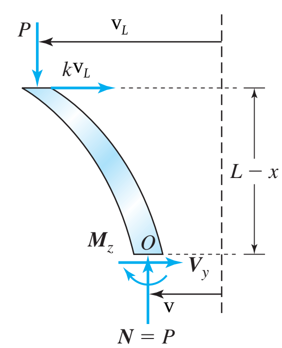


Figure 11.14 Free-body diagram in Example 11.3.

The zero deflection and slope boundary condition are also written to complete the statement of the boundary-value problem,

- **Boundary Conditions:**

$$\mathrm{v}(0) = 0 \tag{E4}$$

$$\frac{d\mathrm{v}}{dx}(0) = 0 \tag{E5}$$

The homogeneous solution v_H to Equation (E3) is given by Equation (11.5). The particular solution is

$$\mathrm{v}_P = \mathrm{v}_L - \frac{k\mathrm{v}_L}{\lambda^2 EI}(L-x) \tag{E6}$$

Thus the total solution $\mathrm{v}_H + \mathrm{v}_P$ can be written as

$$v(x) = A\cos\lambda x + B\sin\lambda x + v_L - \frac{kv_L}{\lambda^2 EI}(L-x) \tag{E7}$$

Substituting $x = 0$ into Equation (E7) and using Equation (E4), we obtain

$$v(0) = A\cos(0) + B\sin(0) + v_L - \frac{kv_L}{\lambda^2 EI}(L-0) = 0 \qquad \text{or} \qquad A = \left(\frac{kL}{\lambda^2 EI} - 1\right)v_L \tag{E8}$$

Differentiating Equation (E7), then substituting $x = 0$ and using Equation (E5), we obtain

$$\frac{dv}{dx}(0) = -\lambda A\sin(0) + B\lambda\cos(0) + \frac{kv_L}{\lambda^2 EI} = 0 \qquad \text{or} \qquad B = -\frac{k}{\lambda^3 EI}v_L \tag{E9}$$

Substituting the values of A and B into Equation (E7), we obtain

$$v(x) = \left[\left(\frac{kL}{\lambda^2 EI} - 1\right)\cos\lambda x - \frac{k}{\lambda^3 EI}\sin\lambda x + 1 - \frac{k}{\lambda^2 EI}(L-x)\right]v_L \tag{E10}$$

Substituting $x = L$ into Equation (E7), we obtain

$$v(L) = \left[\left(\frac{kL}{\lambda^2 EI} - 1\right)\cos\lambda L - \frac{k}{\lambda^3 EI}\sin\lambda L + 1 - 0\right]v_L = v_L \tag{E11}$$

Since v_L is a common factor, Equation (E11) can be simplified to the following *characteristic equation:*

ANS. $$\left(\frac{kL}{\lambda^2 EI} - 1\right)\cos\lambda L - \frac{k}{\lambda^3 EI}\sin\lambda L = 0$$

(b) Substituting $k = 0$ into Equation (E11), we obtain $\cos\lambda L = 0$, which is the characteristic equation for case 2 in Table 11.1. Thus the P_{cr} value corresponding to the smallest root will be as given in Table 11.1 for case 2.
(c) We rewrite Equation (E11) as

$$\tan\lambda L = \lambda L - \frac{\lambda^3 EI}{k} \tag{E12}$$

As k tends to infinity, the second term tends to zero and we obtain $\tan\lambda L = \lambda L$, which is the characteristic equation for case 3 in Table 11.1. Thus the P_{cr} value corresponding to the smallest root will be as given in Table 11.1 for case 3.

COMMENTS

1. This example shows that a spring could simulate an imperfect support that provides some restraint to deflection. The restraining effect is more than zero (free end) but not as much as a roller support.
2. The spring could also represent other beams that are pin connected at the top end. These pin-connected beams provide elastic restraint to deflection but no restraint to the slope. If the beams were welded rather than pin connected, then we would have to include a torsional spring also at the end.
3. The example also demonstrates that the critical buckling loads can be changed by installing some elastic restraints, such as rings, to support the columns of the water tank in Figure 11.9.

EXAMPLE 11.4

Determine the maximum deflection of the column shown in Figure 11.15 in terms of the modulus of elasticity E, the length of the column L, the area moment of inertia I, the axial force P, and the intensity of the distributed force w.

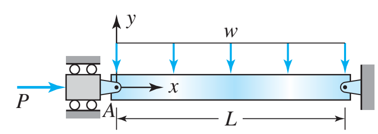


Figure 11.15 Buckling of beam with distributed load in Example 11.4.

PLAN

The moment from the distributed load can be added to the moment for case 1 in Table 11.1 and the differential equation written. The boundary conditions are that the deflection at $x = 0$ and $x = L$ is zero. The boundary-value problem can be solved, and the deflection at $x = L/2$ evaluated to obtain the maximum deflection.

SOLUTION

The reaction force in the y direction is half the total load wL acting on the beam. An imaginary cut at some location x can be made and the free-body diagram of the left part drawn as shown in Figure 11.16. By balancing the moment at point O, we obtain an expression for the moment $\boldsymbol{M_z}$,

$$\boldsymbol{M}_z + P\mathrm{v}(x) - \frac{wL}{2}x + \frac{wx^2}{2} = 0 \qquad \text{(E1)}$$

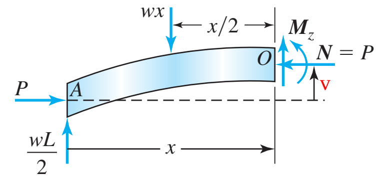


Figure 11.16 Free-body diagram in Example 11.4.

Substituting Equation (E1) into Equation (7.1), we obtain the differential equation

$$EI_{zz}\frac{d^2\mathrm{v}}{dx^2} + P\mathrm{v} = \frac{wL}{2}x - \frac{wx^2}{2} \qquad \text{(E2)}$$

Using Equation (11.3c), Equation (E2) can be written as

- **Differential Equation**

$$\frac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = \frac{wLx}{2EI} - \frac{wx^2}{2EI} \qquad \text{(E3)}$$

The zero-deflection boundary conditions at either end are written to complete the statement of the boundary-value problem.

- **Boundary Conditions**

$$\mathrm{v}(0) = 0 \qquad \text{(E4)}$$

$$\mathrm{v}(L) = 0 \qquad \text{(E5)}$$

To find the particular solution, we substitute $\mathrm{v}_P = a + bx + cx^2$ into Equation (E3) and simplify,

$$2c + \lambda^2(a + bx + cx^2) = \frac{wLx}{2EI} - \frac{wx^2}{2EI} \quad \text{or} \quad (2c + \lambda^2 a) + \left(\lambda^2 b - \frac{wL}{2EI}\right)x + \left(\lambda^2 c + \frac{w}{2EI}\right)x^2 = 0 \qquad \text{(E6)}$$

If Equation (E6) is to be valid for any value of x, then each of the terms in parentheses must be zero and we obtain the values of constants a, b, and c,

$$c = -\frac{w}{2\lambda^2 EI} \qquad b = \frac{wL}{2\lambda^2 EI} \qquad a = -\frac{2c}{\lambda^2} = \frac{w}{\lambda^4 EI} \qquad \text{(E7)}$$

Hence the particular solution is

$$\mathrm{v}_P = \frac{w}{\lambda^4 EI} + \frac{wL}{2\lambda^2 EI}x - \frac{w}{2\lambda^2 EI}x^2 \qquad \text{(E8)}$$

The homogeneous solution v_H to Equation (E3) is given by Equation (11.5). Thus the total solution $\mathrm{v}_H + \mathrm{v}_P$ can be written as

$$\mathrm{v}(x) = A\cos\lambda x + B\sin\lambda x + \frac{w}{\lambda^4 EI} + \frac{wL}{2\lambda^2 EI}x - \frac{w}{2\lambda^2 EI}x^2 \qquad \text{(E9)}$$

Substituting $x = 0$ into Equation (E9) and using Equation (E4), we obtain

$$\mathrm{v}(0) = A\cos(0) + B\sin(0) + \frac{w}{\lambda^4 EI} + 0 - 0 = 0 \quad \text{or} \quad A = -\frac{w}{\lambda^4 EI} \qquad \text{(E10)}$$

Substituting $x = L$ into Equation (E9) and using Equation (E5), we obtain

$$\mathrm{v}(L) = A\cos\lambda L + B\sin\lambda L + \frac{w}{\lambda^4 EI} + \frac{wL^2}{2\lambda^2 EI} - \frac{wL^2}{2\lambda^2 EI} = 0 \quad \text{or} \quad -\frac{w}{\lambda^4 EI}\cos\lambda L + B\sin\lambda L + \frac{w}{\lambda^4 EI} = 0 \qquad \text{(E11)}$$

Since $\sin\lambda L = 2\sin(\lambda L/2)\cos(\lambda L/2)$ and $1 - \cos\lambda L = 2\sin^2(\lambda L/2)$ the above equation can be simplified as

$$B = -\frac{w}{\lambda^4 EI}\,\frac{1-\cos\lambda L}{\sin\lambda L} = -\frac{w}{\lambda^4 EI}\tan\left(\frac{\lambda L}{2}\right) \qquad \text{(E12)}$$

By symmetry the maximum deflection will occur at midpoint. Substituting $x = L/2$, A and B into Equation (E9), we obtain

$$\mathrm{v}_{max} = \mathrm{v}\left(\frac{L}{2}\right) = A\cos\left(\frac{\lambda L}{2}\right) + B\sin\left(\frac{\lambda L}{2}\right) + \frac{w}{\lambda^4 EI} + \frac{wL^2}{4\lambda^2 EI} - \frac{wL^2}{8\lambda^2 EI} \quad \text{or}$$

$$\mathrm{v}_{max} = \frac{w}{\lambda^4 EI}\left[-\cos\left(\frac{\lambda L}{2}\right) - \tan\left(\frac{\lambda L}{2}\right)\sin\left(\frac{\lambda L}{2}\right)\right] + \frac{w}{\lambda^4 EI} + \frac{wL^2}{8\lambda^2 EI} \qquad \text{(E13)}$$

Equation (E13) can be simplified by substituting the tangent function in terms of the sine and cosine functions to obtain

$$v_{max} = -\frac{w}{\lambda^4 EI}\left[\sec\left(\frac{\lambda L}{2}\right) - 1\right] + \frac{wL^2}{8\lambda^2 EI} \qquad \text{(E14)}$$

Substituting for λ, the maximum deflection can be written as

ANS. $$v_{max} = -\frac{wEI}{P^2}\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] + \frac{wL^2}{8P}$$

COMMENTS

1. In Equation (E13), as $\lambda L \to \pi$, the secant function tends to infinity and the maximum displacement becomes unbounded, which means the column becomes unstable. $\lambda L = \pi$ corresponds to the Euler buckling load of Equation (11.9). Thus the *transverse distributed load does not change the critical buckling load* of a column.
2. However the failure mode can be significantly affected by the transverse distributed load. The maximum normal stress will be the sum of axial stress and maximum bending normal stress, $\sigma_{max} = P/A + M_{max} y_{max}/I$. The maximum bending moment will be at $x = L/2$ and can be found from Equation (E1) as $\boldsymbol{M}_{max} = wL^2/8 - Pv_{max}$. Substituting and simplifying gives the maximum normal stress:

$$\sigma_{max} = \frac{P}{A} + \frac{wEy_{max}}{P}\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] \qquad \text{(E15)}$$

By equating the maximum normal stress to the yield stress, we obtain a failure envelope, which clearly depends on the value of *w*.

QUICK TEST 11.1

Time: 15 minutes/Total: 20 points

Answer true or false. If false, give the correct explanation. Each question is worth two points. Use the solutions given in Appendix E to grade yourself.

1. Column buckling can be caused by tensile axial forces.
2. Buckling occurs about an axis with minimum area moment of inertia of the cross section.
3. If buckling is avoided at the Euler buckling load by the addition of supports in the middle, then the column will not buckle.
4. By changing the supports at the column end, the critical buckling load can be changed.
5. The addition of uniform transversely distributed forces decreases the critical buckling load on a column.
6. The addition of springs in the middle of the column decreases the critical buckling load.
7. Eccentricity in loading decreases the critical buckling load.
8. Increasing the slenderness ratio increases the critical buckling load.
9. Increasing the eccentricity ratio increases the normal stress in a column.
10. Material strength governs the failure of short columns and Euler buckling governs the failure of long columns.

PROBLEM SET 11.2

Euler buckling

11.9 A hollow circular steel column ($E = 200$ GPa) is simply supported over a length of 5 m. The inner and outer diameters of the cross section are 75 mm and 100 mm. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load. (d) If roller supports are added at the midpoint, what would be new critical buckling load?

11.10 A 30-ft-long hollow square steel column ($E = 30{,}000$ ksi) is built into the wall at either end. The column is constructed from $\frac{1}{2}$-in.-thick sheet metal and has outer dimensions of 4 in. × 4 in. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load.

11.11 A 10 ft long lumber (E = 1,800 ksi) column with a rectangular cross section of 4 in. x 6 in. is pinned at both ends. (a) Determine the critical buckling load *P.* (b) What is the next higher buckling load?

11.12 A 4 m long column is constructed from a steel (E = 210 GPa) sheet of thickness 10 mm. The sheet metal is bent to form a hollow rectangular cross section with outer dimension of 120 mm x 80 mm. One end of the column is fixed and the other is a free end as in case 2 of Table 11.1 (a) Determine the critical buckling load *P.* (b) What is the next higher buckling load?

11.13 A 12 ft long lumber (E = 1,800 ksi) column with a rectangular cross section of 6 in. x 8 in. is pinned at one end and fixed at the other as in case 3 of Table 11.1. (a) Determine the critical buckling load *P.* (b) What is the next higher buckling load?

11.14 A 5 m long column is constructed from a steel (E = 210 GPa) sheet metal of thickness 15 mm. The sheet metal is bent to form a hollow rectangular cross section with outer dimension of 120 mm x 90 mm. The ends of the column are fixed as in case 4 of Table 11.1. Determine the critical buckling load *P.*

11.15 A 20-ft-long wooden column ($E = 1800$ ksi) has cross-section dimensions of 8 in. × 8 in. The column is built in at one end and simply supported at the other end. Determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load.

11.16 A W12 × 35 steel section (see Appendix E) is used for a 21-ft column that is simply supported at each end. Use $E = 30{,}000$ ksi and determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load. (d) If roller supports are added at intervals of 7 ft, what would be the critical buckling load?

11.17 An S200 × 34 steel section (see Appendix E) is used as a 6-m column that is built in at each end. Use $E = 200$ GPa and determine (a) the slenderness ratio; (b) the critical buckling load; (c) the axial stress at the critical buckling load.

11.18 Columns made from alloy will be used in the construction of a frame. The cross section of the columns is a hollow square of 0.125-in. thickness and outer dimensions of a in. The modulus of elasticity $E = 9000$ ksi and the yield stress $\sigma_{yield} = 90$ ksi. Table 11.18 lists the lengths L and outer square dimensions a. Identify the long and short columns. Assume the ends will be simply supported.

TABLE P11.18 **Column geometric properties**

L **(ft)**	***a*** **(in.)**
1.0	1.125
1.5	1.500
2.0	1.750
2.5	2.750
3.0	3.000
3.5	3.000
4.0	3.000

11.19 Columns made from alloy will be used in the construction of a frame. The cross section of the columns is a hollow cylinder of 10-mm thickness and an outer diameter of d mm. The modulus of elasticity $E = 100$ GPa and the yield stress $\sigma_{yield} = 600$ MPa. Table P11.19 lists the lengths L and outer diameters d. Identify the long and short columns. Assume the ends of the column are built in.

TABLE P11.19 **Column geometric properties**

L **(m)**	***d*** **(mm)**
1	60
2	80
3	100
4	150
5	200
6	225
7	250

11.20 Three column cross sections are shown in Figure P11.20. The area of each of the three cross sections is equal to A. Determine the ratios of critical loads P_{cr1}: P_{cr2}: P_{cr3} assuming (a) the ends are simply supported; (b) the ends are built in. (c) How do you expect the ratios to change if the end conditions were as in cases 2 and 3 of Table 11.1?

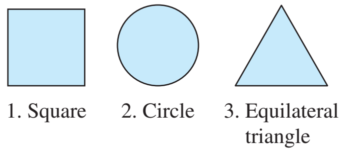


Figure P11.20

11.21 Figure P11.21 shows two steel (E = 30,000 ksi, σ_{yield} = 30 ksi) bars of a diameter $d = \frac{1}{4}$ in. on which a force F = 750 lb is applied. Bars AP and BP have lengths L_{AP} = 8 in. and L_{BP} = 10 in. Determine the factor of safety for the assembly.

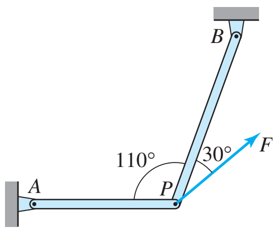


Figure P11.21

11.22 Figure P11.22 shows two steel (E = 30,000 ksi, σ_{yield} = 30 ksi) bars of a diameter $d = \frac{1}{4}$ in. on which a force F = 600 lb is applied. Bars AP and BP have lengths L_{AP} = 7 in. and L_{BP} = 10 in. Determine the factor of safety for the assembly.

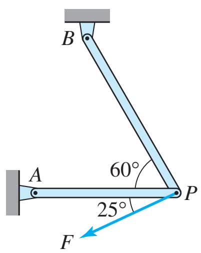


Figure P11.22

11.23 Figure P11.23 shows two copper (E = 15,000 ksi, σ_{yield} = 12 ksi) bars of a diameter $d = \frac{1}{4}$ in. on which a force F = 500 lb is applied. Bars AP and BP have lengths L_{AP} = 7 in. and L_{BP} = 9 in. Determine the factor of safety for the assembly.

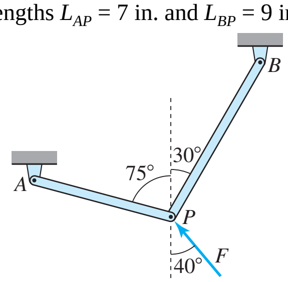


Figure P11.23

11.24 Figure P11.24 shows two (E = 200 GPa, σ_{yield} = 200 MPa) bars of a diameter d =10 mm on which a force F = 10 kN is applied. Bars AP and BP have lengths L_{AP} = 200 mm and L_{BP} = 350 mm. Determine the factor of safety for the assembly.

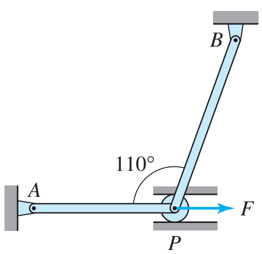


Figure P11.24

11.25 Figure P11.25 shows two (E = 200 GPa, σ_{yield} = 360 MPa) bars of a diameter d =10 mm on which a force F = 10 kN is applied. Bars AP and BP have lengths L_{AP} = 200 mm and L_{BP} = 300 mm. Determine the factor of safety for the assembly.

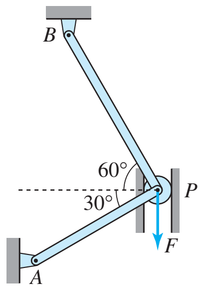


Figure P11.25

11.26 Figure P11.26 shows two (E = 200 GPa, s_{yield} = 200 MPa) bars of a diameter d =10 mm on which a force F = 10 kN is applied. Bars AP and BP have lengths L_{AP} = 200 mm and L_{BP} = *300* mm. Determine the factor of safety for the assembly.

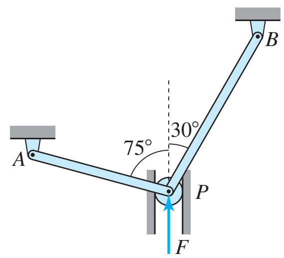


Figure P11.26

Formulation and solutions

11.27 (a) Solve the boundary-value problem for case 2 in Table 11.1 and obtain the critical load value P_{cr} that is given in the table. (b) If buckling in mode 1 is prevented, then what would be the P_{cr} value?

11.28 (a) Solve the boundary-value problem for case 3 in Table 11.1 and obtain the critical load value P_{cr} that is given in the table. (b) If buckling in mode 1 is prevented, then what would be the P_{cr} value?

11.29 (a) Solve the boundary-value problem for case 4 in Table 11.1 and obtain the critical load value P_{cr} that is given in the table. (b) If buckling in mode 1 is prevented, then what would be the P_{cr} value?

11.30 A torsional spring with a spring constant K is attached at one end of a column, as shown in Figure P11.30. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. (b) Show that for $K = 0$ and $K = \infty$ the critical load P_{cr} is as given in Table 11.1 for cases 1 and 3, respectively.

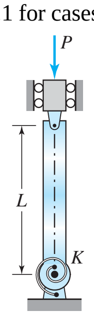


Figure P11.30

11.31 A torsional spring with a spring constant K is attached at one end of a column, as shown in Figure P11.31. Assume that bending about the y axis is prevented. (a) Determine the characteristic equation for this buckling problem. (b) Show that for $K = 0$ the critical load P_{cr} is as given for case 2 in Table 11.1. (c) For $K = \infty$ obtain the critical load P_{cr}.

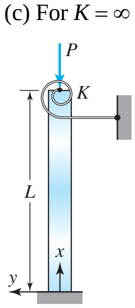


Figure P11.31

11.32 Consider the column shown in Figure P11.32. (a) Determine the critical buckling in terms of E, I, L, and α. (b) Show that when $\alpha = 0.5$, the critical load corresponds to mode 2, as shown in Figure 11.8.

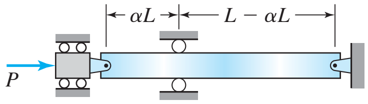


Figure P11.32

11.33 For the column shown in Figure P11.33 determine (a) the deflection at $x = L$; (b) the critical load P_{cr} in terms of the modulus of elasticity E, the column length L, the area moment of inertia I, and the force P.

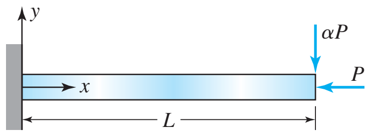


Figure P11.33

11.34 For the column shown in Figure P11.34 determine (a) the deflection at $x = L$; (b) the critical load P_{cr} in terms of the modulus of elasticity E, the column length L, the area moment of inertia I, and the force P.

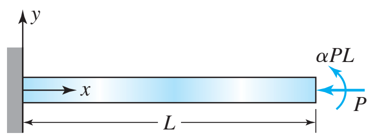


Figure P11.34

11.35 For the column shown in Figure P11.35 determine (a) the deflection at $x = L$; (b) the critical load P_{cr} in terms of the modulus of elasticity E, the column length L, the area moment of inertia I, and the force P.

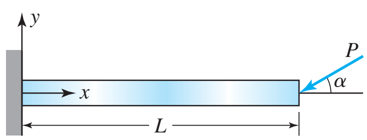


Figure P11.35

Design problems

11.36 Steel (E = 210 GPa) rectangular bars of 15 mm x 25 mm cross section form an assembly shown in Figure P11.36. Determine the maximum load P that can be applied without buckling of any bar. Use a = 1 m, b = 0.7 m, and c = 1 m.

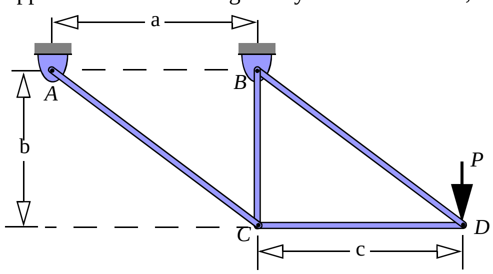


Figure P11.36

11.37 Steel (E = 210 GPa) rectangular bars of 15 mm x 25 mm cross section form an assembly shown in Figure P11.36. Determine the maximum load P that can be applied without buckling of any bar. Use a = 1 m, b = 0.7 m, and c = 1.4 m.

11.38 Steel (E= 30,000 ksi) rectangular bars of 1/2 in. x 1 in. cross section form an assembly shown in Figure P11.38. Determine the maximum load P that can be applied without buckling of any bar.

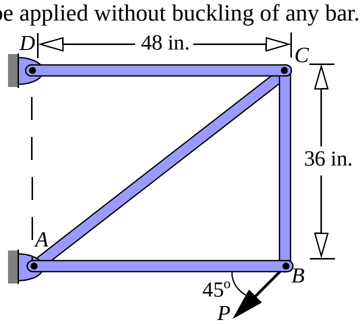


Figure P11.38

11.39 Steel (E= 30,000 ksi) rectangular bars of 1/2 in. x 1 in. cross section form an assembly shown in Figure P11.39. Determine the maximum load P that can be applied without buckling of any bar.

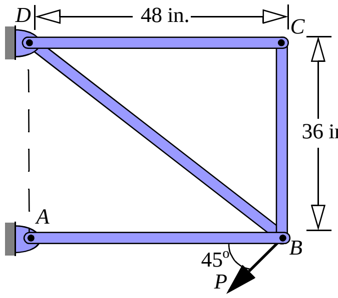


Figure P11.39

11.40 A hoist is constructed using two wooden bars to lift a weight of 5 kips, as shown in Figure P11.40. The modulus of elasticity for wood $E = 1800$ ksi and the allowable normal stress is 3.0 ksi. Determine the maximum value of L to the nearest inch that can be used in constructing the hoist.

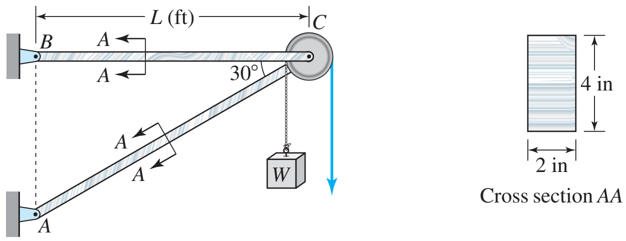


Figure P11.40

11.41 Two steel cylinders ($E = 30{,}000$ ksi and $\sigma_{yield} = 30$ ksi) AB and CD are loaded as shown in Figure P11.41. Determine the maximum load P to the nearest lb, if a factor of safety of 2 is desired. Model the ends of column AB as built in

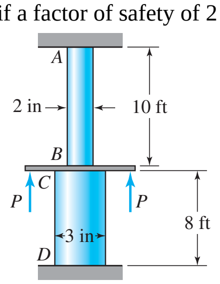


Figure P11.41

11.42 A spreader is to be made from an aluminum pipe ($E = 10{,}000$ ksi) of $\frac{1}{8}$-in. thickness and an outer diameter of 2 in., as shown in Figure P11.42. The pipe lengths available for design start from 4 ft in 6-in. steps up to 8 ft. The allowable normal stress is 40 ksi. Develop a table for the lengths of pipe and the maximum force F the spreader can support.

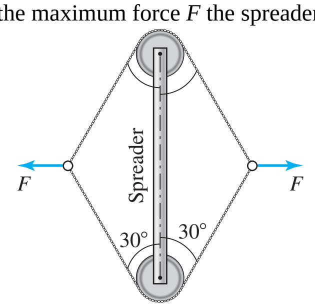


Figure P11.42

11.43 Two 200-mm × 50-mm pieces of lumber ($E = 12.6$ GPa) form a part of a deck that is modeled as shown in Figure P11.43. The allowable stress for the lumber is 18 MPa. (a) Determine the maximum intensity of the distributed load w. (b) What is the factor of safety for column BD corresponding to the answer in part (a)?

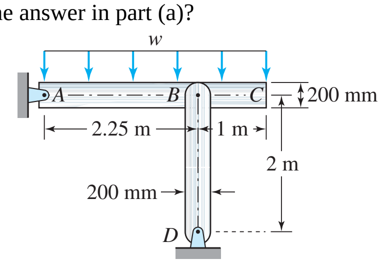


Figure P11.43

11.44 Two 200-mm × 50-mm pieces of lumber ($E = 12.6$ GPa) form a part of a deck that is modeled as shown in Figure P11.44. The allowable stress for the lumber is 18 MPa. (a) Determine the maximum intensity of the distributed load w. (b) What is the factor of safety for column BC corresponding to the answer in part (a)?

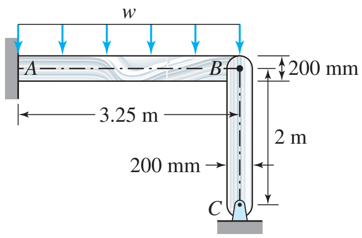


Figure P11.44

11.45 A rigid bar hinged at point O has a force P applied to it, as shown in Figure P11.45. Bars A and B are made of steel with a modulus of elasticity $E = 30{,}000$ ksi and an allowable stress of 25 ksi. Bars A and B have circular cross sections with areas $A_A = 1$ in.^{2} and $A_B = 2$ in.^{2}, respectively. Determine the maximum force P that can be applied.

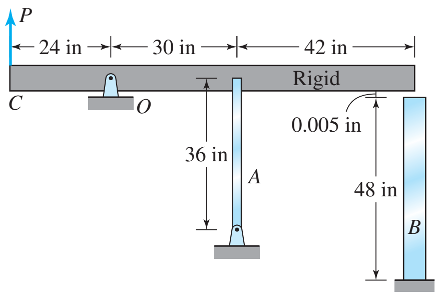


Figure P11.45

Stretch yourself

11.46 Show that for a beam with a constant bending rigidity EI, the fourth-order differential equation for solving buckling problems is given by

$$EI\frac{d^4 v}{dx^4} + P\frac{d^2 v}{dx^2} = p_y \tag{11.13}$$

where P is a compressive axial force and p_y is the distributed force in the y direction.

11.47 Using Equation (11.13), solve Example 11.4.

11.48 Show that the critical change of temperature at which the beam shown in Figure P11.48 will buckle is given by the equation below.

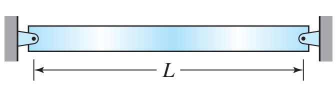


Figure P11.48

$$\Delta T_{crit} = \frac{\pi^2}{\alpha(L/r)^2}$$

where α is the thermal coefficient of expansion and r is the radius of gyration.

11.49 A column with a constant bending rigidity EI rests on an elastic foundation as shown in Figure P11.49. The foundation modulus is k, which exerts a spring force per unit length of kv. Show that the governing differential equation is given by Equation (11.15). (*Hint:* See Problems 7.48 and 11.46.)

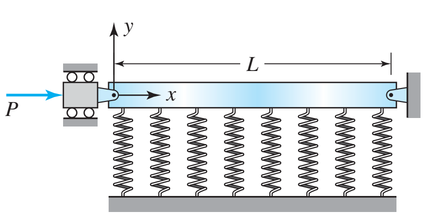


Figure P11.49

$$EI\frac{d^4 v}{dx^4} + P\frac{d^2 v}{dx^2} + kv = 0 \tag{11.14}$$

11.50 Show that the buckling load for the column on an elastic foundation described in Problem 11.49 is given by the eigenvalues

$$P_n = \frac{\pi^2 EI}{L^2}\left[n^2 + \frac{1}{n^2}\left(\frac{kL^4}{\pi^4 EI}\right)\right], \qquad n = 1,2,3,\ldots \tag{11.15}$$

Note: For $n = 1$ and $k = 0$ Equation (11.15) gives the Euler buckling load.

11.51 For a simply supported column with a symmetric composite cross section, show that the critical load P_{cr} is given by

$$P_{cr} = \frac{\pi^2 \sum_{i=1}^{n} E_i I_i}{L_{eff}^2} \tag{11.16}$$

where L_{eff} = the effective length of the column, E_i is the modulus of elasticity for the i^{th} material, I_i is the area moment of inertia about the buckling axis, and n is the number of materials in the cross section. [See Equations (6.36) and (11.3a).]

11.52 A composite column has the cross section shown in Figure P11.52. The modulus of elasticity of the outside material is twice that of the inside material. In terms of *E*, *d*, and *L*, determine the critical buckling load.

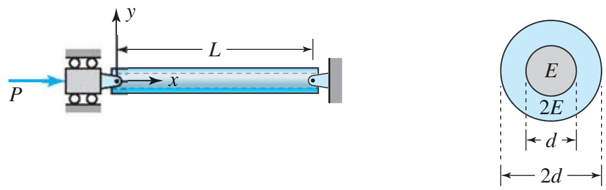


Figure P11.52

11.53 Two strips of material of a modulus of elasticity of 2*E* are attached to a material with a modulus of elasticity *E* to form a composite cross section of the column shown in Figure P11.53. In terms of *E*, *a*, and *L*, determine the critical buckling load. The column is free to buckle in any direction.

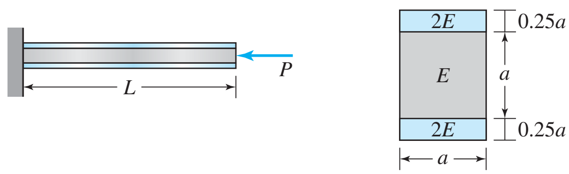


Figure P11.53

11.3* IMPERFECT COLUMNS

In the development of the theory for axial members and the symmetric bending of beams, we obtained that the condition for decoupling axial deformation from bending deformation for linear, elastic, and homogeneous material: the applied loads must pass through the centroid of the cross sections, and the centroids of all cross sections are on a straight line. However, the requirements for decoupling the axial from the bending problem may not be met for a number of reasons, some of which are given here:

- The column material may contain small holes, minute cracks, or other material inclusions. Hence the homogeneity requirement or the requirement that the centroids of all cross sections be on a straight line may not be met.
- The material processing may cause local strain hardening. Hence the condition of linear and elastic material behavior across the entire cross section may not be met.
- The theoretical design centroid and the actual centroid are offset due to manufacturing tolerances.
- Local conditions at the support cause the reaction force to be offset from the centroid.
- The transfer of loads from one member to another may not occur at the centroid.

This partial list can be considered as imperfections in the column, which cause the application of axial loads to be offset from the centroid of the cross section. This offset loading is termed **eccentric loading** on columns. In this section we study the impact of eccentricity in loading on buckling.

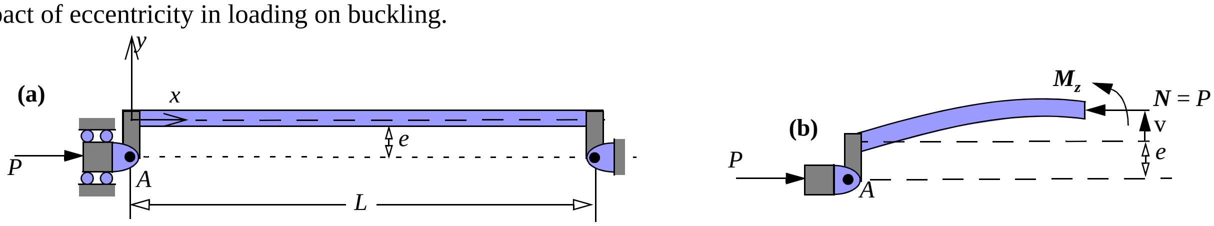


Figure 11.17 Eccentrically loaded column.

Figure 11.17*a* shows a simply supported column on which an eccentric compressive axial load is applied at a distance *e* from the centroid of the cross section. Figure 11.17*b* shows the free-body diagram of the column segment. By balancing the moment at point A we obtain $\boldsymbol{M}_z + P(\mathrm{v} + e) = 0$. Substituting the moment–curvature relationship of Equation (7.1), we obtain the differential equation

$$\frac{d^2\mathrm{v}}{dx^2} + \lambda^2\mathrm{v} = -\frac{Pe}{EI} \tag{11.17}$$

where λ is given by Equation (11.3c). The boundary conditions are that displacements at $x = 0$ and $x = L$ are zero, as given by Equation (11.4a) and (11.4b). The homogeneous solution to Equation (11.17) is given by Equation (11.5), that is, $\mathrm{v}_H(x) = A\cos\lambda x + B\sin\lambda x$. The particular solution to Equation (11.17) is $\mathrm{v}_P(x) = -e$. Thus the total solution $\mathrm{v}_H + \mathrm{v}_P$ is

$$\mathrm{v}(x) = A\cos\lambda x + B\sin\lambda x - e \tag{11.18}$$

From boundary condition (11.4*a*) we obtain

$$\mathrm{v}(0) = A\cos(0) + B\sin(0) - e = 0 \qquad \text{or} \qquad A = e \tag{11.19a}$$

From boundary condition (11.4*b*) we obtain

$$\mathrm{v}(L) = A\cos\lambda L + B\sin\lambda L - e = 0 \qquad \text{or} \tag{11.19b}$$

$$B = \frac{e(1-\cos\lambda L)}{\sin\lambda L} = \frac{e\left(2\sin^2\frac{\lambda L}{2}\right)}{2\sin\frac{\lambda L}{2}\cos\frac{\lambda L}{2}} = e\tan\frac{\lambda L}{2} \tag{11.19c}$$

Substituting for A and B in Equation (11.18), we obtain the deflection as

$$\mathrm{v}(x) = e\left[\cos\lambda x + \tan\left(\frac{\lambda L}{2}\right)\sin\lambda x - 1\right] \tag{11.20}$$

As $\lambda L/2 \to \pi/2$, the function $\tan(\lambda L/2) \to \infty$ and the displacement function $\mathrm{v}(x)$ becomes unbounded. Thus the critical load value can be found by substituting for λ in the equation $\lambda L/2 = \pi/2$ to obtain the same critical value as given by Equation (11.9). In other words, *the buckling load value does not change with the eccentricity of the loading*. We will make use of this observation to extend our formulas to other types of support conditions.

In the eigenvalue approach discussed in Section 11.2, we were unable to determine the displacement function because we had an undetermined constant B in Equation (11.10). But here the displacement function is completely determined by Equation (11.20). The maximum deflection (by symmetry) will be at the midpoint. Substituting $x = L/2$ into Equation (11.20), we obtain

$$\mathrm{v}_{\max} = e\left[\cos\left(\frac{\lambda L}{2}\right) + \tan\left(\frac{\lambda L}{2}\right)\sin\left(\frac{\lambda L}{2}\right) - 1\right] \tag{11.21a}$$

Using trigonometric identities, this equation can be simplified as $\mathrm{v}_{\max} = e[\sec(\lambda L/2) - 1]$. Substituting for λ from Equation (11.3c), we obtain

$$\mathrm{v}_{\max} = e\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right) - 1\right] \tag{11.21b}$$

We can write

$$\sqrt{\frac{P}{EI}} = \sqrt{\frac{PP_{cr}}{P_{cr}EI}} = \frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}} \tag{11.21c}$$

We obtain the maximum deflection equation as

$$\boxed{\mathrm{v}_{\max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right]} \tag{11.22}$$

The maximum normal stress is the sum of compressive axial stress and maximum compressive bending stress:

$$\sigma_{\max} = \frac{P}{A} + \frac{\boldsymbol{M}_{\max}y_{\max}}{I} \tag{11.23a}$$

The maximum bending moment will be at the midpoint of the column, and its value is $\boldsymbol{M}_{max} = P(e + v_{max})$. Substituting for v_{max} we obtain

$$\sigma_{max} = \frac{P}{A} + \frac{Py_{max}}{I}e\left[\sec\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right)\right] \tag{11.23b}$$

Equation (11.23b) was derived for simply supported columns. We can extend the results to other supports by changing the length of the column to the effective length L_{eff}, as given in Table 11.1. We also substitute $y_{max} = c$, where c represents the maximum distance from the buckling (bending) axis to a point on the cross section. Substituting $I = Ar^2$, where A is the cross-sectional area and r is the radius of gyration, we obtain

$$\boxed{\sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\frac{P}{EA}}\right)\right]} \tag{11.24}$$

Equation (11.24) is called the *secant formula*. The quantity ec/r^2 is called the *eccentricity ratio*.

By equating σ_{max} to failure stress σ_{fail} in Equation (11.24), we obtain the failure envelope for an imperfect column. The failure envelope equation can be written in nondimensional form as

$$\frac{P/A}{\sigma_{fail}}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\left(\frac{\sigma_{fail}}{E}\right)\frac{P/A}{\sigma_{fail}}}\right)\right] = 1 \tag{11.25}$$

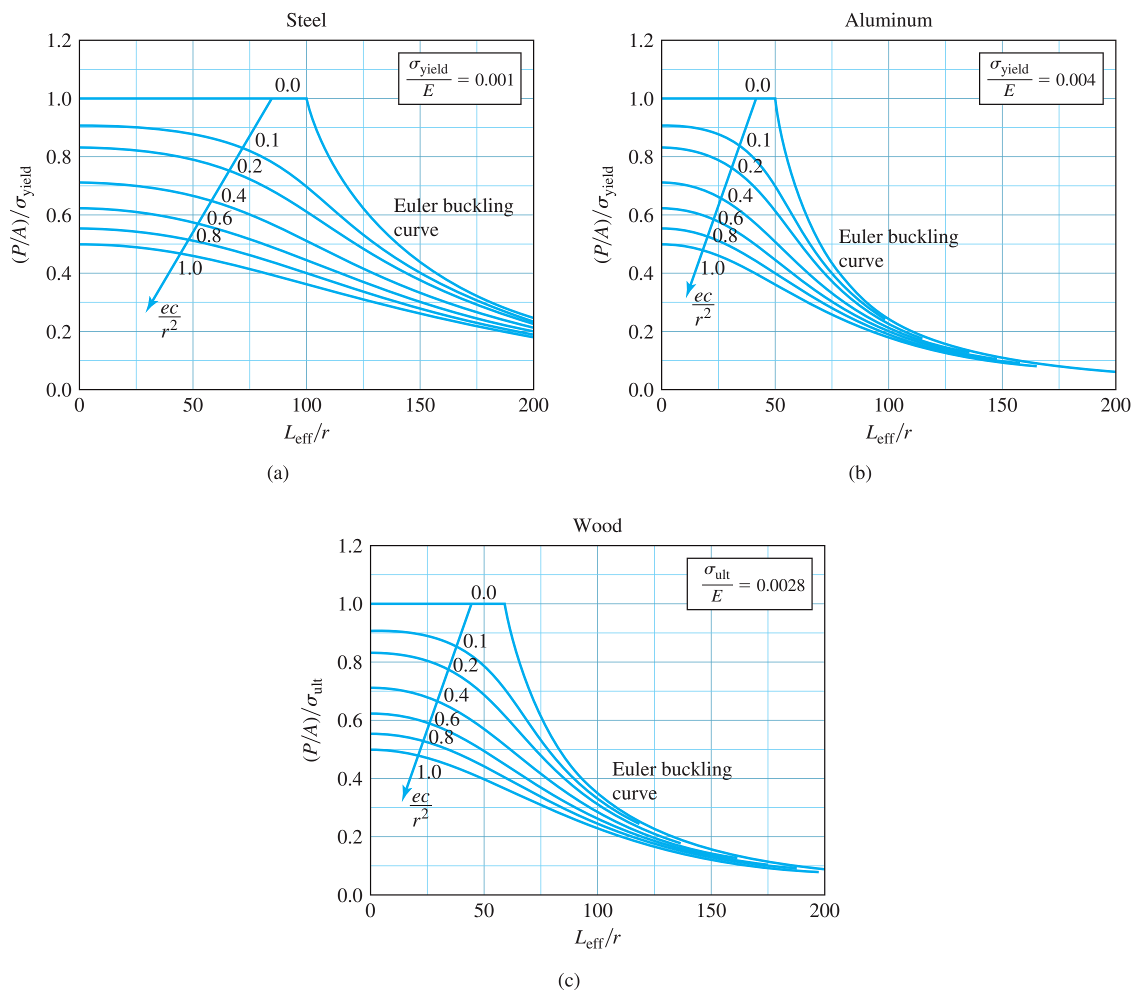


Figure 11.18 Failure envelopes for imperfect columns.

Equation (11.25) can be plotted for different materials, as shown in Figure 11.18. These curves can be used for metric as well for U.S. customary units, since the variables used in creating the plots are nondimensional. The curves can be used for any material that has the same value for σ_{yield}/E. The failure stress in the cases of steel and aluminum would be the yield stress σ_{yield}, whereas for wood it would be the ultimate stress σ_{ult}. The curves can also be used for different end conditions by using the appropriate L_{eff} as given in Table 11.1.

EXAMPLE 11.5

A wooden box column (E = 1800 ksi) is constructed by joining four pieces of lumber together, as shown in Figure 11.19. The load P = 80 kips is applied at a distance of e = 0.667 in. from the centroid of the cross section. (a) If the length is L = 10 ft, what are the maximum stress and the maximum deflection? (b) If the allowable stress is 3 ksi, what is the maximum permissible length L to the nearest inch?

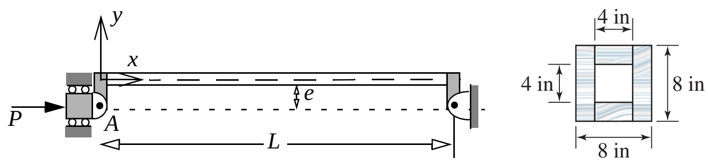


Figure 11.19 Eccentrically loaded box column.

PLAN

The cross-sectional area A, the area moment of inertia I, the radius of gyration r, and the maximum distance c from the bending (buckling) axis can be found from the cross-section dimensions. The effective length is the actual length L as the column is pin held at each end. (a) Substituting L_{eff} = 120 in. and the values of the other variables into Equations (11.22) and (11.24), we can find the maximum stress and the maximum deflection. (b) Equating σ_{max} in Equation (11.24) to 3 ksi and substituting the remaining variables, we find the length L.

SOLUTION

From the given cross section, the cross-sectional area A, the area moment of inertia I, and the radius of gyration r can be found:

$$A = (8\text{ in.})(8\text{ in.}) - (4\text{ in.})(4\text{ in.}) = 48\text{ in.}^2 \qquad I = \frac{1}{12}[(8\text{ in.})^4 - (4\text{ in.})^4] = 320\text{ in.}^4 \tag{E1}$$

$$r = \sqrt{\frac{I}{A}} = 2.582\text{ in.} \tag{E2}$$

(a) Since the column is pinned at both ends, $L_{eff} = L = 10\text{ ft} = 120\text{ in.}$ Substituting L_{eff}, I, and E = 1800 ksi into Equation (11.11) give the critical buckling load:

$$P_{cr} = \frac{\pi^2(1800\text{ ksi})(320\text{ in.}^4)}{(120\text{ in.})^2} = 394.8\text{ kips} \tag{E3}$$

Substituting e = 0.667in., P = 80 kips, and Equation (E3) into Equation (11.22), we obtain the maximum deflection,

$$v_{max} = (0.667\text{ in.})\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{80\text{ kips}}{394.8\text{ kips}}}\right) - 1\right] = 0.2103\text{ in.} \tag{E4}$$

ANS. $v_{max} = 0.21$ in.

Substituting c = 4 in., e = 0.667 in., r = 2.582 in., P = 80 kips, E = 1800 ksi, and A = 48 in.^{2} into Equation (11.24), we obtain the maximum normal stress,

$$\sigma_{max} = \frac{80\text{ kips}}{48\text{ in.}^2}\left[1 + \frac{(0.667\text{ in.})(4\text{ in.})}{(2.582\text{ in.})^2}\sec\left(\frac{120\text{ in.}}{2(2.582\text{ in.})}\sqrt{\frac{80\text{ kips}}{(1800\text{ ksi})(48\text{ in.}^2)}}\right)\right] = 2.544\text{ ksi} \tag{E5}$$

ANS. $\sigma_{max} = 2.5$ ksi (C)

(b) Substituting σ_{max} = 3 ksi, c = 4 in., e = 0.667 in., r = 2.582 in., P = 80 kips, E = 1800 ksi, and A = 48 in.^{2} into Equation (11.24), we can find $L_{eff} = L$ in. can be found,

$$3 = \frac{80\text{ kips}}{48\text{ in.}^2}\left[1 + \frac{(0.667\text{ in.})(4\text{ in.})}{(2.582\text{ in.})^2}\sec\left\{\frac{L\text{ in.}}{2(2.582\text{ in.})}\sqrt{\frac{80\text{ kips}}{(1800\text{ ksi})(48\text{ in.}^2)}}\right\}\right] \tag{E6}$$

$$\sec\{5.892(10^{-3})L\} = 2 \qquad \text{or} \qquad \cos(5.892\times 10^{-3}L) = 0.5 \qquad \text{or} \qquad L = 177.7\text{ in.} \tag{E7}$$

Rounding downward, the maximum permissible length is: thus L = 177 in.

ANS. L = 177 in.

COMMENTS

1. The axial stress $P/A = (80 \text{ kips})/(48 \text{ in.}^2) = 1.667$ ksi, but the normal stress due to bending from eccentricity causes the normal stress to be significantly higher, as seen by the value of σ_{max}.
2. If the right end of the column shown in Figure 11.19 were built in rather than held by a pin, then from case 3 in Table 11.1, $L_{eff} = 0.7L = 84$ in. Using this value, we can find $P_{cr} = 805.7$ kips, $v_{max} = 0.091$ in., and $\sigma_{max} = 2.42$ ksi.
3. In Equation (E7) we rounded downward, as shorter columns will result in a stress that is less than allowable.

EXAMPLE 11.6

A wooden box column ($E = 1800$ ksi) is constructed by joining four pieces of lumber together, as shown in Figure 11.19. The ultimate stress is 5 ksi. Determine the maximum load P that can be applied.

PLAN

The eccentricity ratio and the slenderness ratio can be found using the values of the geometric quantities calculated in Example 11.5. Noting that $\sigma_{ult}/E = 0.0028$, the failure envelopes for wood that are shown in Figure 11.18 can be used and $(P/A)/\sigma_{ult}$ can be found, from which the maximum load P can be determined.

SOLUTION

From Equation (E2) in Example 11.5, $r = 2.582$ in. Thus the slenderness ratio $L_{eff}/r = (120 \text{ in.})/(2.582 \text{ in.}) = 46.48$. From Figure 11.19, $c = 4$ in. and $e = 0.667$ in. Thus the eccentricity ratio $ec/r^2 = 0.400$.

For a slenderness ratio of 46.48 and an eccentricity ratio of 0.4, we estimate the value of $(P/A)/\sigma_{ult} = 0.6$ from the failure envelope for wood in Figure 11.18. Substituting $\sigma_{ult} = 5$ ksi and $A = 48 \text{ in.}^2$, we obtain the maximum load $P_{max} = (0.6)(5 \text{ ksi})(48 \text{in.}^2)$.

ANS. $P_{max} = 144$ kips

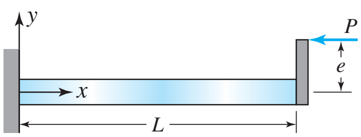
COMMENT

1. If we let x represent $(P/A)/\sigma_{ult}$ and substitute the remaining variables in Equation (11.25), we obtain the following nonlinear equation: $x[1 + 0.4\ \sec(1.2297\sqrt{x})] = 1$. The root of the equation can be found using a numerical method such as discussed in Section B.2.2. The value of the root to the third-place decimal is 0.593, which would yield a value of $P_{max} = 142.3$ kips, a difference of 1.18% from that reported in our example. The difference is small and an acceptable engineering approximation. Use of the plots in Figure 11.18 was a quick way of finding the load value with reasonable engineering approximation.

PROBLEM SET 11.3

Imperfect columns

11.54 A column built in on one end and free at the other end has a load that is eccentrically applied at a distance e from the centroid, as shown in Figure P11.54. Show that the deflection curve is given by the equation below.



$$v(x) = \frac{e(1 - \cos \lambda x)}{\cos \lambda L}$$

Figure P11.54

where λ is as given by Equation (11.3c).

11.55 On the cylinder shown in Figure P11.55 the applied load $P = 3$ kips, the length $L = 5$ ft, and the modulus of elasticity $E = 30{,}000$ ksi. What are the maximum stress and the maximum deflection?

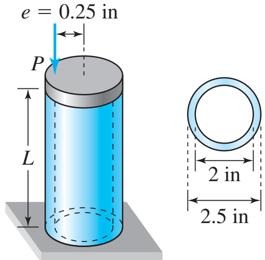


Figure P11.55

11.56 On the cylinder shown in Figure P11.55 the applied load $P = 3$ kips and the modulus of elasticity $E = 30{,}000$ ksi. If the allowable normal stress is 8 ksi, what is the maximum permissible length L of the cylinder?

11.57 The length of the cylinder shown in Figure P11.55 is $L = 5$ ft. The yield stress of steel used in the cylinder is 30 ksi, and the modulus of elasticity $E = 30{,}000$ ksi. Determine the maximum load P that can be applied. Use the plot for steel in Figure 11.18.

11.58 On the column shown in Figure P11.58 the applied load $P = 100$ kN, the length $L = 2.0$ m, and the modulus of elasticity $E = 70$ GPa. What are the maximum stress and the maximum deflection?

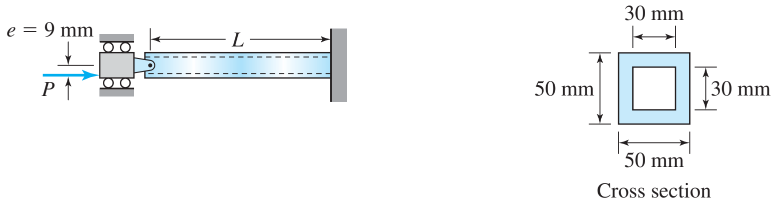


Figure P11.58

11.59 On the column shown in Figure P11.58 the applied load $P = 100$ kN and the modulus of elasticity $E = 70$ GPa. If the allowable normal stress is 250 MPa, what is the maximum permissible length L of the column?

11.60 The length of the column shown in Figure P11.58 is $L = 2.0$ m. The yield stress of aluminum used in the column is 280 MPa, and the modulus of elasticity $E = 70$ GPa. Determine the maximum load P that can be applied. Use the plot for aluminum in Figure 11.18.

11.61 A wide-flange W8 × 18 member is used as a column, as shown in Figure P11.61. The applied load $P = 20$ kips, the length $L = 9$ ft, and the modulus of elasticity $E = 30{,}000$ ksi. What are the maximum stress and the maximum deflection?

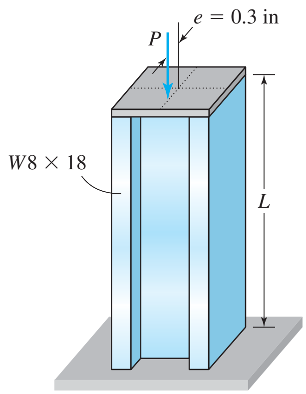


Figure P11.61

11.62 On the column shown in Figure P11.61 the applied load $P = 20$ kips and the modulus of elasticity $E = 30{,}000$ ksi. If the allowable normal stress is 24 ksi, what is the maximum permissible length L of the column?

11.63 The length of the column shown in Figure P11.61 is $L = 9$ ft. The yield stress of steel is 30 ksi, and the modulus of elasticity $E = 30{,}000$ ksi. Determine the maximum load P that can be applied. Use the plot for steel in Figure 11.18.

Stretch yourself

In Problems 11.64 and 11.65, the critical stress in intermediate columns *is between yield stress and ultimate stress. The* tangent modulus theory *of buckling accounts for it by replacing the modulus of elasticity by the tangent modulus of elasticity (see Figure 3.7), that is,*

$$P_{cr} = \frac{\pi^2 E_t I}{L_{eff}^2} \tag{11.26}$$

where E_t is the tangent modulus, which depends on the stress level P_{cr}/A. Using an iterative trial and error procedure and Equation (11.26), the critical buckling load can be determined.

11.64 A simply supported 6-ft pipe has an outside diameter of 3 in. and a thickness of $\frac{1}{8}$ in. The pipe material has the stress–strain curve shown in Figure P11.64. Using Equation (11.26), determine the critical buckling load.

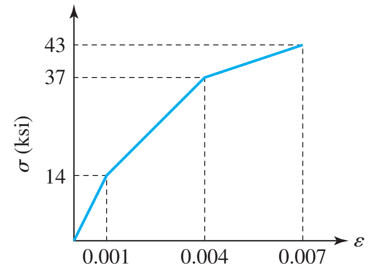


Figure P11.64

11.65 A square box column is constructed from a sheet of 10-mm thickness. The outside dimensions of the square are 75 mm × 75 mm and the column has a length of 0.75 m. The material stress–strain curve is approximated as shown in Figure P11.65. Using Equation (11.26), determine the critical buckling load.

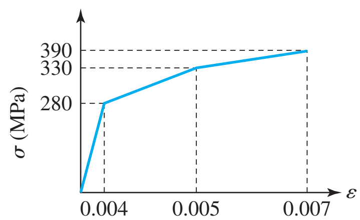


Figure P11.65

11.66 A column that is pin held at its ends has a small initial curvature, which is approximated by the sine function shown in Figure P11.66. Show that the elastic curve of the column is given by the equation below.

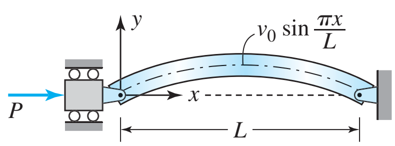


Figure P11.66

$$v(x) = \frac{v_0}{1 - P/P_{cr}} \sin \frac{\pi x}{L}$$

11.67 In *double modulus theory,* also known as *reduced modulus theory* for intermediate columns, it is recognized that the bending action during buckling increases the compressive axial stress on the concave side of the beam but decreases the compressive stress on the convex side of the beam. Thus the use of the tangent modulus of elasticity E_t is appropriate on the concave side, but on the convex side of the beam it may be better to use the original modulus of elasticity. Modeling the cross section material with the two moduli E_t and E and using Equation (11.26), show

$$P_{cr} = \frac{\pi^2 E_r I}{L_{eff}^2} \qquad E_r = E_t \frac{I_1}{I} + E \frac{I_2}{I} \tag{11.27}$$

where E_r is the *reduced modulus of elasticity,* I_1 and I_2 are the moments of inertia of the areas on the concave and convex sides of the axis passing through the centroid, and I is the moment of inertia of the entire cross section.

Computer problems

11.68 A circular marble column of 2-ft diameter and 20-ft length has a load P applied to it at a distance of 2 in. from the center. The modulus of elasticity is 8000 ksi and the allowable stress is 20 ksi. Determine the maximum load P the column can support, assuming that both ends are (a) pinned; (b) built in.

11.69 Determine the maximum load P to the nearest newton in Problem 11.60.

11.70 Determine the maximum load P to the nearest pound in Problem 11.63.

MoM in Action: Collapse of World Trade Center

On September 11, 2001, at 8:46 A.M, five terrorists flew a plane (Figure 11.20*a*) containing 10,000 gallons of fuel into tower 1 of the World Trade Center (WTC 1). Seventeen minutes later, five other terrorists flew a second plane containing 9,100 gallons of fuel into tower 2 (WTC 2). Within an hour, the floors of WTC 2 started collapsing vertically downward, and WTC 1 collapsed just 29 minutes later. A total of 2749 people apart from the terrorists died that day in New York. It is a tragic story of how social forces affect engineering design.

The construction of WTC complex began in 1968. The twin towers were to be the symbol of world commerce and for years the world's tallest buildings, at 110 stories each. Their innovative design maximized usable space by having all supporting columns only on the perimeter of each floor. There were 4 major structural subsystems: (i) the exterior wall (Figure 11.20*b*), with 59 columns on each side; (ii) a rectangular inner core of 47 columns; (iii) a system of bridging steel trusses (Figure 11.20*c*) on each floor, connecting the exterior wall to the inner core using angle clips. Viscoelastic dampers reduced the swaying motion on higher floors due to wind; and (iv) a truss system between 107th and 110th floor—further bridged the inner core to the exterior wall.

The exterior wall (like flanges in beam cross section increase area moment of inertia) was designed to resist the force of 140-mph hurricane winds. The inner core, like an axial column, supported most of the weight of building, equipment, and people. Insulation on the steel and a sprinkler system in the event of fire met building codes at that time. The design even planned for the impact of an airliner lost in fog, and it stood long enough so that most of the 14,500 people in the towers escaped that morning. But it was not designed for a Molotov cocktail of 10,000 gallons of jet fuel.

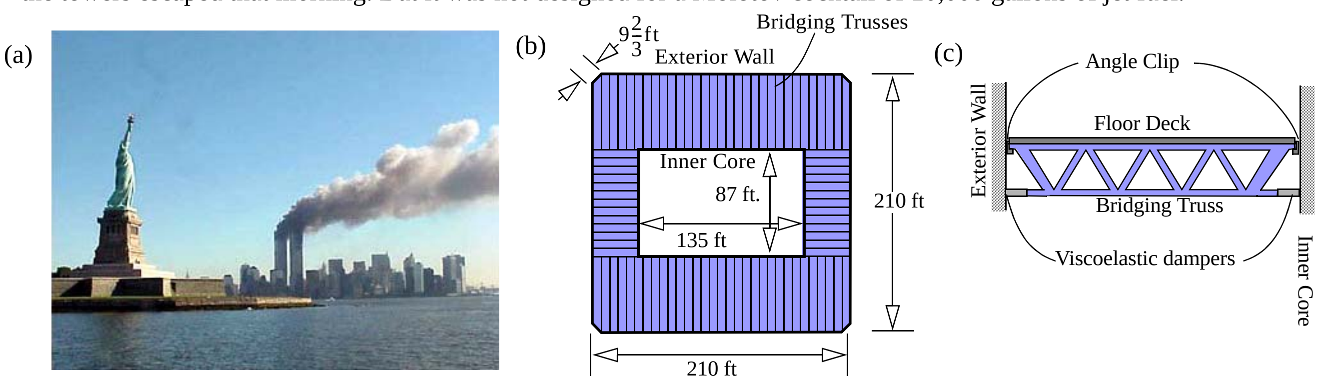


Figure 11.20 (a) World Trade Center Towers; (b) Plan form; (c) Floor.

Even so, it took several factors to initiate the collapse. First, the angle clips on the exterior wall of several floors—at the height of plane—broke, transferring the floors' weight as compressive loads to the inner core. Second, the breaking of the clips in turn removed elastic support from the core column, decreasing the critical buckling loads (see Figure 11.8). Third, the temperature increase from burning fuel introduced another mechanism of buckling failure (see Problem 11.48). Finally, the insulation of the inner core on the floors of impact broke, exposing the steel to high temperatures. This significantly decreased the modulus of elasticity, the critical buckling load, and the ultimate strength. The towers would have survived the first three failures. But the design did not take into account prolonged high temperature and its impact on the stiffness and strength of steel. No one could imagine a fuel-laden plane deliberatively flown into a building.

The floors suffering a direct impact buckled after nearly an hour of intense fire, and the floors above started falling on the weakened floors below. The moving mass of the floors gathered momentum, lending their downward motion to the floors below.

The WTC towers were well designed for the physical forces conceivable at the time. New skyscraper designs will incorporate greater insulation on the steel beams and columns to counter future threats. A collapse happened, but it will happen no more.

*11.4 CONCEPT CONNECTOR

As with the deflection of beams (Chapter 7), mathematicians played a key role in developing the theory of buckling. The history of buckling also shows that original ideas are not enough if the ideas cannot be communicated to others. The importance to engineering of oral and written skills in technical communications is a thus lesson over two hundred years old.

11.4.1 History: Buckling

Leonard Euler (1707–1782) is one of the most prolific mathematicians who ever lived (Figure 11.21). Born in Basel, he went to the University of Basel, then renowned for its research in mathematics. After studying under John Bernoulli (see Section 7.6), he started work in 1727 at the Russian Academy at St. Petersburg, where he developed analytical methods for solving mechanics problems. At the invitation of King Frederick II of Prussia, he moved to Berlin in 1741, where he wrote his book-length *Introduction to Calculus*, *Differential Calculus*, and *Integral Calculus*, in addition to his remarkable original contributions to mathematics. In 1766, Catherine II, the empress of Russia, wooed him back to St. Petersburg. Even as he was going blind from cataract, he continued his prolific publications with the help of assistants. In fact, with a bibliography that runs to 866 entries, one could easily miss his pioneering insight into buckling and the formula he derived [Equation (11.9)].


Leonard Euler.


Joseph-Louis Lagrange.

Figure 11.21 Buckling theory pioneers.

Even after Euler, the early development of buckling was primarily mathematical. Joseph-Louis Lagrange (1736–1813), another pioneer in the establishment of analytical methods for mechanics (Figure 11.21), took the next step. He developed a complete set of buckling loads and the associated buckling modes given by Equations ((11.8) and (11.10). Columns with eccentric loads (Problem 11.54) and columns with initial curvatures (Problem 11.66) were first formulated and studied by Thomas Young (1773–1829). Young was also the first to consider columns of variable cross section. Unfortunately, he was neither a good teacher nor a writer, and much of his work went unappreciated. As his biographer, Lord Rayleigh, said,[3] "Young.... from various causes did not succeed in gaining due attention from his contemporaries. Positions that he had already occupied were in more than one instance reconquered by his successors at great expense of intellectual energy."

There was another reason why in the early 1800s developments in column buckling were unappreciated by the practicing engineer. Euler buckling did not accurately predict compression failure in the structural members then in use. The effects of end conditions and imperfections, as well as the formula's range of validity, were not yet understood It took the experiments of Eaton Hodgkinson in 1840 on cast-iron columns to give new life to the Euler buckling theory. In 1845, Anatole Henri Ernest Lamarle, a French engineer, proposed correctly that the Euler formula should be used below the proportional limit, while experimentally determined formulas should be used for shorter columns.

In 1889 F. Engesser, a German engineer, proposed the *tangent modulus theory* (see Problems 11.64 and 11.65), in which the elastic modulus is replaced by the tangent modulus of elasticity when proportional stress is exceeded. Also in 1889, the French engineer A. G. Considère, based on a series of tests, proposed that if buckling occurs above yield stress, then the elastic modulus in the Euler formula should be replaced by a reduced modulus of elasticity, between the elastic modulus and the tangent modulus. On learning of Considère's work, Engesser incorporated the suggestion into his *reduced modulus theory*, also known as *double modulus theory* (see Problem 11.66). Yet the two approaches competed for almost 50 years. In 1905 J. B. Johnson, C. W. Bryan, and F. E. Turneaure recommended a modification of the Euler formula for steel columns, using an

[3]Quotation is from S. P. Timoshenko, *History of Strength of Materials.*

experimentally determined constant for different supports. It was the beginning of the concept of *effective length* to account for different end conditions, and their text on *Theory and Practice of Modern Framed Structures* remained in print for ten editions. In 1946 F. R. Shanley, an American aeronautical engineering professor, refined these theories and finally resolved “the column paradox,” as he called it, that had separated proponents of the reduced modulus theory and the tangent modulus theory.

For all its refinements and limitations, the Euler buckling formula is still used three centuries later for column design and is still valid for long columns with pin-supported ends. Such is the power of logical thinking.

11.5 CHAPTER CONNECTOR

In past chapters, our analysis was based on the equilibrium of forces and moments. This chapter emphasized that not only equilibrium, but the *stability* of the equilibrium is an important consideration in design. There are many types of instabilities. We studied how coupled axial and bending deformation, for example, can produce buckling in columns. This case emphasizes the need for caution in decoupling phenomena for ease of understanding.

All our theories have relied on an equilibrium approach. An alternative approach can be used to replace the last link in the logic Figure 3.12. Though our theories will have the same assumptions and limitations, the energy method has a very different perspective from equilibrium methods, as discussed in the next and last chapter of this book.

POINTS AND FORMULAS TO REMEMBER

- Buckling is the instability in equilibrium of a structure due to compressive forces or stresses.
- Structural members that support compressive axial loads are called columns.
- Study of buckling as a *bifurcation* problem requires determining the critical buckling load at the point where two or more solutions exist for deformation.
- Study of buckling by the *energy method* requires determining the critical buckling load at the point the potential energy changes from a concave to a convex function.
- Study of buckling as an *eigenvalue* problem requires determining the critical buckling load at the point where a nontrivial solution exists for bending deformation due to axial loading.
- In *snap buckling* the structure snaps(or jumps) from one equilibrium configuration to a very different equilibrium configuration at the critical buckling load.
- *Local buckling* of thin structural members occurs due to compressive stresses.
- Buckling of columns occurs about an axis that has a minimum value of area moment of inertia.
- The Euler buckling load is $P_{cr} = \dfrac{\pi^2 EI}{L^2}$ (11.9)

  where P_{cr} is the critical buckling load, E is the modulus of elasticity, L is the length of the column, and I is the minimum area moment of inertia of the cross section.
- Equation (11.9) is valid only for elastic columns with pin-held ends.
- The effect of supports at the end can be incorporated by defining an effective length L_{eff} for a column and calculating the critical buckling load from

$$P_{cr} = \frac{\pi^2 EI}{L_{eff}^2} \qquad (11.11)$$

- The *Slenderness ratio* is defined as L_{eff}/r, where r is the radius of gyration about the buckling axis.
- The slenderness ratio at which the maximum normal stress is equal to the yield stress separates the short columns from the long columns in Euler buckling.
- The failure of short columns is governed by material strength.
- The failure of long columns is governed by Euler buckling loads.
- Eccentricity in loading does not affect the critical buckling load, but the maximum normal stress becomes significantly larger than the axial stress due to the addition of bending normal stress,

$$v_{max} = e\left[\sec\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] \quad (11.22) \qquad \sigma_{max} = \frac{P}{A}\left[1 + \frac{ec}{r^2}\sec\left(\frac{L_{eff}}{2r}\sqrt{\frac{P}{EA}}\right)\right] \quad (11.24)$$

  where v_{max} is the maximum deflection, e is the eccentricity in loading, P is the applied axial load, P_{cr} is the Euler buckling load for the column, σ_{max} is the maximum normal stress in the column, r is the radius of gyration about the buckling (bending) axis, c is the maximum distance perpendicular to the buckling (bending) axis, A is the cross-sectional area, and L_{eff} is the effective length of the column.
- The *eccentricity ratio* is defined as ec/r^2.

APPENDIX A
STATICS REVIEW

Statics is the foundation course for mechanics of materials. This appendix briefly reviews statics from the perspective of this course. It presupposes that you are familiar with the basic concepts so if you took a course in statics some time ago, then you may need to review your statics textbook along with this brief review. Review exams at the end of this appendix can also be used for self-assessment.

A.1 TYPES OF FORCES AND MOMENTS

We classify the forces and moments into three categories: external, reaction, and internal.

A.1.1 External Forces and Moments

External forces and moments are those that are applied to the body and are often referred to as the *load* on the body. They are assumed to be known in an analysis, though sometimes we carry external forces and moments as variables. In that way we may answer questions such these: How much load can a structure support? What loads are needed to produce a given deformation?

Surface forces and moments are external forces (moments), which act on the surface and are transmitted to the body by contact. Surface forces (moments) applied at a point are called *concentrated* forces (moment or couple). Surface forces (moments) applied along a line or over a surface are called *distributed* forces (moments).

Body forces are external forces that act at every point on the body. Body forces are not transmitted by contact. Gravitational forces and electromagnetic forces are two examples of body forces. A body force has units of force per unit volume.

A.1.2 Reaction Forces and Moments

Other forces and moments are developed at the supports of a body to resist movement due to the external forces (moments). These *reaction* forces (moments) are usually not known and must be calculated before further analysis can be conducted. Three principles are used to decide whether there is a reaction force (reaction moment) at the support:

1. If a point cannot move in a given direction, then a reaction force opposite to the direction acts at that support point.
2. If a line cannot rotate about an axis in a given direction, then a reaction moment opposite to the direction acts at that support.
3. The support in isolation and not the entire body is considered in making decisions about the movement of a point or the rotation of a line at the support. Exceptions to the rule exist in three-dimensional problems, such as bodies supported by balanced hinges or balanced bearings (rollers). These types of three-dimensional problems will not be covered in this book.

Table C.1 shows several types of support that can be replaced by reaction forces and moments using the principles described above.

A.1.3 Internal Forces and Moments

A body is held together by *internal* forces. Internal forces exist irrespective of whether or not we apply external forces. The material resists changes due to applied forces and moments by increasing the internal forces. Our interest is in the resistance the material offers to the applied loads—that is, in the internal forces. Internal forces always exist in pairs that are equal and opposite on the two surfaces produced by an imaginary cut.

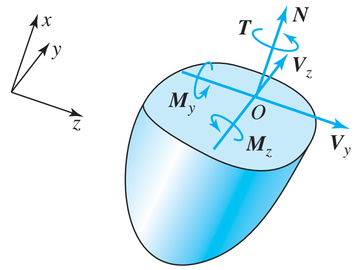


Figure A.1 Internal forces and moments.

The internal forces are shown in Figure A.1. (In this book, all internal forces and moments are printed in bold italics: $\boldsymbol{N}$ = axial force; $\boldsymbol{V}_y$, $\boldsymbol{V}_z$ = shear force; $\boldsymbol{T}$ = torque; $\boldsymbol{M}_y$, $\boldsymbol{M}_z$ = bending moment.) They are defined as follows:

- Forces that are normal to the imaginary cut surface are called **normal forces**. The normal force that points away from the surface (pulls the surface) is called **tensile force**. The normal force that points into the surface (pushes the surface) is called **compressive force**.
- The normal force acting in the direction of the axis of the body is called **axial force**.
- Forces that are tangent to the imaginary cut surface are called **shear forces**.
- Internal moments about an axis normal to the imaginary cut surface are called torsional moments or **torque**.
- Internal moments about an axis tangent to the imaginary cut are called **bending moments**.

A.2 FREE-BODY DIAGRAMS

Newton's laws are applicable only to free bodies. By "free" we mean that if a body is not in equilibrium, it will move. If there are supports, then these supports must be replaced by appropriate reaction forces and moments using the principles described in Section A.1.2. The diagram showing all the forces acting on a free body is called the **free-body diagram**.

Additional free-body diagrams may be created by making imaginary cuts for the calculation of internal quantities. Each imaginary cut will produce two additional free-body diagrams. Either of the two free-body diagrams can be used for calculating internal forces and moments.

A body is in static equilibrium if the vector sum of all forces acting on a free body and the vector sum of all moments about any point in space are zero:

$$\sum \bar{F} = 0 \qquad \sum \bar{M} = 0 \tag{A.1}$$

where $\sum$ represents summation and the overbar represents a vector quantity. In a three-dimensional Cartesian coordinate system the equilibrium equations in scalar form are

$$\begin{aligned} \sum F_x &= 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0 \\ \sum M_x &= 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0 \end{aligned} \tag{A.2}$$

Equations (A.2) imply that there are six independent equations in three dimensions. In other words, we can at most solve for six unknowns from a free-body diagram in three dimensions.

In two dimensions the sum of the forces in the z direction and the sum of the moments about the x and y axes are automatically satisfied, as all forces must lie in the x, y plane. The remaining equilibrium equations in two dimensions that have to be satisfied are

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = 0 \tag{A.3}$$

Equations (A.3) imply that there are three independent equations per free-body diagram in two dimensions. In other words, we can at most solve for three unknowns from a free-body diagram in two dimensions.

The following can be used to reduce the computational effort:

- Balance the moments at a point through which the unknown forces passes. These forces do not appear in the moment equation.

- Balance the forces or moments perpendicular to the direction of an unknown force. These forces do not appear in the equation.

A structure on which the number of unknown reaction forces and moments is greater than the number of equilibrium equations (six in three dimensions and three in two dimensions) is called a **statically indeterminate structure**. Statically indeterminate problems arise when more supports than needed are used to support a structure. Extra supports may be used for safety considerations or for the purpose of increasing the stiffness of a structure. We define the following:

Degree of static redundancy = number of unknown reactions – number of equilibrium equations. (A.4)

To solve a statically indeterminate problem, we have to generate equations on the displacement or rotation at the support points. A mistake sometimes made is to take moments at many points in order to generate enough equations for the unknowns. A statically indeterminate problem cannot be solved from equilibrium equations alone. There are only three independent equations of static equilibrium in two dimensions and six independent equations of static equilibrium in three dimensions. Additional equations must come from displacements or rotation conditions at the support.

The number of equations on the displacement or rotation needed to solve a statically indeterminate problem is equal to the degree of static redundancy. There are two exceptions: (i) With symmetric structures with symmetric loadings by using the arguments of symmetry one can reduce the total number of unknown reactions. (ii) *Pin connections* do not transmit moments from one part of a structure to another. Thus it is possible that a seemingly indeterminate pin structure may be a determinate structure. We will not consider such pin-connected structures in this book.

A structural member on which there is no moment couple and forces act at two points only is called a **two-force member.** Figure A.2 shows a two-force member. By balancing the moments at either point A or B we can conclude that the resultant forces at A and B must act along the line joining the two points. Notice that the shape of the member is immaterial. Identifying two-force members by inspection can save significant computation effort.

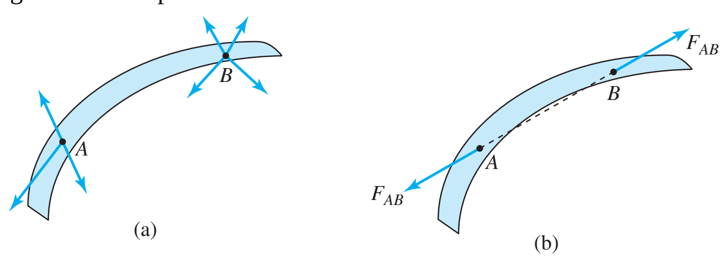


Figure A.2 Two-force member.

A.3 TRUSSES

A **truss** is a structure made up of two-force members. The method of joints and the method of sections are two methods of calculating the internal forces in truss members.

In the method of joints, a free-body diagram is created by making imaginary cuts on all members joined at the pin. If a force is directed away from the pin, then the two-force member is assumed to be in tension; and if it is directed into the pin, then the member is assumed to be in compression. By conducting force balance in two (or three) dimensions two (or three) equations per pin can be written.

In the method of sections an imaginary cut is made through the truss to produce a free-body diagram. The imaginary cut can be of any shape that will permit a quick calculation of the force in a member. Three equations in two dimensions or six equations in three dimensions can be written per free-body diagram produced from a single imaginary cut.

A **zero-force member** in a truss is a member that carries no internal force. Identifying zero-force members can save significant computation time. Zero-force members can be identified by conducting the method of joints mentally. Usually if two members are collinear at a joint *and* if there is no external force, then the zero-force member is the member that is inclined to the collinear members.

A.4 CENTROIDS

The y and z coordinates of the centroid of the two-dimensional body shown in A.3 are defined as

$$y_c = \frac{\int_A y\,dA}{\int_A dA} \qquad z_c = \frac{\int_A z\,dA}{\int_A dA} \tag{A.5}$$

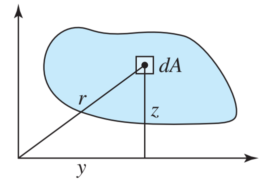


Figure A.3 Area moments.

The numerator in Equations (A.5) is referred to as the first moment of the area. If there is an axis of symmetry, then the area moment about the symmetric axis from one part of the body is canceled by the moment from the symmetric part, and hence we conclude that the centroid lies on the axis of symmetry.

Consider a coordinate system fixed to the centroid of the area. If we now consider the first moment of the area in this coordinate system and it turns out to be nonzero, then it would imply that the centroid is not located at the origin, thus contradicting our starting assumption. We therefore conclude that the first moment of the area calculated in a coordinate system fixed to the centroid of the area is zero.

The *centroid* for a composite body in which the centroids of the individual bodies are known can be calculated from Equations (A.6).

$$y_c = \frac{\sum_{i=1}^{n} y_{c_i} A_i}{\sum_{i=1}^{n} A_i} \qquad z_c = \frac{\sum_{i=1}^{n} z_{c_i} A_i}{\sum_{i=1}^{n} A_i} \tag{A.6}$$

where y_{c_i} and z_{c_i} are the known coordinates of the centroids of the area A_i. Table C.2 shows the locations of the centroids of some common shapes that will be useful in solving problems in this book.

A.5 AREA MOMENTS OF INERTIA

The *area moments of inertia*, also referred to as second area moments, are defined as

$$I_{yy} = \int_A z^2\,dA \qquad I_{zz} = \int_A y^2\,dA \qquad I_{yz} = \int_A yz\,dA \tag{A.7}$$

The polar moment of inertia is defined as in Equation (A.7) with the relation to I_{yy} and I_{zz} deduced using A.3:

$$J = \int_A r^2\,dA = I_{yy} + I_{zz} \tag{A.8}$$

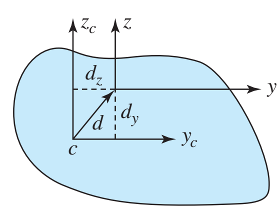


Figure A.4 Parallel-axis theorem.

If we know the area moment of inertia in a coordinate system fixed to the centroid, then we can compute the area moments about an axis parallel to the coordinate axis by the parallel-axis theorem illustrated in Figure A.4 and are given by

$$I_{yy} = I_{y_c y_c} + Ad_y^2 \qquad I_{zz} = I_{z_c z_c} + Ad_z^2 \qquad I_{yz} = I_{y_c z_c} + Ad_y d_z \qquad J = J_c + Ad^2 \tag{A.9}$$

where the subscript c refers to the axis fixed to the centroid of the body. The quantities y^2, z^2, r^2, A, d_y^2, d_z^2, and d^2 are always positive. From Equations (A.7) through (A.9) we conclude that I_{yy}, I_{zz}, and J are always positive and minimum about the axis passing through the centroid of the body. However, I_{yz} can be positive or negative, as y, z, d_y, and d_z can be positive or negative in

Equation (A.7). If either y or z is an axis of symmetry, then the integral in I_{yz} on the positive side will cancel the integral on the negative side in Equation (A.7), and hence I_{yz} will be zero. We record the observations as follows:

- I_{yy}, I_{zz}, and J are always positive and minimum about the axis passing through the centroid of the body.
- If either the y or the z axis is an axis of symmetry, then I_{yz} will be zero.

The moment of inertia of a composite body in which we know the moments of inertia of the individual bodies about its centroid can be calculated from Equation (A.10).

$$I_{yy} = \sum_{i=1}^{n}(I_{y_{c_i}y_{c_i}} + A_i d_{y_i}^2) \qquad I_{zz} = \sum_{i=1}^{n}(I_{z_{c_i}z_{c_i}} + A_i d_{z_i}^2) \qquad I_{yz} = \sum_{i=1}^{n}(I_{y_{c_i}z_{c_i}} + A_i d_{y_i} d_{z_i}) \qquad J = \sum_{i=1}^{n}(J_{c_i} + A_i d_i^2) \tag{A.10}$$

where $I_{y_{c_i}y_{c_i}}$, $I_{z_{c_i}z_{c_i}}$, $I_{y_{c_i}z_{c_i}}$, and J_{c_i} are the area moments of inertia about the axes passing through the centroid of the ith body. Table C.2 shows the area moments of inertia about an axis passing through the centroid of some common shapes that will be useful in solving the problems in this book.

The *radius of gyration* $\hat{r}$ about an axis is defined by

$$\hat{r} = \sqrt{\frac{I}{A}} \qquad \text{or} \qquad I = A\hat{r}^2 \tag{A.11}$$

where I is the area moment of inertia about the same axis about which the radius of gyration $\hat{r}$ is being calculated.

A.6 STATICALLY EQUIVALENT LOAD SYSTEMS

Two systems of forces that generate the same resultant force and moment are called **statically equivalent** load systems. If one system satisfies the equilibrium, then the statically equivalent system also satisfies the equilibrium. The statically equivalent systems simplifies analysis and is often used in problems with distributed loads.

A.6.1 Distributed Force on a Line

Let $p(x)$ be a distributed force per unit length, which varies with x. We can replace this distributed force by a force and moment acting at any point or by a single force acting at point x_c, as shown in Figure A.5.

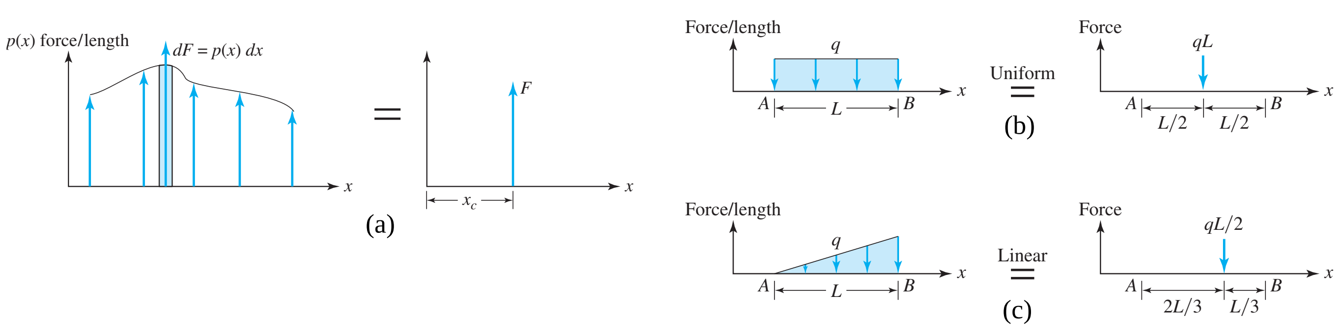


Figure A.5 Static equivalency for (a) distributed force on a line. (b) uniform distribution. (c) linear distribution.

For two systems in Figure A.5*a* to be statically equivalent, the resultant force and the resultant moment about any point (origin) must be the same.

$$F = \int_L p(x)\,dx \qquad x_c = \frac{\int_L x\,p(x)\,dx}{F} \tag{A.12}$$

The force F is equal to the area under the curve and x_c represents the location of the centroid of the *distribution*. This is used in replacing a uniform or a linearly varying distribution by a statically equivalent force, as shown in Figure A.5*b* and *c*.

Two statically equivalent systems are *not* identical systems. The deformation (change of shape of bodies) in two statically equivalent systems is different. The distribution of the internal forces and internal moments of two statically equivalent systems is different. The following rule must be remembered:

- The imaginary cut for the calculation of internal forces and moments must be made on the original body and not on the statically equivalent body.

A.6.2 Distributed Force on a Surface

Let $\sigma(y, z)$ be a distributed force per unit area that varies in intensity with y and z. We would like to replace it by a single force, as shown in Figure A.6.

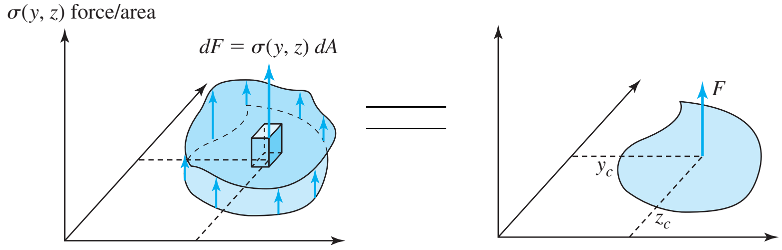


Figure A.6 Static equivalency for distributed force on a surface.

For the two systems shown in Figure A.6 to be statically equivalent load systems, the resultant force and the resultant moment about the y axis and on the z axis must be the same.

$$F = \iint_A \sigma(y, z)\, dy\, dz \qquad y_c = \frac{\iint_A y\,\sigma(y, z)\, dy\, dz}{F} \qquad z_c = \frac{\iint_A z\,\sigma(y, z)\, dy\, dz}{F} \tag{A.13}$$

The force F is equal to the volume under the curve. y_c and z_c represent the locations of the centroid of the *distribution*, which can be different from the centroid of the area on which the distributed force acts. The centroid of the area depends only on the geometry of that area. The centroid of the distribution depends on how the intensity of the distributed load $\sigma(y, z)$ varies over the area.

Figure A.7 shows a uniform and a linearly varying distributed force, which can be replaced by a single force at the centroid of the distribution. Notice that for the uniformly distributed force, the centroid of the distributed force is the same as the centroid of the rectangular area, but for the linearly varying distributed force, the centroid of the distributed force is different from the centroid of the area. If we were to place the equivalent force at the centroid of the area rather than at the centroid of distribution, then we would also need a moment at that point.

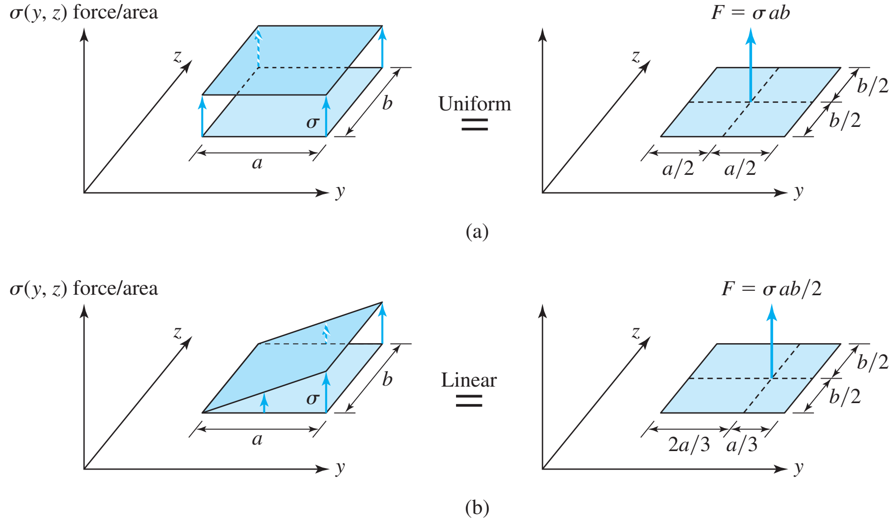
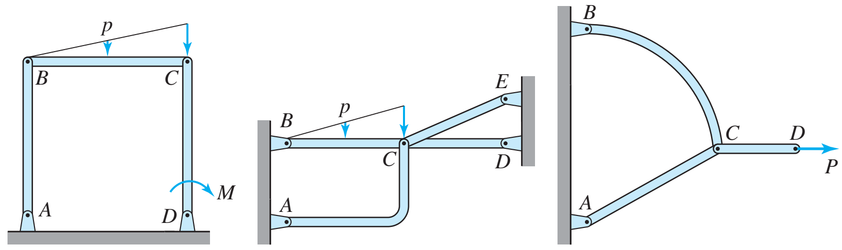


Figure A.7 Statically equivalent force for uniform and linearly distributed forces on a surface.

Quick Test A.1 **Time: 15 minutes/Total: 20 points**

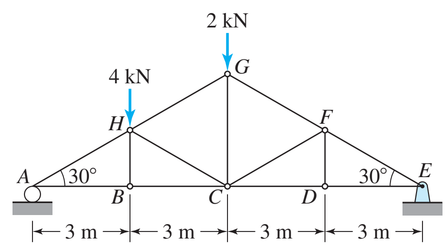
Grade yourself using the answers and points given in Appendix G.

1. Three pin-connected structures are shown: (a) How many two-force members are there in each structure? (b) Which are the two-force member

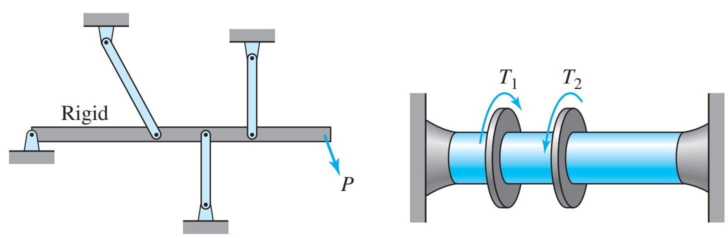


Structure 1 Structure 2 Structure 3

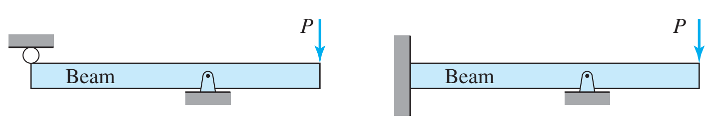
2. Identify all the zero-force members in the truss shown.



3. Determine the degree of static redundancy in each of the following structures and identify the statically determinate and indeterminate structures. Force P, and torques T_1 and T_2 are known external loads.



Structure 1 Structure 2



Structure 3 Structure 4

STATIC REVIEW EXAM 1

To get full credit, you must draw a free-body diagram any time you use equilibrium equations to calculate forces or moments. Grade yourself using the solution and grading scheme given in Appendix D. Each question is worth 20 points.

1. Determine (a) the coordinates (y_c, z_c) of the centroid of the cross section shown in Figure R1.1; (b) the area moment of inertia about an axis passing through the centroid of the cross section and parallel to the z axis.

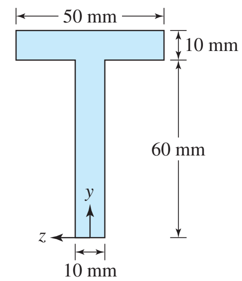


Figure R1.1

2. A linearly varying distributed load acts on a symmetric T section, as shown in Figure R1.2. Determine the force F and its location (x_F, y_F coordinates) that is statically equivalent to the distributed load.

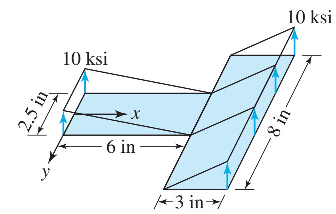


Figure R1.2

3. Find the internal axial force (indicate tension or compression) and the internal torque (magnitude and direction) acting on an imaginary cut through point E in Figure R1.3.

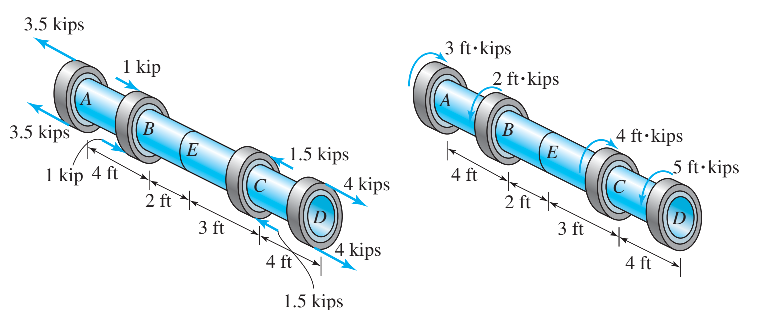


Figure R1.3

4. Determine the internal shear force and the internal bending moment acting at the section passing through A in Figure R1.4

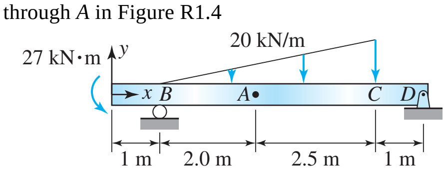


Figure R1.4

5. A system of pipes is subjected to a force P, as shown in Figure R1.5 By inspection (or by drawing a free-body diagram) identify the zero and non-zero internal forces and moments. Also indicate in the table the coordinate directions in which the internal shear forces and internal bending moments act

Internal Force/ Moment	Section *AA* (zero/nonzero)	Section *BB* (zero/nonzero)
Axial force	____________	____________
Shear force	____________ in___ direction	____________ in___ direction
Shear force	____________ in___ direction	____________ in___ direction
Torque	____________	____________
Bending moment	____________ in___ direction	____________ in___ direction
Bending moment	____________ in___ direction	____________in___ direction

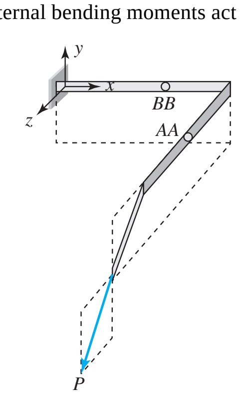


Figure R1.5

STATIC REVIEW EXAM 2

To get full credit, you must draw a free-body diagram any time you use equilibrium equations to calculate forces or moments. Discuss the solution to this exam with your instructor.

1. Determine (a) the coordinates (y_c, z_c) of the centroid of the cross section in Figure R1.6; (b) the area moment of inertia about an axis passing through the centroid of the cross section and parallel to the z axis.

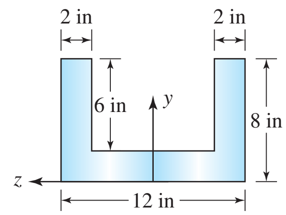


Figure R1.6

2. A distributed load acts on a symmetric C section, as shown in Figure R1.7. Determine the force F and its location (x_F, y_F coordinates) that is statically equivalent to the distributed load.

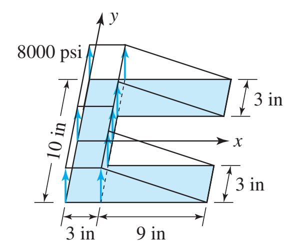


Figure R1.7

3. Find the internal axial force (indicate tension or compression) and the internal torque (magnitude and direction) acting on an imaginary cut through point E in Figure R1.8.

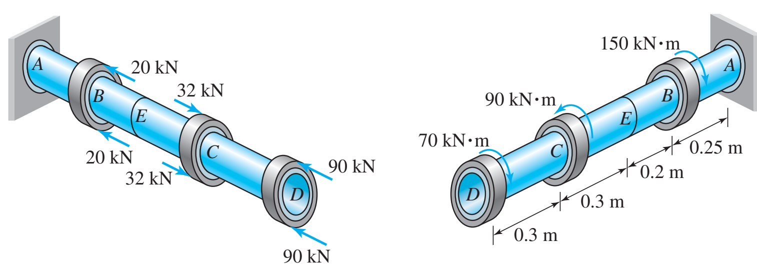


Figure R1.8

4. A simply supported beam is loaded by a uniformly distributed force of intensity 0.1 kip/in. applied at 60°, as shown in Figure R1.9. Also applied is a force F at the centroid of the beam. Neglecting the effect of beam thickness, determine at section C the internal axial force, the internal shear force, and the internal bending moment.

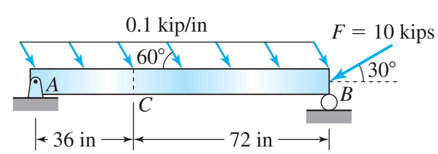


Figure R1.9

5. A system of pipes is subjected to a force P, as shown in Figure R1.10.By inspection (or by drawing a free-body diagram) identify the zero and nonzero internal forces and moments. Also indicate in the table the coordinate directions in which the internal shear forces and internal bending moments act.

Internal Force/Moment	Section *AA* (zero/nonzero)	Section *BB* (zero/nonzero)
Axial force	____________	____________
Shear force	____________ in___ direction	____________ in___ direction
Shear force	____________ in___ direction	____________ in___ direction
Torque	____________	____________
Bending moment	____________ in___ direction	____________ in___ direction
Bending moment	____________ in___ direction	____________ in___ direction

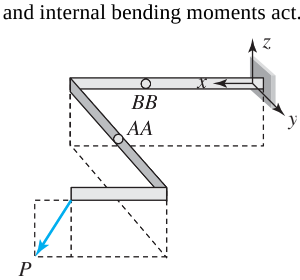


Figure R1.10

POINTS TO REMEMBER

- If a point cannot move in a given direction, then a reaction force opposite to the direction acts at that support point.
- If a line cannot rotate about an axis in a given direction, then a reaction moment opposite to the direction acts at that support.
- The support in isolation and not the entire body is considered in making decisions about the reaction at the support.
- Forces that are normal to the imaginary cut surface are called *normal forces*.
- The normal force that points away from the surface (pulls the surface) is called *tensile force*.
- The normal force that points into the surface (pushes the surface) is called *compressive force*.
- The normal force acting in the direction of the axis of the body is called *axial force*.
- Forces that are tangent to the imaginary cut surface are called *shear forces*.
- The internal moment about an axis normal to the imaginary cut surface is called torsional moment or *torque*.
- Internal moments about axes tangent to the imaginary cut are called *bending moments*.
- Calculation of internal forces or moments requires drawing a free-body diagram after making an imaginary cut.
- There are six independent equations in three dimensions and three independent equations in two dimensions per free-body diagram.
- A structure on which the number of unknown reaction forces and moments is greater than the number of equilibrium equations (6 in 3-D and 3 in 2-D) is called a statically *indeterminate* structure.
- Degree of static redundancy = number of unknown reactions − number of equilibrium equations.
- The number of equations on displacement and/or rotation we need to solve a statically indeterminate problem is equal to the degree of static redundancy.
- A structural member on which there is no moment couple and forces act at two points only is called a *two-force member.*
- The centroid lies on the axis of symmetry.
- The first moment of the area calculated in a coordinate system fixed to the centroid of the area is zero.
- I_{yy}, I_{zz}, and J are always positive and minimum about the axis passing through the centroid of the body.
- I_{yz} can be positive or negative.
- If either the y or the z axis is an axis of symmetry, then I_{yz} will be zero.
- Two systems that generate the same resultant force and moment are called *statically equivalent load systems*.
- The imaginary cut for the calculation of internal forces and moments must be made on the original body and not on the statically equivalent body.

APPENDIX B
ALGORITHMS FOR NUMERICAL METHODS

This appendix describes simple numerical techniques for evaluating the value of an integral, determining a root of a nonlinear equation, and finding constants of a polynomial by the least-squares method. Algorithms are given that can be programmed in any language. Also shown are methods of solving the same problems using a spreadsheet.

B.1 NUMERICAL INTEGRATION

We seek to numerically evaluate the integral

$$I = \int_a^b f(x)\,dx \tag{B.1}$$

where the function $f(x)$ and the limits a and b are assumed known.

This integral represents the area underneath the curve $f(x)$ in the interval defined by $x = a$ and $x = b$. The interval between a and b can be subdivided into N parts, as shown in Figure B.8. In each of the subintervals the function can be approximated by a straight-line segment. The area under the curve in each subinterval is the area of a trapezoid. Thus in the ith interval the area is $(\Delta x_i)[f(x_i) + f(x_{i-1})]/2$. By summing all the areas we obtain an approximate value of the total area represented by the integral in Equation (A.1),

$$I \cong \sum_{i=1}^{N} (\Delta x_i)\,\frac{f(x_i) + f(x_{i-1})}{2} \tag{B.2}$$

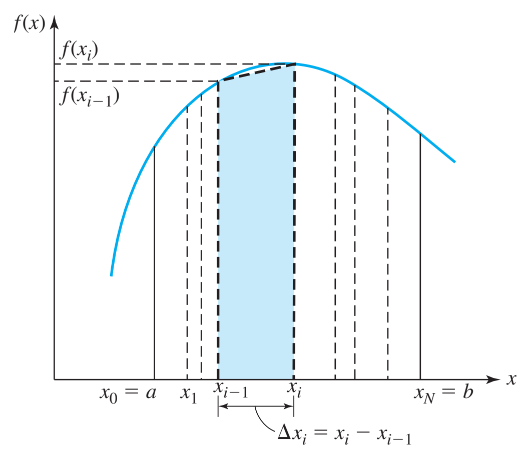


Figure B.8 Numerical integration by trapezoidal rule

By increasing the value of N in Equation (B.2) we can improve the accuracy in our approximation of the integral. More sophisticated numerical integration schemes such as Gauss quadrature may be needed with increased complexity of the function $f(x)$. For the functions that will be seen in this book, integration by the trapezoidal rule given by Equation (B.2) will give adequate accuracy.

B.1.1 Algorithm for Numerical Integration

Following are the steps in the algorithm for computing numerically the value of an integral of a function, assuming that the function value $f(x_i)$ is known at $N + 1$ points x_i, where i varies from 0 to N.

1. Read the value of N.

2. Read the values of x_i and $f(x_i)$ for $i = 0$ to N.
3. Initialize $I = 0$.
4. For $i = 1$ to N, calculate $I = I + (x_i - x_{i-1})[f(x_i) + f(x_{i-1})]/2$.
5. Print the value of I.

B.1.2 Use of a Spreadsheet for Numerical Integration

Figure B.9 shows a sample spreadsheet that can be used to evaluate an integral numerically by the trapezoidal rule given by Equation (B.2). The data x_i and $f(x_i)$ can be either typed or imported into columns A and B of the spreadsheet, starting at row 2. In cells A2 and B2 are the values of x_0 and $f(x_0)$, and in cells A3 and B3 are the values of x_1 and $f(x_1)$. Using these values, the first term ($i = 1$) of the summation in Equation (B.2) can be found, as shown in cell C2. In a similar manner the second term of the summation in Equation (B.2) can be found and added to the result of the first term in cell C2. On copying the formula of cell C3, the spreadsheet automatically updates the column and row entries. Thus in all but the last entry we add one term of the summation at a time to the result of the previous row and obtain the final result.

	A	B	C	D	
1	x_i	$f(x_i)$	I		
2	•	•	=(A3−A2)*(B3+B2)/2		
3	•	•	=C2+(A4−A3)*(B4+B3)/2		
4	•	•	Copy formula in cell C3		
5	•	•			
6	•	•			
7	•	•			

← Comment row

Figure B.9 Numerical integration algorithm on a spreadsheet.

B.2 ROOT OF A FUNCTION

We seek the value of x in a function that satisfies the equation

$$f(x) = 0 \tag{B.3}$$

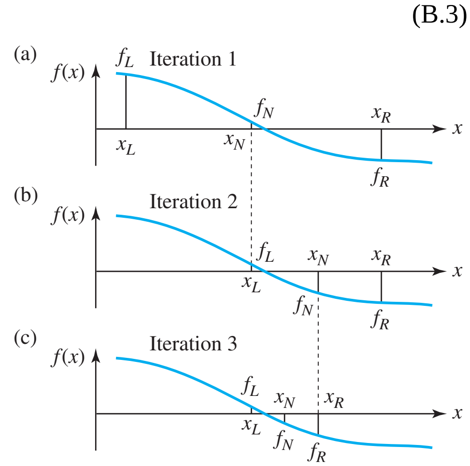


Figure B.10 Roots of an equation by halving the interval.

We are trying to find that value of x at which $f(x)$ crosses the x axis. Suppose we can find two values of x for which the function $f(x)$ has different signs. Then we know that the root of Equation (B.3) will be bracketed by these values. Let the two values of x that bracket the root from the left and the right be represented by x_L and x_R. Let the corresponding function values be $f_L = f(x_L)$

and $f_R = f(x_R)$, as shown in Figure B.10*a*. We can find the mean value $x_N = (x_L + x_R)/2$ and calculate the function value $f_N = f(x_N)$. We compare the sign of f_N to those of f_L and f_R and replace the one with the same sign, as elaborated below.

In iteration 1, f_N has the same sign as f_L; hence in iteration 2 we make the x_L value as x_N and the f_L value as f_N. In so doing we ensure that the root of the equation is still bracketed by x_L and x_R, but the size of the interval bracketing the root has been halved. On repeating the process in iteration 2, we find the mean value x_N, and the corresponding value f_N has the same sign as f_R. Thus for iteration 3, x_R and f_R are replaced by x_N and f_N found in iteration 2. In each iteration the root is bracketed by an interval that is half the interval in the previous iteration. When f_N reaches a small enough value, the iteration is stopped and x_N is the approximate root of Equation (B.3). This iterative technique for finding the root is called *half interval method* or *bisection method.*

B.2.1 Algorithm for Finding the Root of an Equation

The steps in the algorithm for computing the root of Equation (B.3) numerically are listed here. The computation of $f(x)$ should be done in a subprogram, which is not shown in the algorithm. It is assumed that the x_L and x_R values that bracket the root are known, but the algorithm checks to ensure that the root is bracketed by x_L and x_R. Note that if two functions have the same sign, then the product will yield a positive value.

1. Read the values of x_L and x_R.
2. Calculate $f_L = f(x_L)$ and $f_R = f(x_R)$.
3. If the product $f_L f_R > 0$, print “root of equation not bracketed” and stop.
4. Calculate $x_N = (x_L + x_R)/2$ and $f_N = f(x_N)$.
5. If the absolute value of f_N is less than 0.0001 (or a user-specified small number), then go to step 8.
6. If the product $f_L f_N > 0$, then replace x_L by x_N, and f_L by f_N. Go to step 4.
7. If the product $f_R f_N > 0$, then replace x_R by x_N, and f_R by f_N. Go to step 4.
8. Print the value of x_N as the root of the equation and stop.

B.2.2 Use of a Spreadsheet for Finding the Root of a Function

Finding the roots of a function on a spreadsheet can be done without the algorithm described. The method is in essence a digital equivalent to making a plot to find the value of x where the function $f(x)$ crosses the x axis.

To demonstrate the use of a spreadsheet for finding the root of a function, consider the function $f(x) = x^2 - 28.54x + 88.5$. We guess that the root is likely to be a value of x between 0 and 10.

Trial 1: In cell A2 of Figure B.11*a* we enter our starting guess as $x = 0$. In cell A3 we increment the value of cell A2 by 1, then copy the formula in the next nine cells (copying into more cells will not be incorrect or cause problems). In cell B2 we write our formula for finding $f(x)$ and then copy it into the cells below. The results of this trial are shown in Figure B.11*b*. We note that the function value changes sign between $x = 3$ and $x = 4$ in trial 1.

Trial 2: Based on our results of trial 1, we set $x = 3$ as our starting guess in cell D2. In cell D3 we increment the value of cell D2 by 0.1 and then copy the formula into the cells below. We copy the formula for $f(x)$ from cell B2 into the column starting at cell E2. The results of this trial are given in Figure B.11*b*. The function changes sign between $x = 3.5$ and $x = 3.6$.

Trial 3: Based on our results of trial 2, we set $x = 3.5$ as our starting guess in cell G2. In cell G3 we increment the value of cell G2 by 0.01 and then copy the formula into the cells below. We copy the formula for $f(x)$ from cell B2 into the column starting at cell H2. The results of this trial are given in Figure B.11*b*. The function value is nearly zero at $x = 3.54$, which gives us our root of the function.

The starting value and the increments in x are all educated guesses that will not be difficult to make for the problems in this book. If there are multiple roots, these too can be determined and, based on the problem, the correct root chosen.

(a)

	Trial 1			Trial 2			Trial 3	
	A	B	C	D	E	F	G	H
1	x	$f(x)$		x	$f(x)$		x	$f(x)$
2	0	=A2*A2−28.54*A2+88.5		3	Copy formula from cell B2		3.5	Copy formula from cell B2
3	=A2+1	Copy formula from cell B2		=D2+0.1			=G2+0.01	
4	Copy formula from cell A3			Copy formula from cell D3			Copy formula from cell D3	
5								
6								
7								

(b)

Trial 1 x	Trial 1 $f(x)$	Trial 2 x	Trial 2 $f(x)$	Trial 3 x	Trial 3 $f(x)$
0	88.5	3	11.88	3.5	0.86
1	60.96	3.1	9.636	3.51	0.6447
2	35.42	3.2	7.412	3.52	0.4296
$\boxed{3}$	11.88	3.3	5.208	3.53	0.2147
4	−9.66	3.4	3.024	$\boxed{3.54}$	2.84217E-14
5	−29.2	$\boxed{3.5}$	0.86	3.55	−0.2145
6	−46.74	3.6	−1.284	3.56	−0.4288
7	−62.28	3.7	−3.408	3.57	−0.6429
8	−75.82	3.8	−5.512	3.58	−0.8568
9	−87.36	3.9	−7.596	3.59	−1.0705
10	−96.9	4	−9.66	3.6	−1.284

Figure B.11 Roots of an equation using spreadsheet.

B.3 DETERMINING COEFFICIENTS OF A POLYNOMIAL

We assume that at N points x_i we know the values of a function f_i. Often the values of x_i and f_i are known from an experiment. We would like to approximate the function by the quadratic function

$$f(x) = a_0 + a_1 x + a_2 x^2 \tag{B.4}$$

If $N = 3$, then there is a unique solution to the values of a_0, a_1, and a_2. However, if $N > 3$, then we are trying find the coefficients a_0, a_1, and a_2 such that the error of approximation is minimized. One such method of defining and minimizing the error in approximation is the least-squares method elaborated next.

If we substitute $x = x_i$ in Equation (B.4), the value of the function $f(x_i)$ may be different than the value f_i. This difference is the error e_i, which can be written as

$$e_i = f_i - f(x_i) = f_i - (a_0 + a_1 x_i + a_2 x_i^2) \tag{B.5}$$

In the least-squares method an error E is defined as $E = \sum_{i=1}^{N} e_i^2$. This error E is then minimized with respect to the coefficients a_0, a_1, and a_2 and to generate a set of linear algebraic equations. These equations are then solved to obtain the coefficients.

Minimizing E implies setting the first derivative of E with respect to the coefficients equal to zero, as follows. In these equations all summations are performed for $i = 1$ to N.

$$\frac{\partial E}{\partial a_0} = 0 \quad \text{or} \quad \sum 2e_i \frac{\partial e_i}{\partial a_0} = 0 \quad \text{or} \quad \sum 2[f_i - (a_0 + a_1 x_i + a_2 x_i^2)][-1] = 0 \tag{B.6}$$

$$\frac{\partial E}{\partial a_1} = 0 \quad \text{or} \quad \sum 2e_i \frac{\partial e_i}{\partial a_1} = 0 \quad \text{or} \quad \sum 2[f_i - (a_0 + a_1 x_i + a_2 x_i^2)][-x_i] = 0 \tag{B.7}$$

$$\frac{\partial E}{\partial a_2} = 0 \quad \text{or} \quad \sum 2e_i \frac{\partial e_i}{\partial a_2} = 0 \quad \text{or} \quad \sum 2[f_i - (a_0 + a_1 x_i + a_2 x_i^2)][-x_i^2] = 0 \tag{B.8}$$

The equations on the right can be rearranged and written in matrix form,

$$\begin{vmatrix} N & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{vmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \sum f_i \\ \sum x_i f_i \\ \sum x_i^2 f_i \end{Bmatrix} \quad \text{or} \quad \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix} \tag{B.9}$$

The coefficients of the b matrix and the r vector can be determined by comparison to the matrix form of the equations on the left. The coefficients a_0, a_1, and a_2 can be determined by Cramer's rule. Let D represent the determinant of the b matrix. By Cramer's rule, we replace the first column in the matrix of b's by the right-hand side, find the determinant of the constructed matrix, and divide by D. Thus, the coefficients a_0, a_1, and a_2 can be written as

$$a_0 = \frac{\begin{vmatrix} r_1 & b_{12} & b_{13} \\ r_2 & b_{22} & b_{23} \\ r_3 & b_{32} & b_{33} \end{vmatrix}}{D} \qquad a_1 = \frac{\begin{vmatrix} b_{11} & r_1 & b_{13} \\ b_{21} & r_2 & b_{23} \\ b_{31} & r_3 & b_{33} \end{vmatrix}}{D} \qquad a_2 = \frac{\begin{vmatrix} b_{11} & b_{12} & r_1 \\ b_{21} & b_{22} & r_2 \\ b_{31} & b_{32} & r_3 \end{vmatrix}}{D} \tag{B.10}$$

Evaluating the determinant by expanding about the r elements, we obtain the values of a_0, a_1, and a_2

$$D = b_{11}C_{11} + b_{12}C_{12} + b_{13}C_{13} \tag{B.11}$$

$$a_0 = [C_{11}r_1 + C_{12}r_2 + C_{13}r_3]/D \tag{B.12}$$

$$a_1 = [C_{21}r_1 + C_{22}r_2 + C_{23}r_3]/D \tag{B.13}$$

$$a_2 = [C_{31}r_1 + C_{32}r_2 + C_{33}r_3]/D \tag{B.14}$$

where

$$\begin{aligned} C_{11} &= b_{22}b_{33} - b_{23}b_{32} & C_{12} &= C_{21} = -[b_{21}b_{33} - b_{23}b_{31}] \\ C_{13} &= C_{31} = b_{21}b_{32} - b_{22}b_{31} & C_{22} &= b_{11}b_{33} - b_{13}b_{31} \\ C_{23} &= C_{32} = -[b_{11}b_{23} - b_{13}b_{21}] & C_{33} &= b_{11}b_{22} - b_{12}b_{21} \end{aligned} \tag{B.15}$$

It is not difficult to extend these equations to higher order polynomials. However, a numerical method for solving the algebraic equations will be needed as the size of the b matrix grows. For problems in this book a quadratic representation of the function is adequate.

B.3.1 Algorithm for Finding Polynomial Coefficients

The steps in the algorithm for computing the coefficients of a quadratic function numerically by the least-squares method are listed here. It is assumed that x_i and f_i, are known values at N points.

1. Read the value of N.
2. Read the values of x_i and f_i for $i = 1$ to N.
3. Initialize the matrix coefficients b and r to zero.
4. Set $b_{11} = N$.
5. For $i = 1$ to N, execute the following computations:

$$b_{12} = b_{12} + x_i \qquad b_{13} = b_{13} + x_i^2 \qquad b_{23} = b_{23} + x_i^3 \qquad b_{33} = b_{33} + x_i^4$$

$$r_1 = r_1 + f_i \qquad r_2 = r_2 + x_i f_i \qquad r_3 = r_3 + x_i^2 f_i$$

6. Set $b_{21} = b_{12}$, $b_{22} = b_{13}$, $b_{31} = b_{13}$, $b_{32} = b_{23}$.
7. Determine D using Equation (B.11).
8. Determine the coefficients a_0, a_1, and a_2 using Equations (B.12), (B.13), and (B.14).

B.3.2 Use of a Spreadsheet for Finding Polynomial Coefficients

Figure B.12 shows a sample spreadsheet that can be used to evaluate the coefficients in a quadratic polynomial numerically. The data x_i and $f(x_i)$ can be either typed or imported into columns A and B of the spreadsheet, starting at row 2. In cells C2 through G2, the various quantities shown in row 1 can be found and the formulas copied to the rows below. We assume that the data fill up to row 50, that is, $N = 49$. In cell A51 the sum of the cells between cells A2 and A50 can be found using the summation command in the spreadsheet. By copying the formula to cells B51 through G51, the remaining sums in Equation (B.9) can be found. The coefficients in the b matrix and the right-hand-side r vector in Equation (B.9) can be identified as shown in comment row 52. The formulas in Equations (B.11) through (B.14) in terms of cell numbers can be entered in row 53, and D, a_0, a_1, and a_2 can be found.

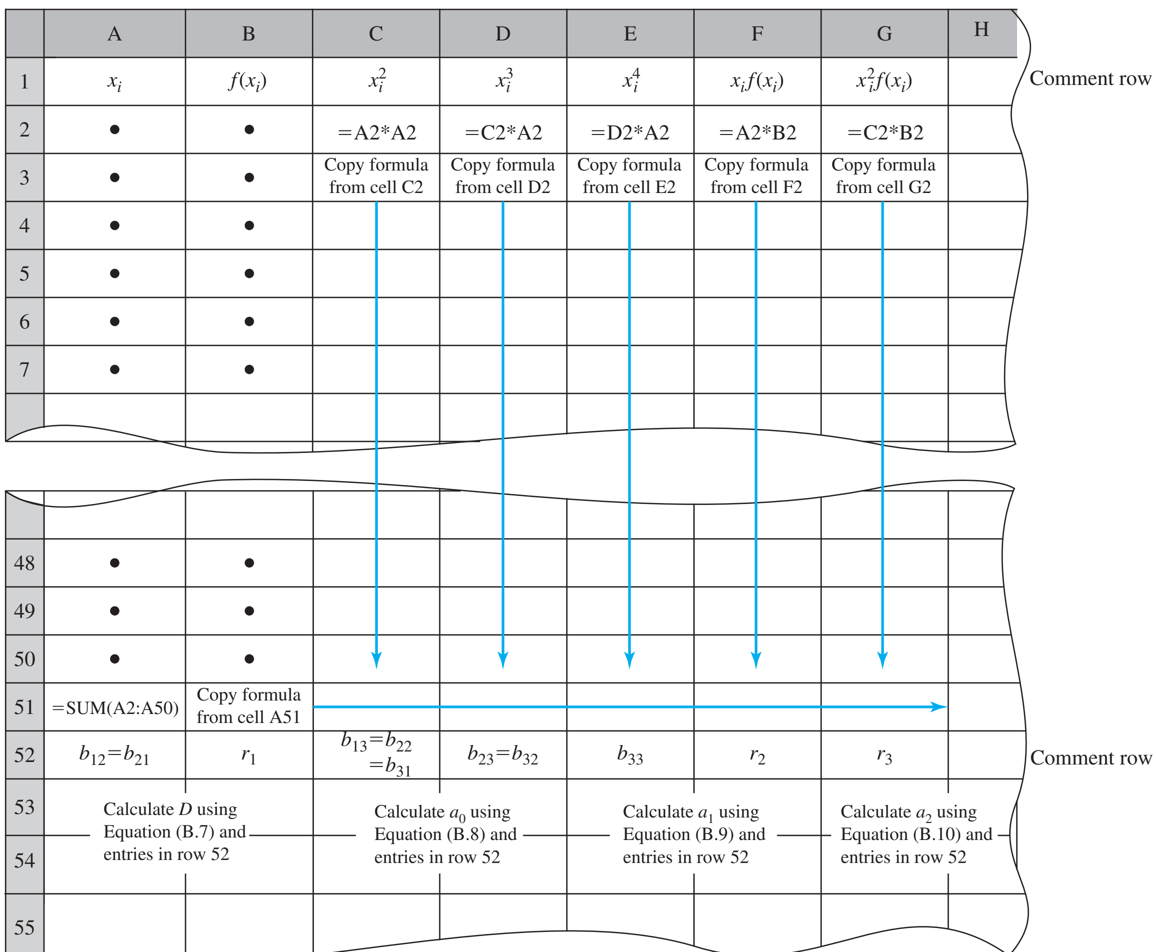
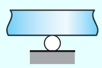
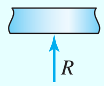
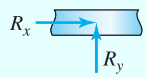
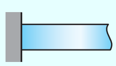
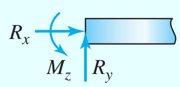
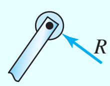
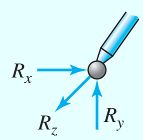
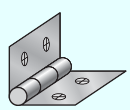
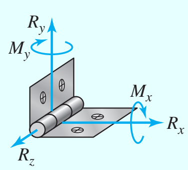
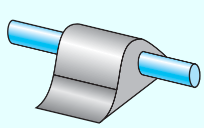
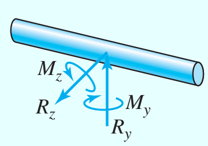
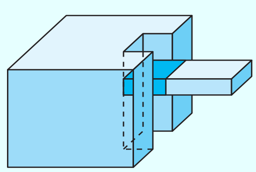
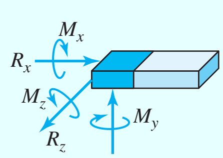


Figure B.12 Numerical evaluation of coefficients in a quadratic function on a spreadsheet.

APPENDIX C
REFERENCE INFORMATION

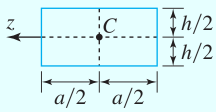
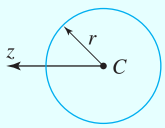
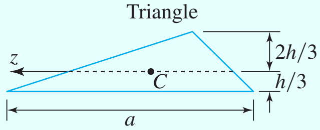
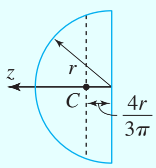
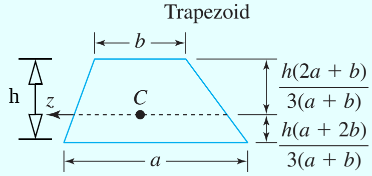
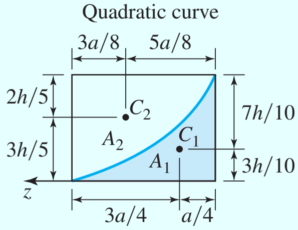
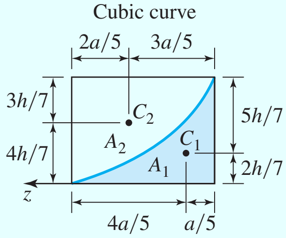
C.1 SUPPORT REACTIONS

TABLE C.1 **Reactions at the support**

Type of Support	Reactions	Comments
 Roller on smooth surface		Only downward translation is prevented. Hence the reaction force is upward.
 Smooth pin		Translation in the horizontal and vertical directions is prevented. Hence the reaction forces R_x and R_y can be in the directions shown, or opposite.
 Fixed support		Beside translation in the horizontal and vertical directions, rotation about the z axis is prevented. Hence the reactions R_x and R_y and M_z can be in the directions shown, or opposite.
 Roller in smooth slot		Translation perpendicular to slot is prevented. The reaction force R can be in the direction shown, or opposite.
 Ball and socket		Translation in all directions is prevented. The reaction forces can be in the directions shown, or opposite.
 Hinge		Except for rotation about the hinge axis, translation and rotation are prevented in all directions. Hence the reaction forces and moments can be in the directions shown, or opposite.
 Journal bearing		Translation and rotation are prevented in all directions, except in the direction of the shaft axis. Hence the reaction forces and moments can be in the directions shown, or opposite.
 Smooth slot		Translation in the z direction and rotation about any axis are prevented. Hence the reaction force R_z and reaction moments can be in the directions shown, or opposite. Translation in the x direction into the slot is prevented but not out of it. Hence the reaction force R_x should be in the direction shown.

C.2 GEOMETRIC PROPERTIES OF COMMON SHAPES

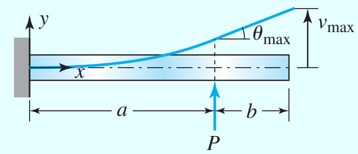
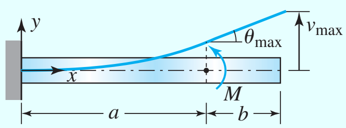
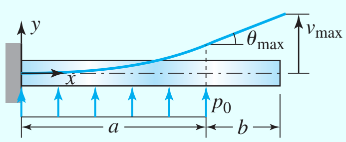
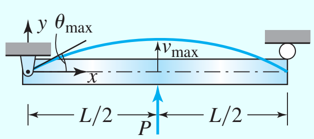
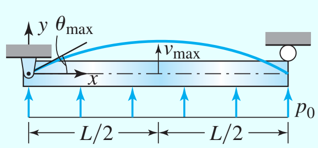
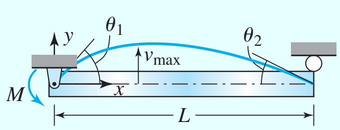
TABLE C.2 **Areas, centroids, and second area moments of inertia**

Shapes[a]	Areas	Second Area Moments of Inertia
Rectangle 	$A = ah$	$I_{zz} = \frac{1}{12}ah^3$
Circle 	$A = \pi r^2$	$I_{zz} = \frac{1}{4}\pi r^4$ $J = \frac{1}{2}\pi r^4$
Triangle 	$A = \frac{ah}{2}$	$I_{zz} = \frac{1}{36}ah^3$
Semicircle 	$A = \frac{\pi r^2}{2}$	$I_{zz} = \frac{1}{8}\pi r^4$
Trapezoid 	$A = \frac{h(a+b)}{2}$	$I_{zz} = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$
Quadratic curve 	$A_1 = \frac{ah}{3}$ $A_2 = \frac{2ah}{3}$	$(I_{zz})_1 = \frac{1}{21}ah^3$ $(I_{zz})_2 = \frac{2}{7}ah^3$
Cubic curve 	$A_1 = \frac{ah}{4}$ $A_2 = \frac{3ah}{4}$	$(I_{zz})_1 = \frac{1}{30}ah^3$ $(I_{zz})_2 = \frac{3}{10}ah^3$

a. *C*-location of centroid.

C.3 FORMULAS FOR DEFLECTION AND SLOPES OF BEAMS

TABLE C.3 **Deflections and slopes of beams**[a]

Case	Beam and Loading	Maximum Deflection and Slope	Elastic Curve
1		$v_{max} = \dfrac{Pa^2}{6EI}(2a+3b)$ $\theta_{max} = \dfrac{Pa^2}{2EI}$	$v = \dfrac{Px^2}{6EI}(3a-x)$ for $0 \le x \le a$ $v = \dfrac{Pa^2}{6EI}(3x-a)$ for $x \ge a$
2		$v_{max} = \dfrac{Ma(a+2b)}{2EI}$ $\theta_{max} = \dfrac{Ma}{EI}$	$v = \dfrac{Mx^2}{2EI}$ for $0 \le x \le a$ $v = \dfrac{Ma}{2EI}(2x-a)$ for $x \ge a$
3		$v_{max} = \dfrac{p_0 a^3(3a+4b)}{24EI}$ $\theta_{max} = \dfrac{p_0 a^3}{6EI}$	$v = \dfrac{p_0 x^2}{24EI}(x^2-4ax+6a^2)$ for $0 \le x \le a$ $v = \dfrac{p_0 a^3}{24EI}(4x-a)$ for $x \ge a$
4		$v_{max} = \dfrac{PL^3}{48EI}$ $\theta_{max} = \dfrac{PL^2}{16EI}$	$v = \dfrac{Px}{48EI}(3L^2-4x^2)$ for $0 \le x \le \dfrac{L}{2}$
5		$v_{max} = \dfrac{5p_0 L^4}{384EI}$ $\theta_{max} = \dfrac{p_0 L^3}{24EI}$	$v = \dfrac{p_0 x}{24EI}(x^3-2Lx^2+L^3)$
6		$v_{max} = \dfrac{ML^2}{9\sqrt{3}EI}$ @ $x = 0.4226L$ $\theta_1 = \dfrac{ML}{3EI}$ $\theta_2 = \dfrac{ML}{6EI}$	$v = \dfrac{Mx}{6EIL}(x^2-3Lx+2L^2)$

[a] These equations can be used for composite beams by replacing the bending rigidity EI by the sum of bending rigidities $\sum E_i I_i$.

C.4 CHARTS OF STRESS CONCENTRATION FACTORS

The stress concentration factor charts given in this section are approximate. For more accurate values the reader should consult a handbook. From Equation (3.25), the stress concentration factor is defined as

$$\boxed{K = \frac{\sigma_{max}}{\sigma_{nom}} \quad \text{or} \quad K = \frac{\tau_{max}}{\tau_{nom}}} \tag{C.1}$$

where σ_{max} and τ_{max} are the maximum normal and shear stress, respectively; σ_{nom} and τ_{nom} are the nominal normal and shear stress obtained from elementary theories.

C.4.1 Finite Plate with a Central Hole

Figure A.13 shows two stress concentration factors that differ because of the cross-sectional area used in the calculation of the nominal stress. If the gross cross-sectional area Ht of the plate is used, then we obtain the nominal stress

$$(\sigma_{\text{nom}})_{\text{gross}} = \frac{P}{Ht} \tag{C.2a}$$

and the top line in Figure A.13 should be used for the stress concentration factor. If the net area at the hole $(H - d)t$ is used, then we obtain the nominal stress

$$(\sigma_{\text{nom}})_{\text{net}} = \frac{P}{(H-d)t} \tag{C.2b}$$

and the bottom line in Figure A.13 should be used for the stress concentration factor. The two stress concentration factors are related as

$$K_{\text{net}} = \left(1 - \frac{d}{H}\right) K_{\text{gross}} \tag{C.2c}$$

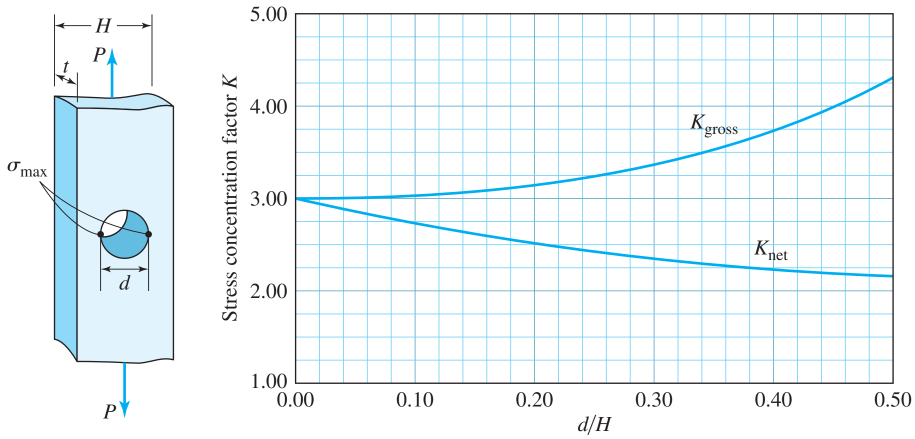


Figure A.13 Stress concentration factor for plate with a central hole.

C.4.2 Stepped axial circular bars with shoulder fillet

The maximum axial stress in a stepped circular bar with shoulder fillet will depend on the values of the diameters D and d of the two circular bars and the radius of the fillet r. From these three variables we can create two nondimensional variables D/d and r/d for showing the variation of the stress concentration factor, as illustrated in Figure A.14. The maximum nominal axial stress will be in the smaller diameter bar and, from Equation (4.8), is given by

$$\sigma_{\text{nom}} = \frac{4P}{\pi d^2} \tag{C.3}$$

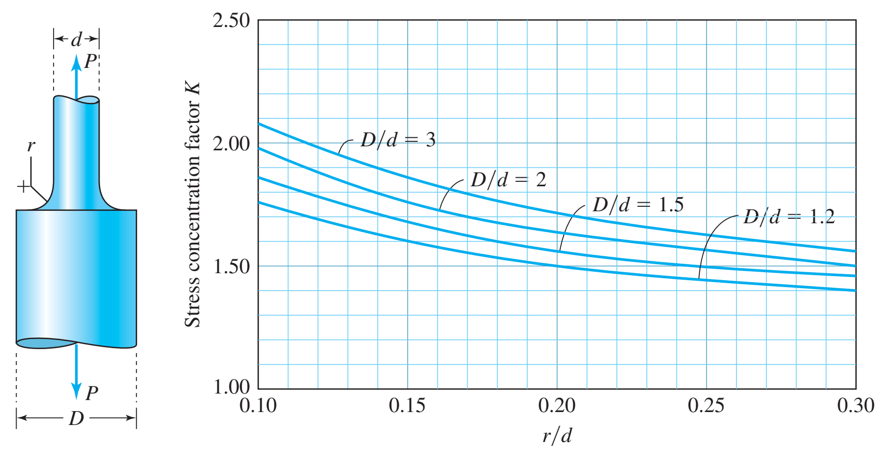


Figure A.14 Stress concentration factor for stepped axial circular bars with shoulder fillet.

C.4.3 Stepped circular shafts with shoulder fillet in torsion

The maximum shear stress in a stepped circular shaft with shoulder fillet will depend on the values of the diameters D and d of the two circular shafts and the radius of the fillet r. From these three variables we can create two nondimensional variables D/d and r/d to show the variation of the stress concentration factor, as illustrated in Figure A.15. The maximum nominal shear stress will be on the outer surface of the smaller diameter bar and, from Equation (5.10), is given by

$$\tau_{nom} = \frac{16T}{\pi d^3} \tag{C.4}$$

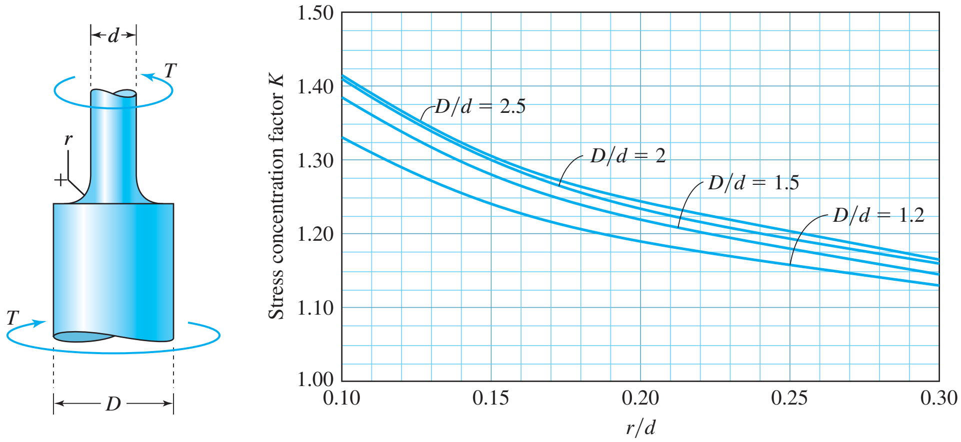


Figure A.15 Stress concentration factor for stepped circular shaft with shoulder fillet.

C.4.4 Stepped circular beam with shoulder fillet in bending

The maximum bending normal stress in a stepped circular beam with shoulder fillet will depend on the values of the diameters D and d of the two circular shafts and the radius of the fillet r. From these three variables we can create two nondimensional variables D/d and r/d to show the variation of the stress concentration factor, as illustrated in Figure A.16. The maximum nominal bending normal stress will be on the outer surface in the smaller diameter bar and, from Equation (6.12), is given by

$$\sigma_{nom} = \frac{32M}{\pi d^3} \tag{C.5}$$

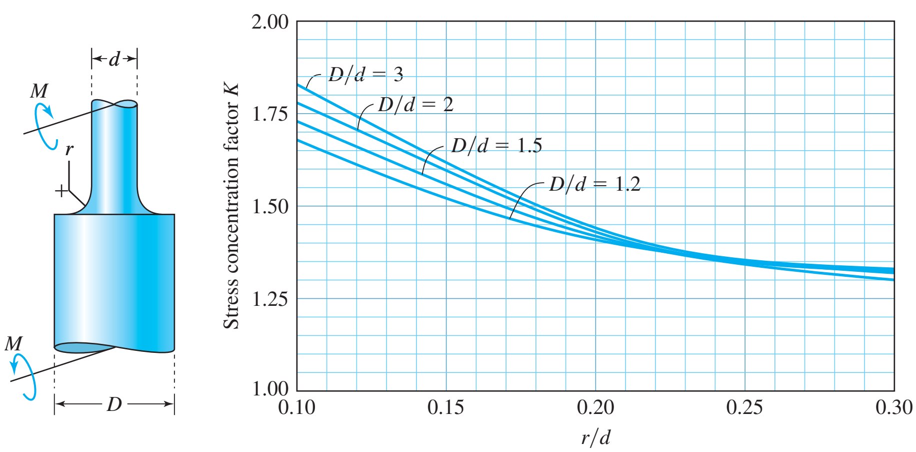


Figure A.16 Stress concentration factor for stepped circular beam with shoulder fillet.

C.5 PROPERTIES OF SELECTED MATERIALS

Material properties depend on many variables and vary widely. The properties given here are approximate mean values. Elastic strength may be represented by yield stress, proportional limit, or offset yield stress. Both elastic strength and ultimate strength refer to tensile strength unless stated otherwise.

TABLE C.4 Material properties in U.S. customary units

Material	Specific Weight (lb/in.^{3})	Modulus of Elasticity E (ksi)	Poisson's Ratio ν	Coefficient of Thermal Expansion α (μ/°F)	Elastic Strength (ksi)	Ultimate Strength (ksi)	Ductility (% elongation)
Aluminum	0.100	10,000	0.25	12.5	40	45	17
Bronze	0.320	15,000	0.34	9.4	20	50	20
Concrete	0.087	4000	0.15	6.0		2*	
Copper	0.316	15,000	0.35	9.8	12	35	35
Cast iron	0.266	25,000	0.25	6.0	25*	50*	
Glass	0.095	7500	0.20	4.5		10	
Plastic	0.035	400	0.4	50		9	50
Rock	0.098	8000	0.25	4	12*	78*	
Rubber	0.041	0.3	0.5	90	0.5	2	300
Steel	0.284	30,000	0.28	6.6	30	90	30
Titanium	0.162	14,000	0.33	5.3	135	155	13
Wood	0.02	1800	0.30	1.9		5*	

*Compressive strength.

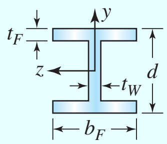
TABLE C.5 Material properties in metric units

Material	Density (mg/m^3)	Modulus of Elasticity E (GPa)	Poisson's Ratio ν	Coefficient of Thermal Expansion α (μ/°C)	Elastic Strength (MPa)	Ultimate Strength (MPa)	Ductility (% elongation)
Aluminum	2.77	70	0.25	22.5	280	315	17
Bronze	8.86	105	0.34	16.9	140	350	20
Concrete	2.41	28	0.15	10.8		14*	
Copper	8.75	105	0.35	17.6	84	245	35
Cast iron	7.37	175	0.25	10.8	175*	350*	
Glass	2.63	52.5	0.20	8.1		70	0
Plastic	0.97	2.8	0.4	90		63	50
Rock	2.72	56	0.25	7.2	84*	546*	
Rubber	1.14	2.1	0.5	162	3.5	14	300
Steel	7.87	210	0.28	11.9	210	630	30
Titanium	4.49	98	0.33	9.5	945	1185	13
Wood	0.55	12.6	0.30	3.4		35*	

*Compressive strength.

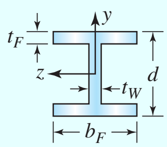
C.6 GEOMETRIC PROPERTIES OF STRUCTURAL STEEL MEMBERS

TABLE C.6 **Wide-flange sections (FPS units)**



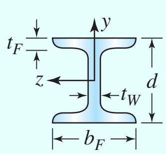
Designation (in. × lb/ft)	Depth d (in.)	Area A (in.^{2})	Web Thickness t_W (in.)	Flange Width b_F (in.)	Flange Thickness t_F (in.)	z Axis I_{zz} (in.^{4})	z Axis S_z (in.^{3})	z Axis r_z (in.)	y Axis I_{yy} (in.^{4})	y Axis S_y (in.^{3})	y Axis r_y (in.)
W12 × 35	12.50	10.3	0.300	6.560	0.520	285.0	45.6	5.25	24.5	7.47	1.54
W12 × 30	12.34	8.79	0.260	6.520	0.440	238	38.6	5.21	20.3	6.24	1.52
W10 × 30	10.47	8.84	0.300	5.81	0.510	170	32.4	4.38	16.7	5.75	1.37
W10 × 22	10.17	6.49	0.240	5.75	0.360	118	23.2	4.27	11.4	3.97	1.33
W8 × 18	8.14	5.26	0.230	5.250	0.330	61.9	15.2	3.43	7.97	3.04	1.23
W8 × 15	8.11	4.44	0.245	4.015	0.315	48	11.8	3.29	3.41	1.70	0.876
W6 × 20	6.20	5.87	0.260	6.020	0.365	41.4	13.4	2.66	13.3	4.41	1.50
W6 × 16	6.28	4.74	0.260	4.03	0.405	32.1	10.2	2.60	4.43	2.20	0.967

TABLE C.7 **Wide-flange sections (metric units)**



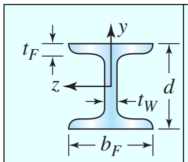
Designation (mm × kg/m)	Depth d (mm)	Area A (mm^2)	Web Thickness t_W (mm)	Flange Width b_F (mm)	Flange Thickness t_F (mm)	z Axis I_{zz} (10^6 mm^4)	z Axis S_z (10^3 mm^3)	z Axis r_z (mm)	y Axis I_{yy} (10^6 mm^4)	y Axis S_y (10^3 mm^3)	y Axis r_y (mm)
W310 × 52	317	6650	7.6	167	13.2	118.6	748	133.4	10.20	122.2	39.1
W310 × 44.5	313	5670	6.6	166	11.2	99.1	633	132.3	8.45	101.8	38.6
W250 × 44.8	266	5700	7.6	148	13.0	70.8	532	111.3	6.95	93.9	34.8
W250 × 32.7	258	4190	6.1	146	9.1	49.1	381	108.5	4.75	65.1	33.8
W200 × 26.6	207	3390	5.8	133	8.4	25.8	249	87.1	3.32	49.9	31.2
W200 × 22.5	206	2860	6.2	102	8.0	20.0	194.2	83.6	1.419	27.8	22.3
W150 × 29.8	157	3790	6.6	153	9.3	17.23	219	67.6	5.54	72.4	28.1
W150 × 24	160	3060	6.6	102	10.3	13.36	167	66	1.844	36.2	24.6

TABLE C.8 **S shapes (FPS units)**



Designation (in. × lb/ft)	Depth d (in.)	Area A (in.^{2})	Web Thickness t_W (in.)	Flange Width b_F (in.)	Flange Thickness t_F (in.)	z Axis I_{zz} (in.^{4})	z Axis S_z (in.^{3})	z Axis r_z (in.)	y Axis I_{yy} (in.^{4})	y Axis S_y (in.^{3})	y Axis r_y (in.)
S12 × 35	12	10.3	0.428	5.078	0.544	229	38.4	4.72	9.87	3.89	0.98
S12 × 31.8	12	9.35	0.350	5.000	0.544	218	36.4	4.83	9.36	3.74	1.0
S10 × 35	10	10.3	0.594	4.944	0.491	147	29.4	3.78	8.36	3.38	0.901
S10 × 25.4	10	7.46	0.311	4.661	0.491	124	24.7	4.07	6.79	2.91	0.954
S8 × 23	8	6.77	0.411	4.171	0.426	64.9	16.2	3.10	4.31	2.07	.798
S8 × 18.4	8	5.41	0.271	4.001	0.426	57.6	14.4	3.26	3.73	1.86	0.831
S7 × 20	7	5.88	0.450	3.860	0.392	42.4	12.1	2.69	3.17	1.64	0.734
S7 × 15.3	7	4.50	0.252	3.662	0.392	36.9	10.5	2.86	2.64	1.44	0.766

TABLE C.9 **S shapes (metric units)**



Designation (mm × kg/m)	Depth d (mm)	Area A (mm^2)	Web Thickness t_W (mm)	Flange Width b_F (mm)	Flange Thickness t_F (mm)	z Axis I_{zz} (10^6 mm^4)	z Axis S_z (10^3 mm^3)	z Axis r_z (mm)	y Axis I_{yy} (10^6 mm^4)	y Axis S_y (10^3 mm^3)	y Axis r_y (mm)
S310 × 52	305	6640	10.9	129	13.8	95.3	625	119.9	4.11	63.7	24.9
S310 × 47.3	305	6032	8.9	127	13.8	90.7	595	122.7	3.90	61.4	25.4
S250 × 52	254	6640	15.1	126	12.5	61.2	482	96.0	3.48	55.2	22.9
S250 × 37.8	254	4806	7.9	118	12.5	51.6	406	103.4	2.83	48.0	24.2
S200 × 34	203	4368	11.2	106	10.8	27.0	266	78.7	1.794	33.8	20.3
S200 × 27.4	203	3484	6.9	102	10.8	24	236	82.8	1.553	30.4	21.1
S180 × 30	178	3794	11.4	97	10.0	17.65	198.3	68.3	1.319	27.2	18.64
S180 × 22.8	178	2890	6.4	92	10.0	15.28	171.7	72.6	1.099	23.9	19.45

C.7 GLOSSARY

The terms used in this book are given in alphabetic order. The third column gives the chapter number in which the term is first introduced, with A representing Appendix A.

Term	Definition	Chapter
Anisotropic material:	A material that has a stress-strain relationships that changes with orientation of the coordinate system at a point.	3
Axial force diagram:	A plot of the internal axial force N versus x.	4
Axial force:	Normal force acting on a surface in the direction of the axis of the body.	A
Axial member:	A long straight body on which the forces are applied along the longitudinal axis.	4
Axial rigidity:	The product of modulus of elasticity (E) and cross-sectional area (A).	4
Axial stress:	The normal stress acting in the direction of the axis of a slender member.	1
Axial template:	An infinitesimal segment of an axial bar constructed by making an imaginary cuts on either side of a supposed external axial force.	4
Axisymmetric body:	A body whose geometry, material properties, and loading are symmetric with respect to an axis.	1
Bauschinger effect:	Material breaking at stress levels smaller than the ultimate stress due to load cycle reversal in the plastic region.	3
Beam template:	An infinitesimal segment of a beam constructed by making an imaginary cuts on either side of a supposed external force or moment.	6
Beam:	A long structural member on which loads are applied perpendicular to the axis.	6
Bearing stress:	The compressive normal stress that is produced when one surface presses against another.	1
Bending moment:	Moments about an axis tangent to a surface of a body.	A
Bending rigidity:	The product of modulus of elasticity (E) and the second area moment of inertia (I_{zz}) about the bending axis.	6
Bifurcation point:	The point at which more then one equilibrium configuration may exist.	11
Body forces:	External forces that act at every point on the body.	A

Term	Definition	Chapter
Boundary-value problem:	The mathematical statement listing of all the differential equations and all the necessary conditions to solve them.	7
Brittle material:	A material that exhibits little or no plastic deformation at failure.	3
Buckling load:	The force (or moment) at which buckling occurs. Also called critical load.	11
Buckling modes:	The deformed shape at buckling load.	11
Buckling:	An instability of equilibrium in structures that occurs from compressive loads or stresses.	11
Centroid:	An imaginary point on a body about which the first area moment is zero.	A
Characteristic equation:	The equation whose roots are the eigenvalues of the problem.	11
Columns:	Axial members that support compressive axial loads.	11
Compatibility equations:	Geometric relationships between the deformations or strains.	4
Complimentary strain energy density:	Complimentary strain energy per unit volume. It is the area between the stress axis and the stress-strain curve at a given value of stress or strain.	3
Complimentary strain energy:	Energy stored in a body due to forces acting on it.	3
Compressive stress:	Normal stress that pushes the imaginary surface into the rest of the material.	1
Concentrated forces (moment).	Surface forces (moments) applied at a point.	A
Continuity conditions:	Conditions that ensure continuity of deformations.	7
Critical load:	The force (or moment) at which buckling occurs. Also called buckling load.	11
Critical slenderness ratio:	The slenderness ratio at which material failure and buckling failure can occur simultaneously. Separates the long from the short columns.	11
Deformation:	The relative movement of a point with respect to another point on the body.	2
Degree of freedom:	The minimum number of displacements / rotations that are necessary to describe the deformed geometry.	4
Degree of static redundancy:	The number of unknown reactions minus the number of equilibrium equations.	A
Delta function:	A function that is zero everywhere except in a small interval where it tends to infinity in such a manner that the area under the curve is 1. It is also called the *Dirac* delta function.	7
Discontinuity functions:	A class of functions that are zero before a point and are non-zero after a point or are singular at the point.	7
Displacement:	The total movement of a point on a body with respect to fixed reference coordinates.	2
Distributed forces (moments):	Surface forces (moments) applied along a line or over a surface.	A
Ductile material:	A material that can undergo a large plastic deformation before fracture.	3
Eccentric loading:	Compressive axial force that is applied at a point that is not on the axis of the column.	11
Elastic curve:	Curve describing the deflection of the beam.	7
Elastic region:	The region of the stress-strain curve in which the material returns to the undeformed state when applied forces are removed.	3
Elastic-plastic boundary:	The set of points forming the boundary between the elastic and plastic regions.	3
Endurance limit:	The highest stress level for which the material would not fail under cyclic loading. Also called fatigue strength.	3
Eulerian strain:	Strain computed from deformation by using the final deformed geometry as the reference geometry.	2
Failure envelope:	The surface (or curve) that separates the acceptable design space from the unacceptable values of the variables affecting design.	10

Term	Definition	Chapter
Failure theory:	A statement on the relationship of the stress components to the characteristic value of material failure	10
Failure:	A component or a structure does not perform the function for which it was designed.	3
Fatigue strength:	The highest stress level for which the material would not fail under cyclic loading. Also called endurance limit.	3
Fatigue:	Failure due to cyclic loading at stress levels significantly lower than the static ultimate stress.	3
Finite element method:	A numerical method used in stress analysis in which the body is divided into elements of finite size.	4
Flexibility coefficient:	The coefficient multiplying internal forces / moments in an algebraic equation.	4
Flexibility matrix:	The matrix multiplying the unknown internal forces / moments in a set of algebraic equations.	4
Fracture stress:	The stress at the point where material breaks.	3
Free surface:	A surface on which there are no forces. Alternatively, a surface that is stress free.	1
Free-body diagram:	A diagram showing all the forces acting on a free body.	A
Gage length:	Length between two marks on a tension test specimen in the central region of uniform axial stress.	3
Generalized Hooke's law:	The equations relating stresses and strains in three dimensions.	3
Global coordinate system:	A fixed reference coordinate system in which the entire problem is described.	8
Hardness:	The resistance of material to indentation and scratches.	3
Homogeneous material:	A material that has same the material properties at all points in the body.	3
Hooke's law:	Equation relating normal stress and strain in the linear region of a tension test.	3
In-plane maximum shear strain:	The maximum shear strain in coordinate systems that can be obtained by rotating about the *z* axis.	9
In-plane maximum shear stress:	The maximum shear stress on a plane that can be obtained by rotating about the *z* axis.	8
Isotropic material:	A material that has a stress-strain relationships independent of the orientation of the coordinate system at a point.	3
Lagrangian strain:	Strain computed from deformation by using the original undeformed geometry as the reference geometry.	2
Load cells:	Any device that measures, controls, or applies a force or moment.	9
Loads:	External forces and moments that are applied to the body.	A
Local buckling:	Buckling that occurs in thin plates or shells due to compressive stresses.	11
Local coordinate system:	A coordinate system that can be fixed at any point on the body and has an orientation that is defined with respect to the global coordinate system.	8
Maximum normal stress theory:	A material will fail when the maximum normal stress at a point exceeds the ultimate normal stress obtained from a uniaxial tension test.	10
Maximum octahedral shear stress theory:	A material will fail when the maximum octahedral shear stress exceeds the octahedral shear stress at the yield obtained from a uniaxial tensile test.	10
Maximum shear strain:	The maximum shear strain at a point in any coordinate system.	9
Maximum shear stress theory:	A material will fail when the maximum shear stress exceeds the shear stress at yield that is obtained from a uniaxial tensile test.	10
Maximum shear stress:	The maximum shear stress at a point that acts on any plane passing through the point.	8
Method of joints:	Analysis is conducted by making imaginary cuts through all the members a the joint.	A

Term	Definition	Chapter
Method of sections:	Analysis is conducted by making an imaginary cut (section) through a member or a structure.	A
Modulus of elasticity:	The slope of the normal stress-strain line in the linear region of a tension test. Also called Young's modulus.	3
Modulus of resilience:	Strain energy density at the yield point.	3
Modulus of rigidity:	Same as shear modulus of elasticity.	3
Modulus of toughness:	The strain energy density at rupture.	3
Mohr's failure theory:	A material will fail if a stress state is on the envelope that is tangent to the three Mohr's circles corresponding to uniaxial ultimate stress in tension, uniaxial ultimate stress in compression, and pure shear.	10
Moment diagram:	A plot of the internal bending moment $\boldsymbol{M}_z$ versus x.	6
Monotonic functions:	Functions that either continuously increases or decreases.	7
Necking:	The sudden decrease in cross-sectional area after ultimate stress.	3
Negative normal strains:	Normal strains from contraction of a line.	2
Negative shear strain:	Shear strain due to a increase of angle between orthogonal lines.	2
Neutral axis:	The line on the cross section where the bending normal stress is zero.	6
Nominal stress:	The stress predicted by theoretical models away from the regions of stress concentration.	3
Normal stress:	Internal distributed forces that are normal to an imaginary cut surface.	1
Offset yield stress:	Stress that would produce a plastic strain corresponding to the specified offset strain.	3
Pitch:	The distance between two adjoining peaks on the threads of a bolt. It is the distance moved by the nut in one full rotation.	4
Plane stress:	A state of stress in which all stress components on the z-plane are zero.	1
Plastic region:	The region in which the material deforms permanently.	3
Plastic strain:	The permanent strain when stresses are zero.	3
Poisson's ratio:	The negative ratio of lateral normal strain to longitudinal normal strain.	3
Positive normal strains:	Normal strains from elongation of a line.	2
Positive shear strain:	Shear strain due to a decrease of angle between orthogonal lines.	2
Principal angle 1:	Angle principal direction one makes with the global coordinate direction x. Counterclockwise rotation from the x axis is defined as positive.	8
Principal angles:	The angles the principal directions makes with the global coordinate system.	8
Principal angles:	The angles the principal axes make with the global coordinate system.	9
Principal axes for strain:	The coordinate axes in which the shear strain is zero.	9
Principal axis for stress:	The normal direction to the principal planes. Also referred to as the principal direction.	8
Principal direction:	The normal direction to the principal planes. Also referred to as the principal axis.	8
Principal planes:	Planes on which the shear stresses are zero.	8
Principal strain 1:	The greatest principal strain.	9
Principal strains:	Normal strains in the principal directions.	9
Principal stress 1:	The greatest principal stress.	8
Principal stress element:	A properly oriented wedge constructed from the principal planes and the plane of maximum shear stress showing all the stresses acting on the respective planes.	8

Term	Definition	Chapter
Principal stress:	Normal stress on a principal plane. Also referred to as maximum or minimum normal stress at a point.	8
Proportional limit:	The point up to which stress and strain are related linearly.	3
Ramp function:	A function whose value is zero before a point and is a linear function after the point.	7
Reaction forces:	Forces developed at the supports that resist translation in a direction.	A
Reaction moment:	Moments developed at the support that resist rotation about an axis.	A
Rupture stress:	The stress at the point where material breaks.	3
Secant modulus:	The slope of the line that joins the origin to the point on the normal stress-strain curve at a given stress value.	3
Second order tensor:	A quantity that requires two directions and obeys certain coordinate transformation properties.	1
Section modulus:	The ratio of the second area moment of inertia about bending axis to the maximum distance from the neutral axis.	6
Shaft:	A long structural member that transmits torque from one plane to another parallel plane.	5
Shear flow:	The product of thickness and tangential shear stress along the center line of a thin cross section.	6
Shear force diagram:	A plot of the internal shear force $\boldsymbol{V}_y$ versus x.	6
Shear force:	Tangential force acting on a surface of a body.	A
Shear modulus of elasticity:	The slope of the shear stress-strain line in the linear region of a torsion test. Also called modulus of rigidity.	3
Shear stress:	Internal distributed forces that are parallel to an imaginary cut surface.	1
Singularity functions:	A class of functions that are zero everywhere except in a small region where they tend towards infinity.	7
Slenderness Ratio:	The ratio of the effective column length to the radius of gyration of the cross section about the buckling axis.	11
SN curve:	A plot of stress versus the number of cycles to failure.	3
Snap buckling:	A structure suddenly jumping (snapping) from one equilibrium position to another very different equilibrium position.	11
Statically equivalent load systems:	Two systems of forces that generate the same resultant force and moment.	A
Statically indeterminate structure:	A structure on which the number of unknown reaction forces and moments is greater than the number of equilibrium equations.	A
Step function:	A function whose value is zero before a point and equal to 1 after the point.	7
Stiffness coefficient:	The coefficient multiplying displacements / rotations in an algebraic equation.	4
Stiffness matrix:	The matrix multiplying the unknown displacements / rotations in a set of algebraic equations.	4
Strain energy density:	Strain energy per unit volume. It is the area under the stress-strain curve at a given value of stress or strain.	3
Strain energy:	Energy stored in a body due to deformation.	3
Strain hardening:	The increase of yield point each time the stress value exceeds the yield stress.	3
Stress concentration:	Large stress gradients in a small region.	3
Stress element:	An imaginary object representing a point that has surfaces with outward normals in the coordinate directions.	1
Tangent modulus:	The slope of the tangent drawn to the normal stress-strain curve at a given stress value.	3

Term	Definition	Chapter
Tensile stress:	Normal stress that pulls the imaginary surface away from the rest of the material.	1
Tension test:	A test conducted to determine mechanical properties by applying tensile forces on a specimen.	3
Thin body:	The thickness of the body is an order of magnitude (factor of 10) smaller than the other dimensions.	1
Timoshenko beam:	Beam in which shear is accounted for by dropping the assumption that planes originally perpendicular remain perpendicular.	6
Torque Diagram:	A plot of the internal torque $\boldsymbol{T}$ versus x.	5
Torque:	Moment about an axis normal to a surface of a body.	A
Torsion template:	An infinitesimal segment of a shaft constructed by making an imaginary cuts on either side of a supposed external torque.	5
Torsional rigidity:	The product of shear modulus of elasticity (G) and the polar moment of inertia (J) of a shaft.	5
Truss:	A structure made up of two-force members.	A
Two-force member:	A structural member on which there is no moment couple and forces act at two points only.	A
Ultimate stress:	The largest stress in the stress-strain curve.	3
Warping:	Axial deformation of shaft cross section due to torque.	5
Yield point:	The point demarcating the elastic from the plastic region.	3
Yield stress:	The stress at yield point.	3
Young's modulus:	Same as modulus of elasticity.	3
Zero-force member:	A two-force member that carries no internal force.	A

C.8 CONVERSION FACTORS BETWEEN U.S. CUSTOMARY SYSTEM (USCS) AND THE STANDARD INTERNATIONAL (SI) SYSTEM

Quantity	USCS to SI	SI to USCS
Length	1 in = 25.400 mm 1 ft = 0.3048	1 m = 39.37 in 1 m = 3.281 ft
Area	1 in^2 = 645.2 mm^2 1 ft^2 = 0.0929 m^2	1 mm^2 = $1.550(10^{-3})$ in^2 1 m^2 = 10.76 ft^2
Volume	1 in^3 = $16.39(10^3)$ mm^3 1 ft^3 = 0.028 m^3	1 mm^3 = $61.02(10^{-6})$ in^3 1 m^3 = 35.31 ft^3
Area Moment of Inertia	1 in^4 = $0.4162(10^6)$ mm^4	1 m^4 = $2.402(10^{-6})$ in^4
Mass	1 slug = 14.59 kg	1 kg = 0.06852 slugs
Force	1 lb = 4.448 N 1 kip = 4.448 kN	1 N = 0.2248 lb 1 kN = 0.2248 kip
Moment	1 in-lb = 0.1130 N-m 1 ft-lb = 1.356 N-m	1 N-m = 8.851 in-lb 1 N-m = 0.7376 ft-lb
Force per unit length	1 lb/ft = 14.59 N/m	1 N/m = 0.06852 lb/ft
Pressure; Stress	1 psi = 6.895 kPa 1 ksi = 6.895 MPa 1 lb/ft^2 = 47.88 Pa	1 kPa = 0.1450 psi 1 MPa = 0.1450 ksi 1 kPa = 20.89 lb/ft^2
Work; Energy	1 lb.ft = 1.356 J	1 J = 0.7376 lb.ft
Power	1 lb.ft/s = 1.356 W 1 hp = 745.7 W	1 W = 0.7376 lb.ft/s 1 kW = 1.341 hp

C.9 SI PREFIXES

Prefix Word	Prefix Symbol	Multiplication Factor
tera	T	10^{12}
giga	G	10^9
mega	M	10^6
kilo	k	10^3
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}

C.10 GREEK ALPHABET

Lowercase	Uppercase	Pronunciation	Lowercase	Uppercase	Pronunciation
α	A	Alpha	ν	N	Nu
β	B	Beta	ξ	ξ	Xi
γ	Γ	Gamma	o	O	Omicron
δ	Δ	Delta	π	Π	Pi
ε	E	Epsilon	ρ	P	Rho
ζ	Z	Zeta	σ	Σ	Sigma
η	H	Eta	τ	T	Tau
θ	Θ	Theta	υ	Y	Upsilon
ι	I	Iota	ϕ	Φ	Phi
κ	K	Kappa	χ	X	Chi
λ	Λ	Lambda	ψ	Ψ	Psi
μ	M	Mu	ω	Ω	Omega

APPENDIX D

SOLUTIONS TO STATIC REVIEW EXAM

D.1 REVIEW EXAM 1

1. As the y axis is the axis of symmetry, the centroid will lie on the y axis. Thus $z_c = 0$. 1 point

Equations (A.6) and (A.9) can be used to find the y coordinate of the centroid and the area moment of inertia. Figure A.17 and Table D.4 show the calculations.

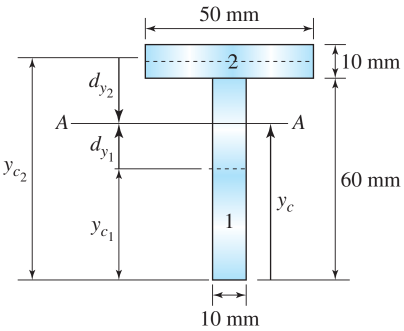


Figure A.17

TABLE D.4 **Calculation of centroid and area moment of inertia.**

	Centroids			Area moment of inertia		
Section	y_{c_i} (mm)	A_i (mm^2)	$y_{c_i}A_i$ (mm^3)	$d_{z_i} = y_c - y_{c_i}$ (mm)	$I_{z_iz_i} = \frac{1}{12}a_ib_i^3$ (mm^4)	$I_{z_iz_i} + A_id_{z_i}^2$ (mm^4)
1	30	60 x 10 = 600	18,000	15.9	10 x 60^3/12 = 180 ¥ 10^3	331.7 x 10^3
2	65	50 x 10 = 500	32,500	19.1	50 x 10^3/12 = 4.2 ¥ 10^3	186.6 x 10^3
Total		1100	50,500			

1 point for each correct entry for a total of 7 points. 1 point for each correct entry. 2 points for each correct entry. 1 point for each correct entry.

From Equations (A.6) and (A.9)we obtain

1 point for each correct answer with units.

$$y_c = \frac{50{,}500}{1100} = 45.9 \text{ mm} \qquad I_{AA} = (331.7 + 186.6)(10^3) = 518.3(10^3)\text{ mm}^4$$

2. We can replace each linear loading by an equivalent force, as shown in Figure A.7, then replace it by a single force. Using Figure A.18 we obtain

3 points/force for correct calculation.

$$F_1 = \frac{10 \times 2.5 \times 6}{2} = 75 \text{ kips} \qquad F_2 = \frac{10 \times 8 \times 3}{2} = 120 \text{ kips}$$

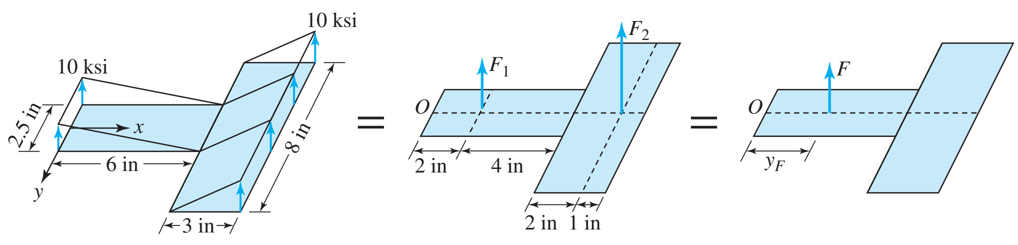


Figure A.18

For correct location of forces F_1 and F_2 3 points/force.

The resultant forces for the two systems on the right in Figure A.18 must be the same. We thus obtain

2 points for correct answer
1 point for correct units.

$$F = F_1 + F_2 = 75 + 120 = 195 \text{ kips}$$

The resultant moments about any point (point O) for the two systems on the right of Figure A.18 must also be the same. We obtain

$$2F_1 + 8F_2 = y_F F \quad \text{or} \quad y_F = \frac{150 + 960}{195} = 5.69 \text{ in}$$

1 point for each correct entry in this equation

1 point for correct answer
1 point for correct units

3. We make an imaginary cut at E and draw the free-body diagrams shown in Figures A.19 and A.20.

Internal axial force calculations (Figure A.19).

Either FBD is acceptable.
For drawing: 7 kip force at A or 8 kips at D--2 marks
2 kips at B or 3 kips at C ----2 marks
Normal force at E---2 marks

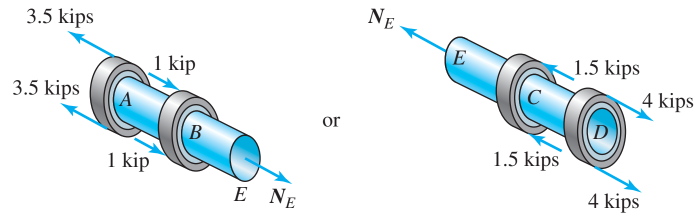


Figure A.19

1 point for correct equation

$$N_E - 7 + 2 = 0 \quad \text{or} \quad N_E - 8 + 3 = 0$$

1 point for correct answer
1 point for correct units
1 point for reporting tension

$$N_E = 5 \text{ kips (T)}$$

Internal torque calculations (Figure A.20)

Either FBD is acceptable.
For drawing: torque at A or D--2 marks
torque at B or at C ----2 marks
torque at E (either direction) ---2 marks

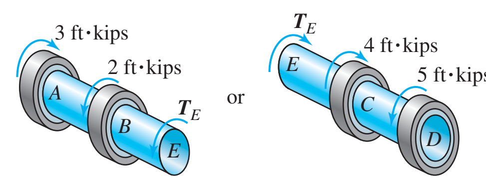


Figure A.20

2 points for correct equation

$$T_E - 3 + 2 = 0 \quad \text{or} \quad T_E - 5 + 4 = 0$$

1 point for correct answer
1 point for correct units

$$T_E = 1 \text{ ft} \cdot \text{kips}$$

4. We can draw the free-body diagram of the entire beam as shown in Figure A.21*a*.

1 mark for each force or moment and 1 mark for correct location of F
Total 5 marks.

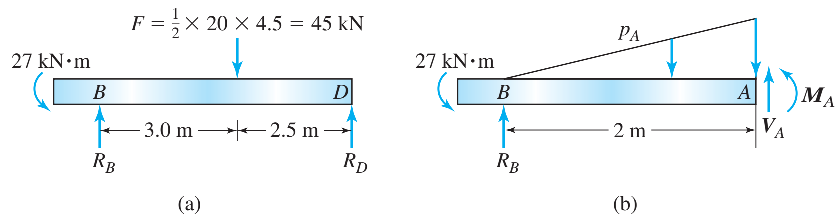


Figure A.21

By balancing the moment at D we find the reaction at B,

1 point for each term in the equation for a total of 3 points.

$$5.5R_B - 27 - 45 \times 2.5 = 0 \quad \text{or} \quad R_B = 25.36 \text{ kN}$$

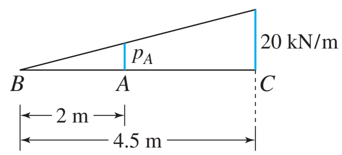
We can then make an imaginary cut at A on the original beam and draw the free-body diagram in Figure A.21*b*. We can find the intensity of the distributed force at A by similar triangles (Figure A.22*a*),

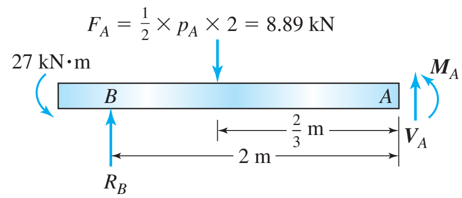
$$\frac{p_A}{2} = \frac{20}{4.5} \quad \text{or} \quad p_A = 8.89 \text{ kN/m}$$

We can then replace the distributed force on the beam that is cut at A and draw the free-body diagram shown in Figure A.22*b*.

1 mark for correct value of F_A
1 mark for correct location of F_A

3 marks for correct calculation of p_A.





2 marks for showing V_A and M_A irrespective of direction.

Figure A.22

Balancing the forces in the *y* direction we obtain

$$V_A + 25.36 - 8.89 = 0 \quad \text{or} \quad V_A = -16.5 \text{ kN}$$

1 point for correct equation correponding to your direction of V_A

1 point for correct answer

Balancing moments about point *A*, we obtain

$$M_A + 27 - 25.36 \times 2 + 8.89 \times 2/3 = 0 \quad \text{or} \quad M_A = 17.8 \text{ kN·m}$$

2 point for correct equation correponding to your direction of M_A

1 point for correct answer

5. By inspection we can write the following answers.

Internal Force/Moment	**Section *AA* (zero/nonzero)**	**Section *BB* (zero/nonzero)**
Axial force	Nonzero	Zero
Shear force	Nonzero in *y* direction	Nonzero in *y* direction
Shear force	Zero in *x* direction	Nonzero in *z* direction
Torque	Zero	Nonzero
Bending moment	Nonzero in *x* direction	Nonzero in *y* direction
Bending moment	Zero in *y* direction	Nonzero in *z* direction

1 point for each correct zero/non-zero entry.

1 point for each correct direction. Total 20 points.

APPENDIX E
ANSWERS TO QUICK TESTS

QUICK TEST 1.1

1. False. Stress is an internal quantity that can only be inferred but cannot be measured directly.
2. True. A surface has a unique normal, and normal stress is the internally distributed force in the direction of the normal.
3. True. Shear stress is an internally distributed force, and internal forces are equal and opposite in direction on the two surfaces produced by an imaginary cut.
4. False. Tension implies that the normal stress pulls the imaginary surface outward, which will result in opposite directions for the stresses on the two surfaces produced by the imaginary cut.
5. False. kips are units of force, not stress.
6. False. The normal stress should be reported as tension.
7. False. 1 GPa equals 10^9 Pa.
8. False. 1 psi nearly equals 7 kPa not 7 Pa.
9. False. Failure stress values are in millions of pascals for metals.
10. False. Pressure on a surface is always normal to the surface and compressive. Stress on a surface can be normal or tangential to it, and the normal component can be tensile or compressive.

QUICK TEST 1.2

1. False. Stress at a point is a second-order tensor.
2. True. Each of the two subscripts can have three values, resulting in nine possible combinations.
3. True. The remaining three components can be found from the symmetry of shear stresses.
4. True. The fourth component can be found from the symmetry of shear stresses.
5. False. A point in plane stress has four nonzero components; thus only five components are zero in general.
6. False. The sign of stress incorporates both the direction of the force and the direction of the imaginary surface.
7. True. A stress element is an imaginary object representing a point.
8. False. The normals of the surface of a stress element have to be in the direction of the coordinate system in which the stress at a point is defined.
9. True. Stress is an internally distributed force system that is equal and in opposite directions on the two surfaces of an imaginary cut.
10. False. The sign of stress incorporates the direction of the force and the direction of the imaginary surface. Alternatively, the sign of stress at a point is independent of the orientation of the imaginary cut surface.

QUICK TEST 2.1

1. Displacement is the movement of a point with respect to a fixed coordinate system, whereas deformation is the relative movement of a point with respect to another point on the body.
2. The reference geometry is the original undeformed geometry in Lagrangian strain and the deformed geometry in Eulerian strain.
3. The value of normal strain is 0.3/100 = 0.003.
4. The value of normal strain is $2000 \times 10^{-6} = 0.002$.
5. Positive shear strain corresponds to a decrease in the angle from right angle.
6. The strain will be positive as it corresponds to extension and is independent of the orientation of the rod.
7. No. We have defined small strain to correspond to strains less than 1%.
8. There are nine strain nonzero components in three dimensions.
9. There are four nonzero strain components in plane strain.
10. There are only three independent strain components in plane strain, as the fourth strain component can be determined from the symmetry of shear strains.

QUICK TEST 3.1

1. The modulus of elasticity has units of pascals or newtons per square meter. For metals it is usually gigapascals (GPa). Poisson's ratio has no units as it is dimensionless.
2. Offset yield stress is the stress value corresponding to a plastic strain equal to a specified offset strain.
3. Strain hardening is the increase in yield stress that occurs whenever yield stress is exceeded.
4. Necking is the sudden decrease in cross-sectional area after the ultimate stress.
5. Proportional limit defines the end of the *linear* region, whereas yield point defines the end of the *elastic* region.
6. A brittle material exhibits little plastic deformation before rupture, whereas a ductile material can undergo large plastic deformation before rupture.
7. A linear material behavior implies that stress and strain be linearly related. An elastic material behavior implies that when the loads are removed, the material returns to the undeformed state but the stress–strain relationship can be nonlinear, such as in rubber.
8. Strain energy is the energy due to deformation in a volume of material, whereas strain energy density is the strain energy per unit volume.
9. The modulus of resilience is a measure of recoverable energy and represents the strain energy density at yield point. The modulus of toughness is a measure of total energy that a material can absorb through elastic as well as plastic deformation and represents the strain energy density at ultimate stress.
10. A strong material has a high ultimate stress, whereas a tough material may not have high ultimate stress but has a large strain energy density at ultimate stress.

QUICK TEST 3.2

1. In an isotropic material the stress–strain relationship is the same in all directions but can differ at different points. In a homogeneous material the stress–strain relationship is the same at all points provided the directions are the same.

or

In an isotropic material the material constants are independent of the orientation of the coordinate system but can change with the coordinate locations. In a homogeneous material the material constants are independent of the locations of the coordinates but can change with the orientation of the coordinate system.

2. There are only two independent material constants in an isotropic linear elastic material.

3. 21 material constants are needed to specify the most general linear elastic anisotropic material.

4. There are three independent stress components in plane stress problems.

5. There are three independent strain components in plane stress problems.

6. There are five nonzero strain components in plane stress problems.

7. There are three independent strain components in plane strain problems.

8. There are three independent stress components in plane strain problems.

9. There are five nonzero stress components in plane strain problems.

10. For most materials E is greater than G as Poisson's ratio is greater than zero and $G = E/2(1 + \nu)$. In composites, however, Poisson's ratio can be negative; in such a case E will be less than G.

QUICK TEST 4.1

1. True. Material models do not affect the kinematic equation of a uniform strain.

2. False. Stress is uniform over each material but changes as the modulus of elasticity changes with the material in a nonhomogeneous cross section.

3. True. In the formulas A is the value of a cross-sectional area at a given value of x.

4. False. The formula is only valid if $\boldsymbol{N}$, E, and A do not change between x_1 and x_2. For a tapered bar A is changing with x.

5. True. The formula does not depend on external load. External loads affect the value of $\boldsymbol{N}$ but not the relationship of $\boldsymbol{N}$ to σ_{xx}.

6. False. The formula is valid only if $\boldsymbol{N}$, E, and A do not change between x_1 and x_2. For a segment with distributed load, $\boldsymbol{N}$ changes with x.

7. False. The equation represents static equivalency of $\boldsymbol{N}$ and σ_{xx}, which is independent of material models.

8. True. The equation represents static equivalency of $\boldsymbol{N}$ and σ_{xx} over the entire cross section and is independent of material models.

9. True. The uniform axial stress distribution for a homogeneous cross section is represented by an equivalent internal force acting at the centroid which will be also collinear with external forces. Thus no moment will be necessary for equilibrium.

10. True. The equilibrium of a segment created by making an imaginary cut just to the left and just to the right of the section where an external load is applied shows the jump in internal forces.

QUICK TEST 5.1

1. True. Torsional shear strain for circular shafts varies linearly.
2. True. The shear strain variation is independent of material behavior across the cross section.
3. False. If the shear modulus of a material on the inside is significantly greater than that of the material on the outside, then it is possible for the shear stress on the outer edge of the inside material to be higher than that at the outermost surface.
4. True. The shear stress value depends on the J at the section containing the point and not on the taper.
5. False. The formula is obtained assuming that J is constant between x_1 and x_2.
6. True. The shear stress value depends on the $\boldsymbol{T}$ at the section. The equilibrium equation relating $\boldsymbol{T}$ to external torque is a separate equation.
7. False. The formula is obtained assuming that $\boldsymbol{T}$ is constant between x_1 and x_2, but in the presence of distributed torque, $\boldsymbol{T}$ is a function of x.
8. False. The equation represents static equivalency and is independent of material models.
9. True. Same reasoning as in question 8.
10. True. Equilibrium equations require that the difference between internal torques on either side of the applied torque equal the value of the applied torque.

QUICK TEST 6.1

1. True. Bending normal strain varies linearly and is zero at the centroid of the cross section. If we know the strain at another point, the equation of a straight line can be found.
2. True. Bending normal stress varies linearly and is zero at the centroid and maximum at the point farthest from the centroid. Knowing the stress at two points on a cross section, the equation of a straight line can be found.
3. False. The larger moment of inertia is about the axis parallel to the 2-in. side, which requires that the bending forces be parallel to the 4-in. side.
4. True. The stresses are smallest near the centroid. Alternatively, the loss in moment of inertia is minimum when the hole is at the centroid.
5. False. y is measured from the centroid of the beam cross section.
6. True. The formula is valid at any cross section of the beam. I_{zz} has to be found at the section where the stress is being evaluated.
7. False. The equations are independent of the material model and are obtained from static equivalency principles, and the bending normal stress distribution is such that the net axial force on a cross section is zero.
8. True. The equation is independent of the material model and is obtained from the static equivalency principle.
9. True. The equilibrium of forces requires that the internal shear force jump by the value of the applied transverse force as one crosses the applied force from left to right.
10. True. The equilibrium of moments requires that the internal moment jump by the value of the applied moment as one crosses the applied moment from left to right.

QUICK TEST 8.1

1. $\theta = 115°$ or $295°$ or $-65°$
2. $\theta = 245°$ or $65°$ or $-115°$
3. $\theta = 155°$ or $-25°$ or $335°$
4. $\sigma_1 = 5$ ksi (T)
5. $\sigma_1 = 5$ ksi (C)
6. $\tau_{max} = 10$ ksi
7. $\tau_{max} = 12.5$ ksi
8. $\tau_{max} = 10$ ksi
9. $\theta_1 = 55°$ or $-125°$
10. $\theta_1 = -35°$

QUICK TEST 8.2

1. *D*
2. *A*
3. *E*
4. 12° ccw or 168° cw
5. 102° ccw or 78° cw
6. 78° ccw or 102° cw
7. *D*
8. *A*
9. *B*
10. $\sigma = 30$ MPa (T), $\tau = -40$ MPa

QUICK TEST 8.3

1. False. There are always three principal stresses. In two-dimensional problems the third principal stress is not independent and can be found from the other two.
2. True.
3. False. Material may affect the state of stress, but the principal stresses are unique for a given state of stress at a point.
4. False. The unique value of the principal stress depends only on the state of stress at the point and not on how these stresses are measured or described.
5. False. Planes of maximum shear stress are always at 45° to the principal planes, and not 90°.
6. True.
7. True.
8. False. Depends on the value of the third principal stress.
9. False. Each plane is represented by a single point on Mohr's circle.
10. False. Each point on Mohr's circle represents a single plane

QUICK TEST 9.1

1. *D*
2. *C*
3. *B*
4. *C*
5. *D*
6. 108° ccw or 72° cw
7. 18° ccw or 162° cw
8. $\varepsilon_1 = 1300\ \mu$, $\gamma_{max} = 2000\ \mu$
9. $\varepsilon_1 = 2300\ \mu$, $\gamma_{max} = 2300\ \mu$
10. $\varepsilon_1 = -300\ \mu$, $\gamma_{max} = 2300\ \mu$

QUICK TEST 9.2

1. (a) $\varepsilon_{yy} = 800\ \mu$; (b) $\varepsilon_{yy} = 800\ \mu$
2. $\theta = +115°$ or $-65°$
3. $\theta = +155°$ or $-25°$
4. $\theta = +25°$ or $-155°$
5. $\gamma_{max} = 2100\ \mu$
6. $\gamma_{max} = 3100\ \mu$
7. $\gamma_{max} = 1700\ \mu$
8. False. There are always three principal strains. In two-dimensional problems the third principal strain is not independent and can be found from the other two.
9. False. The unique value of principal strains depends only on the state of strain at the point and not on how these strains are measured or described.
10. False. Only for isotropic materials are the principal coordinates for stresses and strains the same, but for any anisotropic materials the principal coordinates for stresses and strains are different.

QUICK TEST 11.1

1. False. Only compressive axial forces can cause column buckling.

2. True.

3. False. There are infinite buckling loads. The addition of supports changes the buckling mode to the next higher critical buckling load.

4. True.

5. False. The critical buckling load does not change with the addition of uniform transverse distributed forces, but the increase in normal stress may cause the column to fail at lower loads.

6. False. Springs and elastic supports in the middle increase the critical buckling load.

7. False. The critical buckling load does not change with eccentricity, but an increase in normal stress causes the column to fail at lower loads with increasing eccentricity.

8. False. The critical buckling load decreases with increasing slenderness ratio.

9. True.

10. True.

QUICK TEST A.1

1. Structure 1: One; *AB*.
Structure 2: Three; *AC*, *CD*, *CE*.
Structure 3: Three; *AC*, *BC*, *CD*.

2. *DF*, *CF*, *HB*.

3. Structure 1: Two; indeterminate.
Structure 2: One; indeterminate.
Structure 3: Zero; determinate.
Structure 4: One; indeterminate.

APPENDIX F
ANSWERS TO SELECTED PROBLEMS

CHAPTER 1

1.1 $\sigma = 1019$ psi (T)

1.3 $W_{\text{max}} = 125.6$ lb

1.6 $d_{\text{min}} = 1.5$ mm

1.8 $\sigma = 2.57 MPa(T)$

1.12 (a) $\sigma_{\text{col}} = 232.8$ MPa (C); (b) $\sigma_b = 20 MPa(C)$

1.15 (a) $\sigma_{\text{col}} = 156$ MPa (C); (b) $\sigma_b = 8.33$ MPa (C)

1.19 $\sigma_b = 3$ MPa (C)

1.25 $P_{max} = 10.8 \; kips$

1.28 $P = \tau\pi(d_o + d_i)t$

1.31 $W_{\text{max}} = 125.6$ lb

1.44 $\sigma_{AA} = 3.286$ ksi (T); $\tau_{AA} = 1.53$ ksi

1.51 $\sigma = 11.9$ psi (T); $\boldsymbol{V} = 19$ lbs

1.52 (a) $\sigma_{HA} = 38$ MPa (C); $\sigma_{HB} = 16$ MPa (T); $\sigma_{HG} = 22$ MPa (C); $\sigma_{HC} = 16$ MPa (C)

(b) $(\tau_H)_{\text{max}} = 53.76$ MPa

1.56 $\sigma_{BD} = 100$ MPa (T); $\tau_{\text{max}} = 259$ MPa

1.62 $P_{\text{max}} = 70.6$ kN

1.67 $P_{\text{max}} = 5684$ lb

1.69 $L = 10.4$ in

1.71 (a) $d_{CG} = 30$ mm; $d_{CD} = 27$ mm; $d_{CB} = 23$ mm

(b) $d_C = 22$ mm; sequence: *CB*, *CG*, *CD*

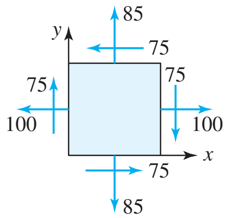
1.74 $\tau = 9947 \; Pa$

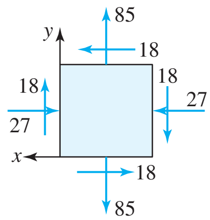
1.75 $P = 3\,aL\tau$

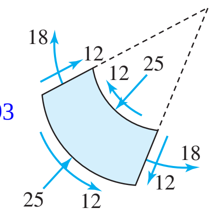
1.77 $\tau = 3.18$ MPa

1.79 $\tau = 226.3$ MPa

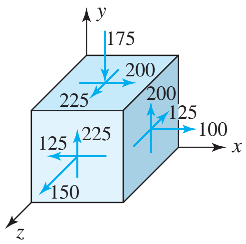
1.82 (a) $\tau = 8.5$ psi; (b) $T = 6.7$ in-lb

1.84 

1.86 

1.93 

1.94



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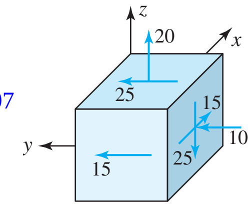

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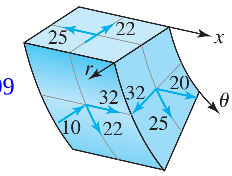

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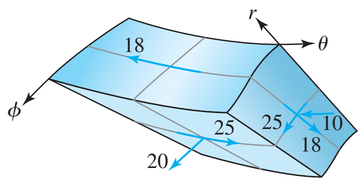

CHAPTER 2

2.1 $\varepsilon = 0.9294$ cm/cm

2.4 $\varepsilon = 0.321$ in/in

2.7 $u_D - u_A = 2.5$ mm

2.8 $\varepsilon_A = 393.3\ \mu$in/in; $\varepsilon_B = -150\ \mu$in/in

2.11 $\varepsilon_A = -0.0125$ in/in

2.13 $\varepsilon_A = -0.0108$ in/in

2.15 $\varepsilon_A = -0.0108$ in/in; $\varepsilon_F = -0.003$ in/in

2.19 $\delta_B = 2$ mm to the left

2.21 $\delta_B = 2.5$ mm to the left

2.22 $\varepsilon_A = -416.7\ \mu$mm/mm; $\varepsilon_F = 400\ \mu$mm/mm

2.29 $\gamma_A = -3000\ \mu$rad

2.32 $\gamma_A = 5400\ \mu$rad

2.34 $\gamma_A = 1296\ \mu$rad

2.38 $\gamma_A = -928\ \mu$rad

2.48 $\gamma_A = -1332\ \mu$rad

2.51 (a) $\varepsilon_{AP} = 1174.7\ \mu$mm/mm; (b) $\varepsilon_{AP} = 1174.6\ \mu$mm/mm; (c) $\varepsilon_{AP} = 1174.6\ \mu$mm/mm

2.54 $\delta_{AP} = 0.0647$ mm extension; $\delta_{BP} = 0.2165$ mm extension

2.57 $\delta_{AP} = 0.0035$ in contraction; $\delta_{BP} = 0.0188$ in contraction

2.61 $\varepsilon_{BC} = 4200\ \mu$mm/mm; $\varepsilon_{CF} = -2973\ \mu$mm/mm; $\varepsilon_{FE} = -2100\ \mu$mm/mm

2.64 $\varepsilon_{BC} = 500\ \mu$mm/mm; $\varepsilon_{CG} = -833\ \mu$mm/mm; $\varepsilon_{GB} = 0$; $\varepsilon_{CD} = 667.5\ \mu$mm/mm

2.68 $\varepsilon_{xx} = -128\ \mu$mm/mm; $\varepsilon_{yy} = -666.7\ \mu$mm/mm; $\gamma_{xy} = 3600\ \mu$rad

2.71 $\varepsilon_{xx} = 1750\ \mu$mm/mm; $\varepsilon_{yy} = -1625\ \mu$mm/mm; $\gamma_{xy} = -1125\ \mu$rad

2.74 $\varepsilon_{xx}(24) = 555\ \mu$in/in

2.77 $u(20) = 0.005$ in

2.80 $u(1250) = 1.516$ mm

2.85 $\varepsilon = 42.2\ \mu$mm/mm

2.87 $\varepsilon = 47\%$

CHAPTER 3

3.1 (a) $\sigma_{ult} = 510$ MPa; (b) $\sigma_{frac} = 480$ MPa (c) $E = 150(7.5)$ GPa; (d) $\sigma_{prop} = 300$ MPa (e) $\sigma_{yield} = 300$ MPa (f) $E_t = 2.5$ GPa; (g) $E_s = 6.5$ GPa

3.2 (a) $P = 23.56$ kN; (b) $P = 35.34$ kN

3.3 $\delta = 3.25$ mm

3.4 $\varepsilon_{total} = 0.065$; $\varepsilon_{elas} = 0.0028$; $\varepsilon_{plas} = 0.0622$

3.5 $P = 36.9$ kN

3.12 (a) $E = 300$ GPa; (b) $\sigma_{prop} = 1022$ MPa;

(c) $\sigma_{yield} = 1060$ MPa; (d) $E_t = 1.72$ GPa; (e) $E_s = 11.2$ GPa; (f) $\varepsilon_{plas} = 0.1203$

3.16 $E = 25{,}000$ ksi; $\nu = 0.2$

3.18 $G = 4000$ ksi

3.25 $P = 70.7$ kN; $\Delta d = -0.008$mm

3.27 0.0327%

3.31 $U = 125$ in.-lbs

3.36 (a) 300 kN-m/m^3; (b) 21,960 kN-m/m^3; (c) 5,340 kN-m/m^3; (d) 57,623 kN-m/m^3.

3.38 (a) 1734 kN-m/m^3; (b) 157 MN-m/m^3; (c) 18 MN-m/m^3; (d) 264 MN-m/m^3

3.41 $F = 22.1$ kN

3.44 $F = 16.7$ kN

3.45 $F = 0.795$ lb; $\theta = 65.96°$

3.50 $P_1 = 0$; $P_2 = 2$ kN

3.53 $h = 4\frac{3}{8}$ in; $d = 1\frac{1}{8}$ in

3.59 $d_{min} = 23$ mm

3.65 $\boldsymbol{N} = 60$ kips; $\boldsymbol{M_z} = 30$ in-kips

3.66 (a) $a = 1062.1$ MPa; $b = 4493.3$ MPa; $c = -12993.1$ MPa; (b) $E_T = 1.621$ GPa

3.68 $P = 70.1$ lbs

3.74 (a) $\sigma_{zz} = 0$; $\varepsilon_{xx} = -3661\ \mu$; $\varepsilon_{yy} = 2589\ \mu$; $\gamma_{xy} = 5357\ \mu$rad; $\varepsilon_{zz} = 357\ \mu$;

(b) $\varepsilon_{zz} = 0$; $\sigma_{zz} = 25$ MPa (C); $\varepsilon_{xx} = -3571\ \mu$; $\varepsilon_{yy} = 2679\ \mu$; $\gamma_{xy} = 5357\ \mu$rad

3.78 (a) $\sigma_{zz} = 0$; $\varepsilon_{xx} = -0.06875$; $\varepsilon_{yy} = 0.0875$; $\varepsilon_{zz} = -0.00625$; $\gamma_{xy} = 0.125$

(b) $\varepsilon_{zz} = 0$; $\sigma_{zz} = 12.50 psi(T)$; $\varepsilon_{xx} = -0.0703$; $\varepsilon_{yy} = 0.08594$; $\gamma_{xy} = 0.125$

3.81 $\sigma_{zz} = 0$; $\sigma_{yy} = 40.9$ ksi (C); $\sigma_{xx} = 36.26$ ksi (C); $\varepsilon_{zz} = 771\ \mu$in/in; $\tau_{xy} = -5.77$ ksi

3.83 $\sigma_{zz} = 0$; $\sigma_{yy} = 60$ MPa (T); $\sigma_{xx} = 60$ MPa (C); $\varepsilon_{zz} = 0$; $\tau_{xy} = 18$ MPa

3.86 $\sigma_{xx} = 16$ ksi (C); $\sigma_{yy} = 4$ ksi (C)

3.92 $a = 50.06$ mm; $b = 50.1725$ mm

3.111 $\varepsilon_{xx} = -936\ \mu$; $\varepsilon_{yy} = -2180\ \mu$; $\gamma_{xy} = -5333\ \mu$

3.115 $\sigma_{xx} = 19.07$ ksi (C); $\sigma_{yy} = 0.99$ ksi (C); $\tau_{xy} = 0.6$ ksi

3.119 $K = 2.4$

3.121 $\sigma_{max} = 45.3$ ksi

3.127 $\theta = 0.34°$

3.130 $\sigma_{xx} = 47.4$ ksi (C); $\sigma_{yy} = 52.02$ ksi (C); $\varepsilon_{zz} = 1254\ \mu$; $\tau_{xy} = -5.77$ ksi

3.137 (a) $T = 33.33$ hours; (b) $T = 133.33$ hours; (c) $T = \infty$

3.139 $n = 400{,}000$ cycles

3.142 (a) $E_1 = 15{,}000$ ksi; (b) $E_2 = 64.15$ ksi; (c) $n = 0.1694$; $E = 56.2$ ksi

CHAPTER 4

4.2 $F_1 = 108.5$ kN; $F_2 = 45.2$ kN; $F_3 = 94.3$ kN

4.4 $F = 11.25$ kips

4.9 $u_D - u_A = -0.175$ in

4.13 (a) $u_D - u_A = -0.0234$ in; (b) $\sigma_{max} = 3.75$ ksi (C)

4.18 $u_B - u_A = 0.126$ mm

4.20 $u = 0.4621\ P/EK$

4.21 (a) $u_C - u_A = 0.034$ in; (b) $\sigma_{max} = 33.95$ ksi (T)

4.24 $u_B = -\gamma L^2/2E$

4.27 $\delta = 0.045$ in

4.31 $F_{max} = 4886$ lb

4.33 $d_p = 0.5$ in; $a_b = 1\frac{1}{8}$ in; $b_s = 1\frac{5}{16}$ in

4.41 (a) $\Delta u = 0.60$ mm; (b) $\sigma_{max} = 62.2$ MPa (T)

4.44 $a = 224.40$; $b = -23.60$; $c = -0.40$; $u_A = 0.017$ in to the left

4.46 $F = 46.9$ kips

4.50 $\delta_P = 0.23$ mm

4.51 $\sigma_A = 8.0$ ksi (C); $\delta_B = 0.0021$ in

4.61 $\delta_P = 0.24$ mm; $\sigma_A = 118$ MPa (C)

4.65 (a) $\delta_p = 0.0265$ in; (b) $\Delta d_s = 0.00074$ in; $\Delta d_{al} = -0.00066$ in

4.67 $\sigma_A = 22.5$ ksi (C); $\sigma_B = 17.2$ ksi (T)

4.70 $F_{max} = 555$ kN

4.74 $F_{max} = 17.2$ kN

4.77 $w_{max} = 9.4$ MPa

4.83 $P_{max} = 106.7$ kips

4.85 $A_{BC} = 1.1\ \text{in}^2$; $d = 1.3$ in

4.87 $F_{max} = 148.6$ kN

4.89 $F_{max} = 181.9$ kN

4.90 $\sigma_A = 5.2$ ksi (T); $\sigma_B = 3.5$ ksi (T)

4.94 $\sigma_{xx} = 0$; $u(L/2) = \alpha T_L L/24$

4.95 $\sigma_{xx} = E\alpha T_L/3$ (C); $u(L/2) = -\alpha T_L L/8$

4.99 $\sigma_A = 25.70$ ksi (T)

4.100 $\sigma_{\theta\theta} = 10$ MPa (T); $\tau_r = 40$ MPa

4.104 $t_{min} = 0.05$ in; $d_{noz} = 0.206$ in

4.106 $p_{max} = 500$ psi; $d_{riv} = 0.85$ in

CHAPTER 5

5.1 $\gamma_D = 2400\ \mu$rad

5.2 $T = 64.8$ in-kips

5.7 $\phi_1 = 0.0400$ rad; $\phi_2 = 0.0243$ rad; $\phi_3 = 0.0957$ rad

5.11 $\boldsymbol{T} = -495.2$ in-kips

5.13 $\boldsymbol{T} = 10.9$ kN-m

5.23 (a) $(\tau_{xy})_A > 0$; (b) $(\tau_{xy})_B < 0$

5.26 (a) $(\tau_{xy})_A > 0$; (b) $(\tau_{xy})_B < 0$

5.29 $\phi_D - \phi_A = 0.00711$ rads CW

5.32 $\phi_D = 0.0163$ rads CW; $\gamma_{max} = -1094\mu$; $(\tau_{x\theta})_E = -4.4$ ksi

5.35 $\phi_A = 1676$ μrads CW; $(\tau_{x\theta})_E = 15.1$ MPa

5.38 (a) $\phi_B = 0.1819(T_{\text{ext}}L/Gr^4)$ CCW; (b) $\tau_{\max} = 0.275T_{\text{ext}}/r^3$

5.40 $\phi_A = (qL^2/GJ)$ CW

5.41 $T = 69.2$ in-kips

5.43 $(r_i)_{\max} = 24$ mm

5.47 $d_{\min} = 21$ mm; $\tau_{AB} = 52.5$ MPa

5.57 $R_o = 2\frac{3}{8}$ in

5.59 $\Delta\phi = 0.085$ rad; $\tau_{\max} = 172$ MPa

5.60 $\Delta\phi = 0.088$ rad

5.63 $\phi_B = 0.0516$ rads ccw; $\tau_{\max} = 25.8$ ksi

5.66 $\phi_C = 0.006$ rads CCW; $T = 200.5$ in-kips

5.67 $\phi_B = 0.0438$ rads CW

5.71 $\phi_B = 5.659\dfrac{TL}{Gd^4}$ CCW; $\tau_{\max} = 2.83\ \dfrac{T}{d^3}$

5.74 $T_{\max} = 32$ kN-m; $\phi_B = 0.048$ rads CCW; $\tau_{\max} = 130.4$ MPa

5.75 $d_{\min} = 89$ mm; $\phi_B = 0.0487$ rads CCW; $\tau_{\max} = 116$ MPa

5.76 $d_{\min} = 108$ mm; $\phi_B = 0.025$ rads CCW; $\tau_{\max} = 58.62$ MPa

5.95 $|\tau_{\max}| = 10.8$ MPa

5.101 $|\tau_{\max}| = 21.65$ MPa

CHAPTER 6

6.1 $\psi = 2.41°$

6.3 $\varepsilon_1 = 182$ μm/m; $\varepsilon_3 = -109.1$ μm/m; $\varepsilon_4 = -654$ μm/m; $\varepsilon_6 = 393$ μm/m

6.6 $P = 1454N$; $M_z = 123.6$ N-m

6.7 $P_1 = 14.58$ kN; $M_1 = 130.3$ N-m; $P_2 = 9.88$ kN; $M_2 = 64.0$ N-m

6.12 $\boldsymbol{M}_z = 9.13$ in-kips

6.14 $\boldsymbol{M}_z = -2134$ kN-m

6.25 $\sigma_T = 3.73ksi(T)$; $\sigma_C = 6.93ksi(C)$

6.29 $\sigma_A = 1224$ psi (C); $\sigma_B = 735$ psi (C); $\sigma_D = 1714$ ksi (T)

6.35 σ_A is (C); σ_B is (T)

6.38 σ_A is (T); σ_B is (C)

6.42 (a) $\sigma_{3.0} = 2.96$ ksi (C); (b) $\sigma_{\max} = 6.93$ ksi (C) or (T)

6.45 $\sigma_A = 4.17$ ksi (C); $\sigma_{\max} = 12.5$ ksi (C) or (T)

6.49 $\sigma_A = 6.68$ ksi (C); $\sigma_{\max} = 28.9$ ksi (T)

6.51 $\varepsilon_A = -1500\ \mu$

6.53 $\varepsilon_A = -327\ \mu$

6.60 (a) $V_y = 3(72 - x)$ kips; (b) $M_z = 1.5(72 - x)^2$ in-kips

6.62 (a) $V_y = [108 - \frac{1}{48}x^2]$ kips; (b) $M_z = [5184 - 108x + \frac{1}{144}x^3]$ in-kips

6.64 $V_y = -wL$ kips $\quad 0 \le x < L$; $M_z = (wLx - wL^2)$ in-kips $\quad 0 \le x < L$;

$V_y = [w(x-L) - wL]$ kips $\quad L < x \le 2L$; $M_z = [wLx - \frac{w}{2}(x-L)^2 - wL^2]$ in-kips $\quad L < x \le 2L$

6.68 $V_y = (76 - 12x)$ kN $\quad 5\text{ m} < x < 9\text{ m}$; $M_z = (6x^2 - 76x + 154)$ kN-m $\quad 5\text{ m} < x < 9\text{ m}$;

$V_y = -20$ kN $\quad 9\text{ m} < x < 12\text{ m}$; $M_z = (20x - 240)$ kN-m $\quad 9\text{ m} < x < 12\text{ m}$

6.69 $V_y = -6x$ kN $\quad 0 \le x < 3\text{ m}$; $M_z = 3x^2$ kN-m $\quad 0 \le x < 3\text{ m}$;

$V_y = -8$ kN $\quad 3\text{ m} < x < 5\text{ m}$; $M_z = (8x - 7)$ kN-m $\quad 3\text{ m} < x < 5\text{ m}$

6.74 $(V_y)_{\max} = \pm 7.5$ kN; $(M_z)_{\max} = 5.625$ kN-m

6.79 $(V_y)_{\max} = 36$ kN; $(M_z)_{\max} = -86.67$ kN-m

6.83 $(V_y)_{\max} = 9$ kN; $(M_z)_{\max} = -23.625$ kN-m

6.86 $(V_y)_{\max} = \pm 6$ kips; $(M_z)_{\max} = -16$ in-kips

6.92 $w_{\max} = 154.3$ lb/in

6.99 $a_i = 11\frac{7}{8}$ in

6.100 $r = 3.75$ mm

6.102 $P = 165.7$ N

6.107 Point A: negative τ_{xz}; Point B: positive τ_{xy}; Point C: negative τ_{xz}; Point D: positive τ_{xz}

6.112 Point A: positive τ_{xy}; Point B: negative τ_{xz}; Point C: zero; Point D: positive τ_{xy}

6.117 $|\sigma_{\max}| = 348.4$ MPa; $|\tau_{\max}| = 6.84$ MPa; $(\sigma_{xx})_A = 48$ MPa (C); $(\tau_{xy})_A = -3.2$ MPa

6.120 $V_{AB} = 614.4$ lbs; $V_{BC} = 921.6$ lbs

6.124 $M_{\text{ext}} = 8333.33$ in-lbs

6.125 $P_{\max} = 202$ N; $\Delta s = 16$ cm

6.137 $R_O = 2\frac{3}{16}$ in

6.138 $|\sigma_{\max}| = 9185$ psi; $|\tau_{\max}| = 295$ psi

CHAPTER 7

7.2 $v(x) = -[wx/(24EI)](x^3 - 2Lx^2 + L^3)$; $v(L/2) = -[5wL^4/(384EI)]$

7.4 $v(x) = -[wx^2/(24EI)](x^2 - 4Lx + 6L^2)$; $v(L) = -[wL^4/(8EI)]$

7.9 $v_A = PL^3/(3EI)$

7.17 $v(x) = \begin{cases} [wLx/(9EI)](x^2 - 5L^2) & 0 \le x \le L \\ [wLx/(9EI)](x^2 - 5L^2) - [wL/(6EI)](x-L)^3 & L \le x \le 3L \end{cases}$; $\quad v(L) = -(4wL^4/9EI)$

7.19 $v(x) = \begin{cases} (wLx/48EI)(2x^2 - 7L^2) & 0 \le x \le L \\ (w/EI)(2Lx^3 - 7L^3x)/48 - (w/24EI)(x-L)^4 & L \le x \le 2L \end{cases}$; $\quad v(L) = -[5wL^4/(48EI)]$

7.25 $h(x) = \sqrt{6Px/(b\sigma)}$; $v_{\max} = -\sqrt{8b\sigma^3L^3/(27PE^2)}$

7.28 $\sigma_{\max} = 128PL/(27\pi d_0^3)$; $v_{\max} = -8PL^3/(3E\pi d_0^4)$

7.32 $R_A = 16.2$ kips up; $M_A = 10.8$ in-kips CCW

7.34 $R_A = 5P/2$; $\quad v(x) = \begin{cases} (P/12EI)[2(x-2L)^3 - 5(x-L)^3 - 9L^2x + 11L^3] & 0 \le x \le L \\ (P/12EI)[2(x-2L)^3 - 9L^2x + 11L^3] & L \le x \le 2L \end{cases}$

7.36 $\frac{dv}{dx}(L) = \frac{wL^3}{80EI}$; $R_A = \frac{61wL}{120}$ up; $M_A = 11wL^2/120$ CW

7.57 $v_A = -41wL^4/24EI$

7.61 $v_A = -PL^3/96EI$

7.64 $v_A = -wL^4/136EI$; $R_C = 11wL/17$; $M_C = 5wL^2/34$

7.67 $v(x) = (w/18EI)[2Lx^3 - 3L\langle x-L\rangle^3 - 10L^3x]$; $v(L) = -4wL^4/9EI$

7.71 $v(x) = (w/24EI)[x^4 - \langle x-L\rangle^4 - 4L\langle x-2L\rangle^3 - 12L^2\langle x-2L\rangle^2 - 40L^3x + 71L^4]$; $v(2L) = (wL^4)/(4EI)$

7.72 $v(x) = (P/12EI)[3Lx^2 - 3x^3 + 5\langle x-L\rangle^3]$; $R_A = 5P/2$

7.76 $v'(x_A) = -(wL^3/6EI)$; $v(x_A) = -(wL^4/8EI)$

CHAPTER 8

8.1 σ_{nn} is (C); τ_{nt} is positive

8.6 σ_{nn} is (C); τ_{nt} can't say

8.12 $\sigma_{nn} = 8.66$ ksi (C) $\tau_{nt} = 5.0$ ksi

8.15 Compression

8.19 $\sigma_{nn} = 50$ MPa (C); $\tau_{nt} = 40$ MPa

8.24 $\sigma_{nn} = 45.36$ ksi (C); $\tau_{nt} = 1.84$ ksi

8.26 $P_{\max} = 84.9$ lb

8.33 (a) $\sigma_{nn} = \sigma$ (T); $\tau_{nt} = 0$; (b) $\sigma_{nn} = 0$; $\tau_{nt} = -\sigma$

8.42 $\sigma_{nn} = 7$ MPa (T); $\tau_{nt} = -59.7$ MPa; $\sigma_1 = 75.2$ MPa (T); $\sigma_2 = 45.2$ MPa (C); $\sigma_3 = 0$; $\theta_1 = 69.2°$; $\tau_{\max} = 60.2$ MPa

8.44 $\sigma_{nn} = 45.4$ ksi (C); $\tau_{nt} = 1.84$ ksi; $\sigma_1 = 15.4$ ksi (T); $\sigma_2 = 45.4$ ksi(C); $\sigma_3 = 0$; $\theta_1 = 40.3°$; $\tau_{\max} = 30.4$ ksi

8.47 $\sigma_{nn} = 0.63$ ksi(C); $\tau_{nt} = -7.06$ ksi; $\sigma_1 = 0.62$ ksi (T); $\sigma_2 = 40.62$ ksi (C); $\sigma_3 = 12$ ksi (C) $\theta_1 = 128°$; $\tau_{\max} = 20.62$ ksi

8.50 $\sigma_1 = 67.9$ MPa (T); $\sigma_2 = 207.9$ MPa(C); $\sigma_3 = 0$; $\theta_1 = 78°$; $\tau_{\max} = 137.9$ MPa

8.54 $\sigma_{xx} = 7.54$ ksi(C); $\sigma_{yy} = 9.46$ ksi (C); $\tau_{xy} = 1.15$ ksi

8.60 $\sigma_{nn} = 16.5$ ksi (C); $\tau_{nt} = -9.55$ ksi

8.71 $P_{\max} = 30.6$ kN

CHAPTER 9

9.3 $\varepsilon_{nn} = -234.7\ \mu$; $\phi = 196.96\ \mu$rad CW

9.6 $\varepsilon_{nn} = 150\ \mu$; $\varepsilon_{tt} = 450\ \mu$; $\gamma_{nt} = -519.6\ \mu$

9.8 $\varepsilon_{nn} = -70.2\ \mu$; $\varepsilon_{tt} = -529.8\ \mu$; $\gamma_{nt} = 385.67\ \mu$

9.13 $\varepsilon_{nn} = -295.4\ \mu$; $\varepsilon_{tt} = 295.4\ \mu$; $\gamma_{nt} = -104.2\ \mu$

9.16 Sectors 8 and 2 or Sectors 4 and 6

9.31 $\varepsilon_1 = 659\ \mu$; $\varepsilon_2 = -459\ \mu$; $\varepsilon_3 = 0$; $\gamma_{\max} = 1118\ \mu$; $\theta_1 = 103.3°$; $\varepsilon_{nn} = 643.7\ \mu$; $\varepsilon_{tt} = -443.7\ \mu$; $\gamma_{nt} = -259.8\ \mu$

9.37 $\varepsilon_1 = 1246.5\ \mu$; $\varepsilon_2 = -196.5\ \mu$; $\varepsilon_3 = 0$; $\gamma_{\max} = 1443\ \mu$;

$\varepsilon_{xx} = 1027\ \mu$; $\varepsilon_{yy} = 23\ \mu$; $\gamma_{xy} = -1037\ \mu$

9.42 $\varepsilon_{xx} = -466\ \mu$; $\varepsilon_{yy} = 1266\ \mu$; $\gamma_{xy} = -1000\ \mu$

9.44 $\varepsilon_1 = 767.9\ \mu$; $\varepsilon_2 = -125\ \mu$; $\varepsilon_3 = -214.3\ \mu$; $\gamma_{\max} = 982.2\ \mu$; $\theta_1 = -26.57^o$

9.46 $\varepsilon_1 = 681.4\ \mu$; $\varepsilon_2 = -604.3\ \mu$; $\varepsilon_3 = 0$; $\gamma_{\max} = 642.9\ \mu$; $\theta_1 = 26.57^o$

9.49 $(\theta_1)_{strain} = 103.3^o$; $(\theta_1)_{stress} = 98.8°$

9.54 $\varepsilon_a = 33.49\ \mu$; $\varepsilon_b = 400\ \mu$; $\varepsilon_c = 166.5\ \mu$

9.58 $\varepsilon_a = 687.5\ \mu$; $\varepsilon_b = -406.3\ \mu$; $\varepsilon_c = -656.9\ \mu$

9.62 $\varepsilon_1 = 685.9\ \mu$; $\varepsilon_2 = -185.9\ \mu$; $\varepsilon_3 = -166.7$; $\gamma_{\max} = 871.8\ \mu$; $\theta_1 = 48.3°$

9.64 $\varepsilon = 392.9\ \mu$

9.68 $\varepsilon = 716.7\ \mu$

9.74 $\varepsilon = -112.5\ \mu$

CHAPTER 10

10.1 $\sigma_{nn} = 4.6$ ksi (C); $\tau_{nt} = -16.4$ ksi

10.4 $P = 60.76$ kN

10.6 $\varepsilon_a = 1696\ \mu$; $\varepsilon_b = -1176\ \mu$

10.12 $\varepsilon_a = 1333\ \mu$; $\varepsilon_b = -666.66\ \mu$

10.24 $(\sigma_{xx})_A = 0$; $(\sigma_{xx})_B = -\sigma_{\text{bend-}y} = 85.39MPa(C)$; $(\tau_{xz})_A = \tau_{\text{tor}} + \tau_{\text{bend-}y}$=42.89$MPa$;
$(\tau_{xy})_B = -\tau_{\text{tor}} = -25.62MPa$

10.27 $(\sigma_{xx})_A = 0$; $(\sigma_{xx})_B = -\sigma_{\text{bend-}y} = 222MPa(C)$; $(\tau_{xz})_A = \tau_{\text{bend-}y} = 17.27MPa$; $(\tau_{xy})_B = 0$

10.30 $(\sigma_{max})_A = 102.7MPa(T)\ or\ (C)$; $(\sigma_{max})_B = 137.33MPa(C)$; $(\tau_{max})_A = 51.35MPa$;
$(\tau_{max})_B = 91.79MPa$;

10.35 $(\sigma_{xx})_A = 23.1$ ksi (C); $(\tau_{xy})_A = -7.2$ ksi

10.36 $\sigma_{nn} = 8219$ psi (C); $\tau_{nt} = 13180$ psi

10.44 $P_{\max} = 4.3$ kN

10.48 $w = 791.2$ N/m

10.53 $\sigma_{BD} = \sigma_{CE} = 5.13$ psi (C); $\sigma_{BC} = 10$ psi (T); $\sigma_{AB} = 115.4$ psi (C)

10.55 $W_{\max} = 67$ lb

10.62 $R_o = 2.405$ in

10.65 (a) $P_{\max} = 5$ kN; (b) $P_{\max} = 5.75$ kN

10.66 $P_{\max} = 9.5$ kips

10.69 $K = 1.22$

CHAPTER 11

11.2 $P_{cr} = 5/4$ kL

11.6 $P_{cr} = 153.3$ lb

11.15 $L/r = 72.7$; $P_{cr} = 215.4$ kip; $\sigma_{cr} = 3.36$ ksi (C)

11.21 $K = 1.106$

11.25 $K = 3.633$

11.40 $L_{\max} = 42$ in

11.43 $w_{\max} = 12$ kN/m; $K_{BD} = 2.3$

11.55 $\sigma_{\max} = 2.68$ ksi (C); $v_{\max} = 0.0458$ in

11.59 $L_{\max} = 2.09$ m

11.63 $P_{\max} = 39.45$ kip

11.64 $P_{cr} = 17.0$ kip

FORMULA SHEET

$$\sigma_{av} = \boldsymbol{N}/A \qquad \tau_{av} = \boldsymbol{V}/A \qquad \sigma_{ij} = \lim_{\Delta A_i \to 0}\left(\frac{\Delta F_j}{\Delta A_i}\right)$$

$$\varepsilon = \frac{L_f - L_o}{L_o} \qquad \varepsilon = \frac{\delta}{L_o} \qquad \varepsilon = \frac{u_B - u_A}{x_B - x_A} \qquad \gamma = \pi/2 - \alpha \qquad \varepsilon_{xx} = \frac{du(x)}{dx}$$

$$\varepsilon_{xx} = [\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})]/E \qquad \gamma_{xy} = \tau_{xy}/G \qquad G = \frac{E}{2(1+\nu)}$$

$$\sigma_{xx} = [\varepsilon_{xx} + \nu\varepsilon_{yy}]\frac{E}{(1-\nu^2)} \qquad \varepsilon_{zz} = -\left(\frac{\nu}{1-\nu}\right)(\varepsilon_{xx} + \varepsilon_{yy})$$

$$\frac{du}{dx} = \frac{\boldsymbol{N}}{EA} \qquad u_2 - u_1 = \frac{\boldsymbol{N}(x_2 - x_1)}{EA} \qquad \delta = \frac{\boldsymbol{N}L}{EA} \qquad \sigma_{xx} = \frac{\boldsymbol{N}}{A}$$

$$\sigma_{\theta\theta} = \frac{pR}{t} \qquad \sigma_{xx} = \frac{pR}{2t} \qquad \sigma = \frac{pR}{2t}$$

$$\frac{d\phi}{dx} = \frac{\boldsymbol{T}}{GJ} \qquad \phi_2 - \phi_1 = \frac{\boldsymbol{T}(x_2 - x_1)}{GJ} \qquad \tau_{x\theta} = \frac{\boldsymbol{T}\rho}{J}$$

$$\boldsymbol{M}_z = EI_{zz}\frac{d^2v}{dx^2} \qquad \sigma_{xx} = -\left(\frac{\boldsymbol{M}_z y}{I_{zz}}\right) \qquad \tau_{xs} = -\left(\frac{\boldsymbol{V}_y Q_z}{I_{zz}t}\right)$$

$$\sigma_{xx} = -\left(\frac{\boldsymbol{M}_y z}{I_{yy}}\right) \qquad \tau_{xs} = -\left(\frac{\boldsymbol{V}_z Q_y}{I_{yy}t}\right)$$

$$\boldsymbol{V}_y = -\boldsymbol{V} \qquad \frac{d\boldsymbol{V}}{dx} = p \qquad \frac{d\boldsymbol{M}_z}{dx} = V \qquad \boldsymbol{V}_2 = \boldsymbol{V}_1 + \int_{x_1}^{x_2} p\,dx \qquad \boldsymbol{M}_2 = \boldsymbol{M}_1 + \int_{x_1}^{x_2} \boldsymbol{V}\,dx$$

$$\sigma_{nn} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \qquad \tau_{nt} = -\sigma_{xx}\cos\theta\sin\theta + \sigma_{yy}\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{(\sigma_{xx} - \sigma_{yy})} \qquad \sigma_{1,2} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} \qquad \tau_{max} = \max\left(\left|\frac{\sigma_1 - \sigma_2}{2}\right|, \left|\frac{\sigma_2 - \sigma_3}{2}\right|, \left|\frac{\sigma_3 - \sigma_1}{2}\right|\right)$$

$$\varepsilon_{nn} = \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \qquad \gamma_{nt} = -2\varepsilon_{xx}\sin\theta\cos\theta + 2\varepsilon_{yy}\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\tan 2\theta_p = \frac{\gamma_{xy}}{(\varepsilon_{xx} - \varepsilon_{yy})} \qquad \varepsilon_{1,2} = \frac{(\varepsilon_{xx} + \varepsilon_{yy})}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} \qquad \frac{\gamma_{max}}{2} = \max\left(\left|\frac{\varepsilon_1 - \varepsilon_2}{2}\right|, \left|\frac{\varepsilon_2 - \varepsilon_3}{2}\right|, \left|\frac{\varepsilon_3 - \varepsilon_1}{2}\right|\right)$$

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$\eta_C = \frac{4r}{3\pi} \qquad I = \frac{1}{12}ab^3 \qquad I = \frac{1}{4}\pi r^4 \qquad J = \frac{1}{2}\pi r^4$$